Dispersion of Stoneley waves through the irregular common interface of two hydrostatic stressed MTI media

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Abstract: The present work deals with the mathematical inspection of Stoneley wave propagation through the corrugated irregular common interface of two dissimilar magneto-elastic transversely isotropic (MTI) half-space media under the impression of hydrostatic stresses. For the enumeration of the Lorentz’s force besmeared in the structure, generalized Ohm’s law and Maxwell’s equation have been considered. The interior deformations are calculated analytically to obtain the wave frequency equation using prescribed boundary conditions. To investigate the impacts of irregularity and various affecting parameters such as magnetic couplings and hydrostatic stresses on the wave propagation, frequency curves are framed-up for the phase velocity of the wave.

Keywords Stoneley wave; magneto-elasticity; hydrostatic stress; transversely isotropic; corrugation; Maxwell’s equation.

Nomenclature:

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1. Introduction

Earth is the bizarre collection of elastic media. For a greater extent information on the composition of Earth’s interior has been procured through seismic body waves. Therefore for the theoretical inspection, one has taken into account numerous numbers of appropriate models. These models provide indirect data of internal structure of Earth and may help seismologists, geophysicists to figure out various physical properties, state and configuration of the interior of Earth’s medium and to explore the valuable and unknown materials beneath the Earth.

Stoneley [1] elucidated the conceivable presence of another type of waves (after him known as Stoneley waves) identical to surface waves propagating through interfaces of either solid and liquid medium or two elastic media and obtained the secular relation for the wave and inferred that the condition of existence of these kinds of waves. The variation in the propagation attributes of Stoneley waves in a fluid filled media has been well examined by Ashour [2]. Stoneley wave dispersion equation is determined by Abo-Dahab [3] in magneto-thermoelastic materials. Tiwana et al. [4] investigated the consequences of spherical wave diffraction due to a point source from a PEMC half plane.

The Earth’s internal layer is made of various types of substance having non-identical type of properties like homogeneous, heterogeneous, transversely isotropic and orthotropic etc. Transversely isotropic substances are the particular type of orthotropic substances having symmetry about an axis that is normal to a plane of isotropy. The hexagonal crystals exist in the collection of transversely isotropic solids. Investigations over the wave regulations in transversely isotropic and orthotropic (fiber-reinforced [5]) materials are quite important in the improvement of many principal works on the mechanics. Recently, Alam et al. [6] and

\[ \tilde{h} \text{: Perturbed magnetic field vector} \]

Realistically, the Earth’s surface is not continuous as well as regular plane. The interfaces of two adjacent layers are irregular in nature and complicated too. Irregular interfaces are in many forms i.e. corrugation (undulated), parabolic, rectangular etc. Hence, it becomes mandatory to consider the impact of irregularity while studying the propagation of Stoneley waves. Various authors have been investigated seismic waves through irregular media, notable among them are Vishwakarma and Xu [9], Singh [10], Alam et al. [11] and Saroj et al. [12].


The magneto-elastic materials exhibit the magneto and elastic properties. The Earth’s layer consists of various kinds of sedimentary, volcanic and metamorphic rocks. Some ferromagnetic minerals like nickel, iron, cobalt etc., having the ability to generate magnetic field. Rayleigh wave Propagation in magneto-elastic materials is entirely interesting due to its numerous application and various properties in a large extent of fields: geophysics, geotectonics, seismology, astrophysics, acoustic, defectoscopy, optics. Said [16], Majhi et al. [17], Shaw et al. [18], Sahu et al. [19] carried considerable amount of research work in this field.

In the present paper, we investigate Stoneley wave propagation through the corrugated irregular common interface of two dissimilar magneto-elastic, transversely isotropic half-space media. In the 2nd section, basic equations are elaborated using Lame’s potential method. To get the general solution, those equations are enlightened. In 4th, 5th and 6th section, we obtain the particular solutions. We apply those solutions at the corrugated joint between two dissimilar media with the suitable boundary condition to get the dispersion relation of Stoneley waves in magneto-elastic, transversely isotropic media. The outcomes acquired are computed numerically and illuminated graphically in 8th section. Inferring observations are delivered in the last section.

2. Basic Equations
The equations of motion for a transversely isotropic elastic half-space with magnetic effect, having hydrostatic state stress are given as follows by Biot [20]:
\[
\tau_{i,j} + (\vec{J} \times \vec{B})_i - p_i \nabla^2 u_i = \rho_i \frac{\partial^2 u_i}{\partial t^2}
\]  
(1)

Taking into account that the assumed half space is perfect electric conductor, we consider the Maxwell’s equations in linear form have been taken into account for the electromagnetic field in the absence of displacement current (Mukhopadhyay [21])
\[ \text{curl} \vec{h} = j, \quad \text{curl} \vec{E} = -\mu_e \frac{\partial \vec{h}}{\partial t}, \quad \text{div} \vec{h} = 0 \]  \tag{2}

Where

\[ \vec{h} = \text{curl} \left( \vec{u} \times \vec{H}_0 \right) \quad \text{and} \quad \vec{H} = \vec{H}_0 + h(x,z,t) \]  \tag{3}

Here we consider that MTI half-space is under constant \( H_0 \) acts on y axis.

This wave is polarised in the xz-plane. Therefore we have \( u \) and \( w \) are not zero but \( v = 0 \). Also we have \( u \) and \( w \) are independent of \( y \) i.e.

\[ u = u(x,z,t), \quad w = w(x,z,t), \quad v = 0 \quad \text{and} \quad \frac{\partial}{\partial y} \equiv 0 \]  \tag{4}

Using Eqs. (4) and (1) into two dimensional form, we have

\[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - p_i \nabla^2 u + F_1 = \rho \frac{\partial^2 u}{\partial t^2} \]  \tag{5}

and

\[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - p_i \nabla^2 w + F'_3 = \rho \frac{\partial^2 w}{\partial t^2} \]  \tag{6}

Where \( F_i = (J \times B)_i \)

The non-vanishing stress components for transversely isotropic media in xz-plane are (Anderson [22])

\[ \tau_{xx} = c_{11} \frac{\partial u}{\partial x} + c_{13} \frac{\partial w}{\partial z}, \quad \tau_{xz} = c_{44} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \tau_{zz} = c_{13} \frac{\partial u}{\partial x} + c_{33} \frac{\partial w}{\partial z} \]  \tag{7}

Substituting Eqs. (2), (3) and (7) into Eqs. (5) and (6), we obtain

\[ \left( c_{11} - p_i + 2\mu_e H_0^2 \right) \frac{\partial^2 u}{\partial x^2} + \left( c_{13} + c_{44} + 2\mu_e H_0^2 \right) \frac{\partial^2 w}{\partial x \partial z} + \left( c_{44} - p_i \right) \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial t^2} \]  \tag{8}

and

\[ \left( c_{44} - p_i \right) \frac{\partial^2 w}{\partial x^2} + \left( c_{13} + c_{44} + 2\mu_e H_0^2 \right) \frac{\partial^2 u}{\partial x \partial z} + \left( c_{33} - p_i + 2\mu_e H_0^2 \right) \frac{\partial^2 w}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2} \]  \tag{9}

3. Formulation of the problem

For the study of Stoneley wave, we consider a mathematical model in Fig.1 consisting of a transversely magneto elastic isotropic semi-infinite media under hydrostatic stress \( M_1 : -\infty \leq z \leq \xi (x) \) and \( M_2 : \xi (x) \leq z \leq \infty \) having different elastic properties and corrugation at interface between two semi-infinite media as displayed. We choose such a Cartesian system where the x-axis is in the parallel direction of the wave traversal and z-axis is vertically downwards. \( \xi (x) \) is periodic and continuous function of \( x \). To exhibit the common surface as represented in Fig.1, the suitable Fourier series expansion of the function are given by Singh [10]
\[ \xi(x) = \sum_{n=1}^{\infty} \left( \xi_n e^{inx} + \xi_{-n} e^{-inx} \right) \]

where \( \xi_n \) and \( \xi_{-n} \) are the coefficients of \( n \)th order Fourier series expansion such that

\[
\xi_{\pm n} = \begin{cases} 
\frac{a}{2}, & \text{for } n=1 \\
\frac{A_n + B_n}{2}, & \text{for } n=2,3,4,\ldots
\end{cases}
\]

With help of aforesaid declarations of \( \xi_n \) and \( \xi_{-n} \), \( \xi(x) \) are given by Singh [10]

\[ \xi(x) = a \cos(\alpha x) + \sum_{n=2}^{\infty} \left[ A_n \cos(n\alpha x) + B_n \sin(n\alpha x) \right] \]

4. Solution of the upper half-space

According to Helmholtz’s theorem, the displacement vector \( \ddot{u} \) can be inscribed in respect of the displacement potentials \( \varphi \) and \( \psi \) are given by

\[ \ddot{u} = \nabla \varphi + \nabla \times \psi \]

Eq. (10) reduces to

\[ u_1(x,z,t) = \frac{\partial \varphi_1}{\partial x} - \frac{\partial \psi_1}{\partial z} \quad \text{and} \quad w_1(x,z,t) = \frac{\partial \varphi_1}{\partial z} + \frac{\partial \psi_1}{\partial x} \]

Substituting Eq. (11) into Eqs. (8) and (9), we get the following equations

\[
\begin{align*}
(c_{13} - p_i + 2\mu_i H_0 \cdot \xi) \frac{\partial^2 \varphi_1}{\partial x^2} + (c_{13} + 2c_{44} - p_i + 2\mu_i H_0 \cdot \xi) \frac{\partial^2 \varphi_1}{\partial z^2} &= \rho_i \frac{\partial^2 \varphi_1}{\partial t^2} \\
(c_{13} - c_{44} - p_i) \frac{\partial^2 \psi_1}{\partial x^2} + (c_{44} - p_i) \frac{\partial^2 \psi_1}{\partial z^2} &= \rho_i \frac{\partial^2 \psi_1}{\partial t^2} \\
(c_{44} - p_i) \frac{\partial^2 \psi_1}{\partial x^2} + (c_{33} - c_{44} - p_i) \frac{\partial^2 \psi_1}{\partial z^2} &= \rho_i \frac{\partial^2 \psi_1}{\partial t^2} \\
(c_{13} + 2c_{44} - p_i + 2\mu_i H_0 \cdot \xi) \frac{\partial^2 \varphi_1}{\partial x^2} + (c_{33} + 2\mu_i H_0 \cdot \xi) \frac{\partial^2 \varphi_1}{\partial z^2} &= \rho_i \frac{\partial^2 \varphi_1}{\partial t^2}
\end{align*}
\]

Eqs. (12) and (15) express the compressive wave associated with \( x \) and \( z \) directions respectively, and Eqs. (13) and (14) express the shear wave associated with \( x \) and \( z \) directions respectively. Without a doubt, the body wave velocity are different in \( x \) and \( z \) directions. Since we consider stoneley wave propagation in \( x \) direction only, we confine our focus only to Eqs. (12) and (14).

For the harmonic travelling wave, propagating along \( x \) direction, we pursue analytical solution of Eqs. (12) and (14) in the following form

\[ \left[ \varphi_1, \psi_1 \right](x,z,t) = \left[ \varphi_1(z), \psi_1(z) \right] \exp \left[ ik(x-ct) \right] \]

Substituting from Eq. (16) into Eqs. (12) and (14), we obtain
\[ (D^2 - s_1^2) \phi_1(z) = 0 \]  
and  
\[ (D^2 - s_2^2) \psi_1(z) = 0 \]

where, \( D^2 = \frac{d^2}{dz^2} \) and \( s_1, s_2 \) are described in Appendix.

Therefore, the solution of Eqs. (17) and (18) are given by

\[ \phi_1(z) = A \exp(s_1 z) + B \exp(-s_1 z) \]  
and  
\[ \psi_1(z) = C \exp(s_2 z) + D \exp(-s_2 z) \]

Therefore the potential functions \( \phi_1 \) and \( \psi_1 \) are given by

\[ \phi_1(x, z, t) = \left\{ A \exp(s_1 z) + B \exp(-s_1 z) \right\} \exp[i k (x - ct)] \]  
and  
\[ \psi_1(x, z, t) = \left\{ C \exp(s_2 z) + D \exp(-s_2 z) \right\} \exp[i k (x - ct)] \]

For upper half-space, we have

\[ \phi_1(x, z, t) = A \exp(s_1 z) \exp[i k (x - ct)] \]  
and  
\[ \psi_1(x, z, t) = C \exp(s_2 z) \exp[i k (x - ct)] \]

Therefore for the upper half-space, the solutions are given by

\[ u_1(x, z, t) = \left[i k A + s_1 C \exp(s_2 z) - s_2 C \exp(s_1 z) \right] \exp[i k (x - ct)] \]  
and  
\[ w_1(x, z, t) = \left[ s_1 A + i k C \exp(s_2 z) + s_2 C \exp(s_1 z) \right] \exp[i k (x - ct)] \]

5. Solution of the lower half-space

On solving the lower half-space similarly, we get the potential function as

\[ \phi_2(x, z, t) = \left\{ A \exp(r_1 z) + B \exp(-r_1 z) \right\} \exp[i k (x - ct)] \]  
and  
\[ \psi_2(x, z, t) = \left\{ C \exp(r_2 z) + D \exp(-r_2 z) \right\} \exp[i k (x - ct)] \]

where, \( r_1, r_2 \) are given in Appendix.

For lower half-space, we have

\[ \phi_2(x, z, t) = B \exp(-r_1 z) \exp[i k (x - ct)] \]  
and  
\[ \psi_2(x, z, t) = D \exp(-r_2 z) \exp[i k (x - ct)] \]

Therefore for the lower half-space, the solutions are given by

\[ u_2(x, z, t) = \left[i k B + r_1 D \exp(-r_2 z) - r_2 D \exp(-r_1 z) \right] \exp[i k (x - ct)] \]  
and  
\[ w_2(x, z, t) = \left[ -r_1 B + i k D \exp(-r_2 z) + r_2 D \exp(-r_1 z) \right] \exp[i k (x - ct)] \]
6. Boundary Conditions

The suitable boundary conditions must be satisfied at the common corrugated interface [i.e., at \( z = \xi(x) = a \cos(\alpha x) \)]

a) Displacement components are continuous
\[ u_1 = u_2 \quad \text{and} \quad w_1 = w_2 \]

b) Stresses are continuous
\[ (\tau_{zz}^1 + \tau_{zz}^{M_1}) - \xi'(x)(\tau_{xx}^1 + \tau_{xx}^{M_1}) = (\tau_{zz}^2 + \tau_{zz}^{M_2}) - \xi'(x)(\tau_{xx}^2 + \tau_{xx}^{M_2}) \]
\[ (\tau_{xz}^1 + \tau_{xz}^{M_1}) - \xi'(x)(\tau_{xx}^1 + \tau_{xx}^{M_1}) = (\tau_{xz}^2 + \tau_{xz}^{M_2}) - \xi'(x)(\tau_{xx}^2 + \tau_{xx}^{M_2}) \]

7. Dispersion relation

Using the value of \( u_1, w_1, u_2, w_2 \) into the aforesaid BCs, we get following homogeneous algebraic system of equations for unknowns \( A_1, B_2, C_1, D_2 \). These algebraic system of equations can be expressed with help of matrix-vector term as follows

\[ Z_{4 \times 4} Y_{4 \times 1} = 0 \]  (33)

where \( Z_{4 \times 4} \) is a four by four matrix given by \( Z_{4 \times 4} = [\alpha_{ij}], (i,j=1,2,...,6) \); \( Y_{4 \times 1} \) is a four by one column matrix, described as \( Y_{4 \times 1} = [A_1 \ B_2 \ C_1 \ D_2]^T \); \( \alpha_{ij} \)'s values are given in the appendix.

For non-trivial solution of aforementioned homogeneous system of equations, the determinant of given matrix should be zero that is \( |\alpha_{ij}| = 0 \) and this gives the required dispersion relation of stoneley waves in presumed structure.

8. Numerical Conclusion and discussion

An interpretive inspection is made to emphasize the effects of various elastic parameters against the velocity of Stoneley wave propagation. The dispersion Eq. (33) is deployed for numerical interpretation and graphical depiction.

i. For the upper half-space \( M_1 \), we have taken example of Magnesium material (Rehman [23])
\[ c_{11} = 0.597 \times 10^{11} \text{ N/m}^2, \quad c_{13} = 0.217 \times 10^{11} \text{ N/m}^2, \quad c_{33} = 0.617 \times 10^{11} \text{ N/m}^2, \quad c_{44} = 0.164 \times 10^{11} \text{ N/m}^2, \quad \rho_1 = 1740 \text{ kg/m}^3 \]

ii. For the lower half-space \( M_2 \), Beryllium material has been taken (Ding et al. [24])
\[ \tilde{c}_{11} = 2.293 \times 10^{11} \text{ N/m}^2, \quad \tilde{c}_{13} = 0.14 \times 10^{11} \text{ N/m}^2, \quad \tilde{c}_{33} = 3.364 \times 10^{11} \text{ N/m}^2, \quad \tilde{c}_{44} = 1.625 \times 10^{11} \text{ N/m}^2, \quad \rho_2 = 1040 \text{ kg/m}^3 \]

An aim is to made to study the consequences of magneto-elastic coupling parameter for upper and lower half-space
\[ m_u = \frac{\mu_l H_0^2}{c_{11}}, \quad l_u = \frac{\bar{H}_0^2}{\bar{c}_{11}} \] respectively, hydrostatic stress for upper
\[ P_1 = \frac{p_1}{c_{11}}, \quad P_2 = \frac{p_2}{\bar{c}_{11}} \] respectively and position parameter \( t = \frac{x}{H} \) for our assumed model, we have plotted dimensionless phase velocity \((c/c_l)\) against
dimensionless common surface corrugation parameter \((\alpha a)\) in each Figs. 2-6 for their various values. The vertical axis is represented by dimensionless phase velocity \((c/c_1)\) and dimensionless common surface corrugation parameter \((\alpha a)\) is represented by horizontal axis in every Figure.

In Fig. 2, the consequence of \(m_H\) on the Stoneley waves phase velocity are demonstrated. Fig. 2 provides the fluctuation of the phase velocity of Stoneley wave against the dimensionless common surface corrugation parameter for different values for \(m_H\). Considering different values of \(m_H\), we can see from these curves that phase velocity of Stoneley wave in Magnesium material is increasing with increasing value of \(m_H\).

The curves of Figure 3 are plotted for dimensionless phase velocity against the dimensionless common surface corrugation parameter for the different values of magneto-elastic coupling parameter \((l_H)\) for lower half-space. We can see from the set of curves (1, 2, 3, 4) that the magneto-elastic parameter \((l_H)\) has the proportional impacts on phase velocity in Beryllium material.

In Figure 4, the effect of hydrostatic stress parameter \((P_t)\) has been revealed. From this figure it has been noticed that for a particular dimensionless corrugation parameter, the hydrostatic stress parameter has discouraging effects on the dimensionless phase velocity in Magnesium material.

Figure 5 elaborates the effect of hydrostatic stress parameter \((P_2)\) associated with lower half-space. From the figure, we observed that for the frequency region \(\alpha a < 1.3\) phase velocity increases uniformly, whereas for the frequency region \(\alpha a > 1.3\) phase velocity decreases as the magnitude of hydrostatic stress parameter increases.

The curves of figure 6 illuminates the dispersion curves of Stoneley wave when position parameter \((t)\) has been taken into account for the variation. The meticulous observations of the figure depicts that the phase velocity decreases.

A correlative study of dispersion curves due to different cases is illustrated in Figure 7. Curve 1 depicts the case when the upper half-space is free from magnetic field \((m_H=0)\). Curve 2 implicates the case when the lower half-space is free from magnetic field \((l_H=0)\). Curve 3 elucidates to the case when the upper half-space is free from hydrostatic state of stress \((P_1=0)\). Curve 4 clarifies the case when the lower half-space is free from hydrostatic state of stress \((P_2=0)\). It can be perceived that Curve 3 \((m_H=0)\) supports more phase velocity, whereas the Curve 1 \((P_1=0)\) supports less phase velocity as compare to all other cases.

Considering the dependence of phase velocity \((c/c_1)\) on magneto-elasticity and hydrostatic stresses, surface plot of phase velocity against varying hydrostatic stress parameters \((P_1\) and \(P_2\)) and magneto-elasticity parameters \((m_H\) and \(l_H\)) have been shown in Figs. 8 and 9 respectively.

9. Conclusions
An analytical approach is accustomed to inspect the Stoneley wave propagation through the corrugated interface of two MTI half-spaces. The dispersion equation which figure out the
velocity of Stoneley waves has been obtained. All the wave fronts are significantly exerted by different parameters and are exhibited graphically using 2D and 3D plots. From the theoretical and numerical discussion we can remark as follows:

i. The magneto-elastic coupling parameters for both half-spaces have increasing trend on the phase velocity. Absence of magneto-elastic parameter for the upper half-space has a great effect on phase velocity whereas absence of magneto-elastic coupling parameter has a less effect on phase velocity.

ii. The hydrostatic stress parameter for upper half-space has inverse effect on the phase velocity whereas the hydrostatic stress parameter for lower half-space has the mixed variation on the phase velocity.

References:


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**Appendix:**

\[ c_1 = \sqrt{\frac{\alpha_1}{\rho_1}}; \quad s_1^2 = k^2 \frac{c_{11} - p_1 + 2\mu H_0^2 - \rho c^2}{c_{13} + 2c_{44} - p_1 + 2\mu H_0^2}; \quad r_1^2 = k^2 \frac{\bar{c}_{11} - p_1 + 2\bar{\mu} \bar{H}_0^2 - \rho_c c^2}{\bar{c}_{13} + 2\bar{c}_{44} - p_2 + 2\bar{\mu} \bar{H}_0^2} \]

\[ c_2 = \sqrt{\frac{\alpha_2}{\rho_2}}; \quad s_2^2 = k^2 \frac{c_{44} - p_1 - \rho c^2}{c_{33} - c_{13} - c_{44} - p_1}; \quad r_2^2 = k^2 \frac{\bar{c}_{44} - p_2 - \rho c^2}{\bar{c}_{33} - \bar{c}_{13} - \bar{c}_{44} - p_2} \]

\[ \alpha_{11} = ik \exp(s, a \cos(\alpha x)); \quad \alpha_{12} = -ik \exp(-r, a \cos(\alpha x)); \quad \alpha_{13} = -s \exp(s, a \cos(\alpha x)) \]

\[ \alpha_{14} = -r \exp(-r, a \cos(\alpha x)); \quad \alpha_{21} = s \exp(s, a \cos(\alpha x)); \quad \alpha_{22} = r \exp(-r, a \cos(\alpha x)) \]

\[ \alpha_{31} = \left[s_1^2 c_{33} - k^2 c_{13} + \mu H_0^2 (s_1^2 - k^2) + 2iks aac_{44} \sin(\alpha x)\right] \exp(s, a \cos(\alpha x)) \]

\[ \alpha_{32} = \left[\bar{c}_{13} k^2 - \bar{c}_{33} r_1^2 - \bar{\mu} \bar{H}_0^2 (r_1^2 - k^2) + 2ikaar \bar{c}_{44} \sin(\alpha x)\right] \exp(-r, a \cos(\alpha x)) \]

\[ \alpha_{33} = \left[iks_2 (c_{33} - c_{13}) - aac_{44} (s_2^2 + k^2) \sin(\alpha x)\right] \exp(s, a \cos(\alpha x)) \]

\[ \alpha_{34} = \left[ikr_2 (c_{33} - c_{13}) + aac_{44} (r_2^2 + k^2) \sin(\alpha x)\right] \exp(-r, a \cos(\alpha x)) \]

\[ \alpha_{41} = \left[2iks c_{44} + aa \sin(\alpha x) \left(s_1^2 c_{13} - k^2 c_{11} + \mu H_0^2 (k^2 - s_1^2)\right)\right] \exp(s, a \cos(\alpha x)), \]

\[ \alpha_{42} = \left[2ikaar \bar{c}_{44} - aa \sin(\alpha x) \left(r_1^2 \bar{c}_{13} - \bar{c}_{11} k^2 + \bar{\mu} \bar{H}_0^2 (k^2 - r_1^2)\right)\right] \exp(-r, a \cos(\alpha x)), \]

\[ \alpha_{43} = \left[ikaaas \sin(\alpha x) (c_{13} - c_{11}) - c_{44} (s_2^2 + k^2)\right] \exp(s, a \cos(\alpha x)), \]

\[ \alpha_{44} = \left[\bar{c}_{44} (r_2^2 + k^2) - ikaar \sin(a x) (\bar{c}_{11} - \bar{c}_{13})\right] \exp(-r, a \cos(\alpha x)) \]
Figure 1: Geometry of the problem
Figure 2: Influence of Stoneley wave phase velocity \((c/c_1)\) for increasing values of \(m_H\) against corrugation parameter \((aa)\).

Figure 3: Influence of Stoneley wave phase velocity \((c/c_1)\) for various values of \(l_H\) against corrugation parameter \((aa)\).

Figure 4: Stoneley wave phase velocity \((c/c_1)\) for different values of \(P_1\) against corrugation parameter \((aa)\).

Figure 5: Stoneley wave phase velocity \((c/c_1)\) for increasing values of \(P_2\) against corrugation parameter \((aa)\).

Figure 6: Stoneley wave phase velocity \((c/c_1)\) for different values of position parameter \((t)\) against corrugation parameter \((aa)\).

Figure 7: Stoneley wave phase velocity \((c/c_1)\) in the absence of different parameters against corrugation parameter \((aa)\).
**Figure 8:** Variation of Stoneley wave velocity \((c/c_1)\) with respect to \(P_1\) and \(P_2\)

**Figure 9:** Variation of Stoneley wave velocity \((c/c_1)\) with respect to \(m_H\) and \(l_H\)