Measuring skewness: we don not assume much

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Abstract: Skewness plays a vital role in different engineering phenomena so it is desired to measure this characteristic accurately. Several measures to quantify the extent of skewness in distributions have been developed over the course of history but each measure has some serious limitations. Therefore, in this article, we propose a new skewness measuring functional, based on distribution function evaluated at mean with minimal assumptions and limitations. Four well recognized properties for an appropriate measure of skewness are verified and demonstrated for the new measure. Comparisons with the conventional moment-based measure are carried out by employing both functionals over range of distributions available in literature. Furthermore, the robustness of the proposed measure against unusual data points is explored through the application of influence function. The Mathematical findings are verified through meticulous simulation studies and further verified by real data sets coming from diverse fields of inquiries. It is witnessed that the suggested measure passes all the checks with distinction while comparing to the classical moment-based measure. Based on computational simplicity, applicability in more general environment and preservation of *c-ordering* of distribution, it may be considered as an attractive addition to the family of skewness measures.

Keywords: distribution function, mean, moment, influence function, skewness.

1. Introduction

As a pioneering effort, Pearson [1] introduced the concept of skewness and proposed a measure for skewness by standardizing the difference between the mean and the mode of a distribution. The comprehension of the idea of skewness immediately earned a core position in statistics and allied literature (see Kendall & Stuart [2]) and this concept has been extensively used in different fields (see for example: Firat e al. [3], Harvey and Sucarrat [4], Amaya et al.[5], Åstebro et al. [6], Ho and Yu [7], Decker et al. [8], Colacito [9], Ensthaler et al. [10], Amjadzadeh and Ansari-Asl [11], Taylor and Fang [12] and references cited therein). Zwet [13] introduced the concept of ordering two functions in relevance to skewness and brought the idea of *c-preceding* or *c-ordering* into the lime light. Till date, many researchers have paid tribute to the vibrancy of the concept of skewness by employing it to (i) develop tests of normality, (ii) investigate robustness of the standard normal theory procedure and, (iii) select a member of family such as from the Karl Pearson family (see Goldstein et al. [14] and Mardia [15]). A number of recent high-profile academic articles, including Aucremanne et al. [16], Brys et al. [17], Doane & Seward [18], Hosking [19] and Li et al. [20], from various fields of inquiry, witnessed the ongoing glamour of skewness in research community. It has been widely accepted that an appropriate measure of skewness, say $\gamma_{.}(X)$, must satisfy the following characteristics (see Arnold & Groeneveld [21].

i- For a symmetric distribution, $\gamma_{.}(X)=0$.

ii- An appropriate skewness measure is insensitive to linear transformation, that means, when dealing with transformation of the form $Y=cX+d$, we have $\gamma_{.}(cX+d)=\gamma_{.}(X)$, where c and d are subject to the conditions that $c>0$ and $-\infty<d<+\infty$.

iii- For $Y=-X$, $\gamma_{.}(Y)=-\gamma_{.}(X)$.

iv- While comparing two distributions with respect to skewness, if, $F_X(.)$ *c − preceedes* $G_Y(.)$, that is, if $F_X(.)<_c G_Y(.)$ then $\gamma_{.}(F)\le\gamma_{.}(G)$.

Several skewness functionals and their assessed performances under various situations are available in multidisciplinary research literature. For example, Brys et al. [22] highlighted the shortcomings of using the moment-based measures of skewness in presence of outliers. They compared several alternative robust measures of skewness based on quantiles, which are less sensitive to outliers. The authors rather suggested using the median, double median and triple median of actual data points in the range of median and quantiles to measure skewness of the data. A comparison revealed that the skewness measure, *"medcouple"* based on double median, is less sensitive to other moment or quantile-based measures of skewness.

Kim & White [23] compared the single-outlier robust quantile-based and octile-based measures of skewness, using the stock market SP & 500 index data. The authors, however, ranked their effort as a starting point for measuring skewness in financial market data and for further modeling of asset prices. Holgersson [24] proposed a modified version of the conventional measure of skewness based on third central moment. The author suggested using the difference between mean and median as a base for calculating the third central moment instead of using only the mean (which is extremely sensitive to outliers). Moreover, Tajuddin [25] extended the work done by Brys et al. [23] and investigated the *medcouple* as a robust measure of skewness. However, mixed results were found regarding the robustness of the *medcouple* based on the severity of skewness. Yet another evidence of ongoing efforts, based on midrange, mean, median and mode, can be witnessed in Altinay [26] for comparing a simple class of measures of skewness. Although, the use of range and midrange in the proposed skewness coefficient guarantees the insensitivity to the changes in location or scale of the data distribution, but in presence of outliers, the results might be misleading.

For discussion purpose, we list some of the well known measures suggested in literature.

$$S_k = (\mu - M)/\sigma$$

$$\gamma_1 = E(X-\mu)^3/\sigma^3$$

$$\gamma'_m = (\mu - m)/\sigma$$

$$\gamma_m = (\mu - m)/E|X-m|$$

$$\gamma_M = 1 - 2F(M).$$

Under the notion of simplicity, we use conventional notations of μ, m and M for mean, median and mode of distribution and σ, Q_1 and Q_3 for standard deviation, 1st and 3rd quartiles, respectively. Even being premium skewness functionals, their performance has been questionable with respect to certain features. For example, S_k and γ'_m fail to maintain the feature of *c-preceding* (see Arnold & Groeneveld [21]). Li et al. [20] noted that, in some scenarios, γ_1 can result in misguiding estimates altering the direction of skewness. The mode-based functional, γ_M, assumes uni-model distribution which limits its utility in real situations, whereas, γ_m is often not expressible as a simple function of parameters of distribution.

In this study, we propose a new skewness functional, based on cumulative distribution function (cdf) in relation to mean of a distribution. The inspiration of using distribution function as base of our measure is consistent with statistical intuitions. Ideally, a skewness measure should be a functional involving parameter(s) which influence shape of the distribution. It is worth noting here that distribution function alone can offer us different dynamics of distribution under consideration. Also, the arithmetic mean is a well celebrated measure of central tendency because of its capacity of using all the available information and amenability to further mathematical treatments. Furthermore, being a proxy of central tendency of data, it is also related to the skewness of data. Thus, using a skewness measure based on the cdf and the mean of the distribution is more appealing since both the mean and cdf of a distribution may help in explaining the shape of the distribution. Motivated by the work of above referred knowledgeable peers, this is what we intend to do in the sections as our contribution towards the body of skewness-oriented research.

In the following sections, we propose a skewness measure and establish its appropriateness as a valid measure of skewness by mathematically proving above designated characteristics (i)-(iv). By taking full advantage of available computational facilities, intense simulation-based investigation is carried out to verify the statistical strength of the new measure. For demonstration purposes, eight most commonly used distributions (Cauchy, F, T, Chi-square, Gamma, Rayleigh, Weibull and Beta) in statistical research are considered. Readers may notice the notion of choosing these distributions; they are general, offer wide range of applications spanning from data modeling to asymptotic theory, essential for hypothesis testing and for some of these distributions existing measures are either not calculable or have limited applicability. Additionally, performance of the proposed functional is assessed by using real data set on

Algeria's yearly fatality counts for the period 1997-2017 and survey data, collected by Pakistan Bureau of Statistics (PBS), on the respondents' degree of co-operation (non-co-operation) at district level covering 78,635 households, for the year 2014-205. For comparison purposes, we consider commonly practiced measure γ_1. Superiority of the measure γ_1 in comparison to others is well noticed in Arnold & Groeneveld [21] and Zwet [13] and we expect better performance of the proposed measure in comparison to the γ_1, thus scaling its utility higher in literature.

2. The Measure

Assume a random variable, X, having cdf $F(x)$, pdf $f(x)$ and mean μ. We define our functional as;

$$\gamma_p = 2F(\mu) - 1 \tag{1}$$

One may notice that our proposition of using value of cdf at mean as a major component of skewness quantifying measure stays consistent with the inherent tendency of mean to slide in the direction of skewness. To further strengthen the candidature of γ_p, we highlight that unlike other members of family of skewness measures, it neither assumes uni-modality of the distribution nor calculation of higher order moments. We only need the existence of mean of a distribution. Now, let us demonstrate that γ_p is an appropriate skewness measure by verifying that it holds all four properties (i)-(iv) mentioned in Arnold & Groeneveld [21] and Oja [27].

i- In case of symmetry, mean and median will be positioned at same place in population which leads to the fact that $F(\mu) = 0.5$. Based on this argument, it remains trivial to verify that for symmetric distributions, proposed skewness functional, γ_p, takes the value equal to zero.

ii- Let $Y = cX + d \Rightarrow \mu_y = c\mu_x + d$. Now, skewness functional in terms of Y is

$$\gamma_p(Y) = 2F(\mu_y) - 1 = 2\Pr[Y < \mu_y] - 1 = 2\Pr[cX + d < c\mu_x + d] - 1$$
$$= 2\Pr[X < \mu_x] - 1 = \gamma_p(X).$$

iii- Let $Y = -X \Rightarrow \mu_y = -\mu_x$, the skewness functional for variable Y is then written as, $\gamma_p(Y) = 2F(\mu_y) - 1 \Rightarrow \gamma_p(Y) = 2F(-\mu_x) - 1$. It is trivially verifiable that $F(-\mu_x) = 1 - F(\mu_x)$ and by using this argument, we get $\gamma_p(Y) = 2\left[1 - F(\mu_x)\right] - 1 = -\left[2F(\mu_x) - 1\right] = -\gamma(X)$.

iv- To prove this property, we need to show that $G(\mu_y) > F(\mu_x)$. It is easy to perceive that if $G_Y(.)$ is more skewed to the right than $F_X(.)$, then $\mu_y < \mu_x$. Keeping this argument in mind, Figure 1 reveals the following inequality.

$$G_Y(\mu_x) - G_Y(\mu_{y)} \le G_Y(\mu_x) - F_X(\mu_x)$$

$$\frac{d}{d\mu_y}\left(G_Y(\mu_x) - G_Y(\mu_y)\right) \le \frac{d}{d\mu_y}\left(G_Y(\mu_x) - F_X(\mu_x)\right)$$

$$\frac{d}{d\mu_y}\left(G_Y(\mu_x)\right) - \frac{d}{d\mu_y}\left(G_Y(\mu_y)\right) \le \frac{d}{d\mu_y}\left(G_Y(\mu_x)\right) - \frac{d}{d\mu_y}\left(F_X(\mu_x)\right)$$

$$-\frac{d}{d\mu_y}(G_Y(\mu_y)) \le -\frac{d}{d\mu_y}\left(F_X(\mu_x)\right)$$

$$\frac{d}{d\mu_y}(G_Y(\mu_y)) \ge \frac{d}{d\mu_y}\left(F_X(\mu_x)\right)$$

$$\frac{d}{d\mu_y}(G_Y(\mu_y)) \ge 0,$$

which is obvious. Hence our supposition is true. That is, we have $G_Y(\mu_x) - G_Y(\mu_y) \le G_Y(\mu_x) - F_X(\mu_x)$, and hence $G_Y(\mu_y) \ge F_X(\mu_x)$. This implies that $\gamma_p(F) = 2F_X(\mu_x) - 1 \le 2G_Y(\mu_y) - 1 = \gamma_p(G)$, which verifies the property (iv). Figure 1 below, depicts the notion of mathematical proof.

[**Insert Figure 1**]

In addition to the above discussed attributes, for positive skewness, $F(\mu) > 0.5 \Rightarrow 2F(\mu) - 1 > 0$ thus ensuring a positive value of the skewness measure, which at its extreme can take a value of $+1$. Likewise, dealing with negatively skewed distribution, $F(\mu) < 0.5 \Rightarrow 2F(\mu) - 1 < 0$ and therefore, resulting a negative value, with -1 indicating the extreme negatively skewed scenario. This discussion establishes well-defined, interpretable, and comparable skewness functional as we always have $-1 < \gamma_p < +1$ and $\gamma_p = 0$ when symmetry is witnessed, while taking *c-ordering* of distributions into account.

3. Finite Sample Behavior

This section is dedicated to explore the behavior of γ_p in finite sample along with the verification of the mathematical facts derived in the above section. We demonstrate the sustainability, interpretability, and comprehension of the proposed skewness functional by employing extensive simulations study. The notations used in this section are appended below.

$s(\bar{x})$: Empirical cdf at $\bar{x}$

$\overline{s(\bar{x})}$: Mean of the 10,000 empirical cdfs at $\bar{x}$

$\hat{\gamma}_p = 2s(\bar{x}) - 1$: Estimate of γ_p

$\bar{\hat{\gamma}}_p$: Mean of the 10,000 $\hat{\gamma}_p$

$\hat{\gamma}_1$: Estimate of γ_1

$\bar{\hat{\gamma}}_1$: Mean of the 10,000 $\hat{\gamma}_1$

We generate $10,000$ samples of different sizes $(n = 30, 50, 100 \text{ and } 500)$ for all the distributions mentioned in the introduction section. Under the notion of generality, various combinations of parameter(s) driving the extent of skewness are taken into account to evaluate the performance of the proposed measure in the situations of moderate to extreme skewness. To document the comparative spirit of γ_p, the finite sample behavior of existing measure of skewness, γ_1, is also presented, under the same above-mentioned environment. Tables 1-8, showcase the values averaged over $10,000$ repetitions highlighting the features mathematically established in section (2). In addition, interesting behavior of variance of the estimate of the proposed functional in comparison to that of γ_1 alongside the well celebrated *consistency* remains overwhelming throughout the study.

The results are compiled and interpreted in four sub-sections assembling the selected distributions with respect to their inherent attributes.

3.1. The Chi-square and Gamma Distributions

We start with documenting the computational findings for well recognized Chi-square and Gamma (both positively skewed) distributions. Tables 1 and 2, are designated to this purpose. As an obvious fact, with the increase in value of shape parameter, the extent of skewness decreases for both distributions. It is witnessed that regardless of the extent of skewness, both functionals (proposed and existing) result in their permissible ranges and thus remain interpretable. Further, the signs of estimates are consistent with the direction of skewness that is exhibited in population. Aligned with mathematical findings of section 2, the *c-preceding* characteristic is maintained by both the measures in every case without exception. Interesting and distinctive feature of the proposed statistic is the variability control which becomes more prominent with increased sample size. To make it vivid, we highlighted the variance of both measures for $n = 500$. One exception is observed in the case of Gamma distribution for $\alpha = 3$ and $\beta = 1$.

[**Insert Table 1**]

[**Insert Table 2**]

3.2. The Weibull and Beta Distributions

The altering effect of the parameters on skewness while working with these two distributions makes them very informative candidates to assess the performance of skewness functionals. Results are offered in Tables 3 and 4 for Weibull and Beta distributions, respectively.

Interpretability and *c-preceding* are maintained by both the functionals regardless of the change direction and extent of skewness. The variance behavior of proposed estimator is again appreciable in comparison with existing measure.

[**Insert Table 3**]

[**Insert Table 4**]

3.3. The Student's t and F Distributions

These two distinguished distributions are major players in hypotheses testing domain of statistics and their dynamic nature limits the applicability of γ_1 in certain conditions, such as, it is only estimable if t-distribution's $\text{d.f.} > 3$,whereas, F-distribution requires denominator $\text{d.f.} > 6$. These inherent functional complexities provide ideal grounds to test the utility of γ_p and thus demonstrating the superiority of this new functional.

Table 5 comprehends the simulated results in the case of t-distribution for various degrees of freedom dictating the calculability of γ_1. For comparison purposes, we report the $\bar{\hat{\gamma}}_1$ based on empirical evaluation even in the situations, where, γ_1 does not exist for population. Deteriorated performance of $\hat{\gamma}_1$ is evident through estimated average value for $\text{d.f.} \leq 3$ (highlighted variances are foretelling this fact). For $\text{d.f.} > 3$, our proposed functional still outperforms the existing one. The rapid convergence of $\bar{\hat{\gamma}}_p$ towards its true value is worth cheering, especially in feasibly large samples.

[**Insert Table 5**]

The F-distribution demands a more elaborative account. We comprehend our discussion of results in the light of four motivational factors; i- the γ_1 is not estimable, ii- the numerator and denominator degrees of freedom are equal, iii- the numerator d.f. are greater than denominator's d.f. $(v_1 > v_2)$ and iv- vice versa $(v_2 > v_1)$. In general, recommended *c-preceding* attribute is carried by both measures in all cases.

In first case, the estimability and closeness of $\hat{\gamma}_p$ to its population counterpart γ_p is noticeable and becomes more obvious as the sample size increases. The statistical strength can be assessed by comparing the variances of both functionals at every point.

In second case, other than variance comparison (which is consistent with the usual findings), the convergence of both functionals towards their respective population parameters is thought provoking. For explanatory purposes, consider the case when $v_1 = v_2 = 7$ for $n = 500$. The true value of $\gamma_1 = 10.1559$ and its estimated simulated value is, $\bar{\hat{\gamma}}_1 = 4.2547$, whereas, $\gamma_p = 0.3318$ and corresponding $\bar{\hat{\gamma}}_p = 0.3299$. This is yet another evidence in the favor of γ_p projecting it as a serious member of the family of skewness measures. Similar patterns are observed in third and fourth cases; the glimpses of the performance are offered by highlighting the scenarios where $v_1 = 3, v_2 = 7$ and $v_1 = 9, v_2 = 7$.

[**Insert Table 6**]

3.4. The Rayleigh and Cauchy Distributions

Both of these distributions are interesting members of our study, as extent of skewness in Rayleigh distribution remains unchanged with respect to any amendment in parameter while Cauchy distribution does not allow the existence of usual skewness measure, γ_1, in any circumstances. The findings related to Rayleigh distribution are offered in Table 7 below. In increased sample sizes, the estimates approach to population measures with consistent manner. The variance behavior of $\hat{\gamma}_p$ is again impressive in comparison to $\hat{\gamma}_1$.

[**Insert Table 7**]

[**Insert Table 8**]

For Cauchy distribution, the true value of γ_p is deducted by replacing $F(\mu)$ with $F(\text{median})$ (conceptually it is not forbidden because of the symmetry of the distribution). The evaluation of $\hat{\gamma}_p$ (and resultant $\bar{\hat{\gamma}}_p$), is still based on empirical cdf encapsulating mean of the data. The results in Table 8 highlight the superiority of the proposed functional in terms of stability and interpretability. In every considered situation, the estimated value of $\hat{\gamma}_p$ remains very close to population value projecting inherited symmetry of the distribution. The behavior of variance is, however, altered; it increases with increase in sample size (surely, enormously less than that of $\hat{\gamma}_1$). This fact at this stage is attributed to the well-recognized complex functional form of the Cauchy distribution and is left for future inquiries.

4. Influence Function Comparison

The influence function, being the directional derivative of functional, is usually utilized to compare the extent of robustness inherent in functionals against unusual points (see Arnold & Groeneveld [21]). In its general form, for a functional $T(.)$, we can write influence function as;

$$T(F;G) = \lim_{\epsilon \downarrow 0} \frac{T(F_\epsilon) - T(F)}{\epsilon},$$

where, $F_\epsilon = \epsilon G + (1-\epsilon)F$ and $T(F;G)$ is directional derivative of functional T at F in the direction of G. Under the assumption of symmetry of distribution, say F around '0' with differentiable density function, let us consider $G(x) = (1/2b\sigma)\left[x + (b-a)\sigma\right]$ where, σ^2 is the variance of F and X is bounded between $\left[(a-b)\sigma, (a+b)\sigma\right]$ with $-\infty < a < \infty$ and $b > 0$. The authors provided the general form of F_ϵ as,

$$F_\epsilon = \begin{cases} (1-\epsilon)F(x), & x < (a-b)\sigma \\ \epsilon(1/2b\sigma)\left[x+(b-a)\sigma\right] + (1-\epsilon)F(x), & (a-b)\sigma \le x \le (a+b)\sigma \\ \epsilon + (1-\epsilon)F(x), & x > (a+b)\sigma. \end{cases} \tag{2}$$

The resultant influence functions for γ_1 and γ_p are,

$$IF(a,b,F,\gamma_1) = a^3 - a(3-b^2) \tag{3}$$

and

$$\mathrm{IF}\left(\mathrm{a},\mathrm{b},\mathrm{F},\gamma_{\mathrm{p}}\right)=\begin{cases}-1, & \mathit{for}\ a>b\\ -\mathrm{a}/\mathrm{b}, & \mathit{for}\ -b<a<b\\ 1, & \mathit{for}\ a<-b,\end{cases} \tag{4}$$

respectively. For comparison purposes, a graphical display of influence functions of both measures is offered in Figure 2, fixing the value of b at $1/2$, whereas, "a" can take values over the permissible range. It reveals that influence function of γ_{p} is bounded in contrast to that of γ_1. It is observable that both influence functions behave almost identically in the range of $0<\mathrm{a}<1/2$, whereas, for $\mathrm{a}>1/2$, distinctive features of $\mathrm{IF}\left(\mathrm{a},\mathrm{b},\mathrm{F},\gamma_1\right)$ and $\mathrm{IF}\left(\mathrm{a},\mathrm{b},\mathrm{F},\gamma_{\mathrm{p}}\right)$ are witnessed. The influence function of proposed functional stays constant and remains negative highlighting the resilience against contamination after certain level. On the other hand, $\mathrm{IF}\left(\mathrm{a},\mathrm{b},\mathrm{F},\gamma_1\right)$ endorses its reliability when $0<\mathrm{a}<1.65$ (the upper limit 1.65 can easily be verified by putting $\mathrm{b}=1/2$ in Equation (3)) but after that point, the influence function explodes. The altered behavior of $\mathrm{IF}\left(\mathrm{a},\mathrm{b},\mathrm{F},\gamma_1\right)$ for $\mathrm{a}>1.65$ is not only counter intuitive but also pose serious threats to the reliability of estimate in the presence of outliers.

[**Insert Figure 2**]

5. Applications

5.1. Algeria's Fatality Count Data (1997 – 2017)

In this section, we demonstrate the applicability of proposed functional by examining yearly data providing the Algeria's fatality counts from year 1997 to 2017. The data are borrowed from the website of Armed Conflict Location & Event Data Project (ACLED). The histogram presented in Figure 3 shows positive skewness in the data which is rightly projected in the magnitudes of both estimates (see Table 9).

[**Insert Figure 3**]

After estimating the degree of skewness for complete data set, we remove outliers, visible in box plot (Figure 3), and then estimate γ_1 and γ_{p}. Table 9 comprehends the results for readers.

The robustness of γ_p against outliers is self evident under the heading of absolute relative percentage change (RC). The results contained in Table 9 depict moderate extent of skewness.

[**Insert Table 9**]

5.2. PBS Data on Respondents' Extent of Co-operation (non-cooperation) in PSLM (2014-15) Survey

We next explore the utility of the proposed measure in comparison to the usual moment-based measure of skewness by studying Pakistan Social and Living Standard Measurements (PSLM) survey (2014-15) data compiled by PBS. The histogram and box plot given in Figure 4 present the pictorial display of percentages of non-cooperative respondents across 114 districts of the country.

[**Insert Figure 4**]

Table 10 documents the robustness of the proposed and commonly used moment-based skewness measure against existing outliers in the data. We initiate the skewness estimation, employing γ_1 and γ_p, by first considering complete data set and then through reduced data set (by dropping outliers). The absolute relative percentage change prominently reveals the robustness of the proposed functional in comparison to the usual measure. In case of our proposition, outliers derive almost 12% change in the value of the proposed skewness functional, whereas, a change of almost 55% (more than 4 times of that of the proposed functional) is associated with the value of usual measure when outliers are active. The overall results of Table 10 show greater extent of skewness in the data.

[**Insert Table 10**]

Moreover, the values of proposed measure remain comparable over the consistent range of [-1, 1]. Such as, for Table 9 that showcases results related to the Algeria data, the $\gamma_p = 0.3636$ and $\gamma_p = 0.2613$ for the PBS data of Table 10, it remains plausible to conclude that the Algeria data reveals higher extent of skewness (to the right) than the PBS data.

6. Discussion

In this article, a new measure of skewness, γ_p, is introduced based on the distribution function and mean of the distribution. The novelty of the contribution is demonstrated on three fronts (i) development, (ii) establishment and, (iii) assessment. Being distinctive from existing measures, the proposed measure does not require the knowledge of higher order moments or uniqueness of mode of distribution. It is shown that, γ_p, is a proper skewness measuring functional satisfying all essential characteristics recommended in literature. Defined over interpretable range, i.e. ‘0’ projecting the symmetry of distribution and a value of ‘+1 (-1)’ indicating extreme right (left) skewness, γ_p is straight forward to calculate. Following the spirit of competition, intense comparative study is conducted with respect to γ_1; the most commonly used and comprehensive skewness measure among existing ones. Superiority of the proposed measure is witnessed without exceptions. The influence function pleads the resistance of the proposed measure towards unusual points in data. In reasonably large sample sizes, it converges to its population value with minimal amount of variability. The computational ease, interpretability, and rigorous use of available information project γ_p as a potential candidate for future research. Especially, variance behavior for increasing sample sizes makes it worth studying to develop tests of skewness exploiting asymptotic theory. Its utility in higher dimensional data is yet another research venue. As such, it merits attention as a skewness measure with respect to mean.

Lastly, it is appropriate to recognize some complexities, especially, in the case of working with discrete data set. It is to recall that our proposed measure depends on the empirical evaluation of the cdf at mean, and for discrete data, the mean may not always be the member of the data, see for example, the data of Table 3 in Altinay [26]. In such scenario, a cautious use of proposed measure is suggested. In our understanding, in the above given situation, the empirical cdf at mean fails to efficiently estimate population cdf at mean. This can lead towards the misguided results. In future, it will be interesting to explore the possible amendments in the proposed framework to handle the aforementioned complication.

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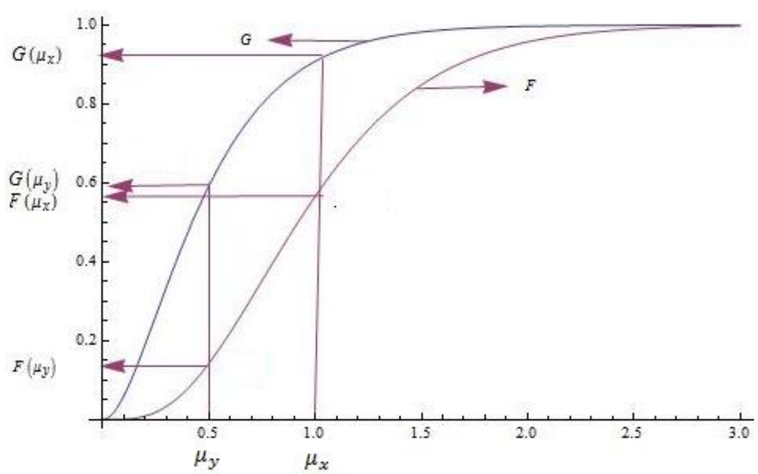

Figure 1: Display of distribution functions with regards to the concept of *c-ordering* of distributions.

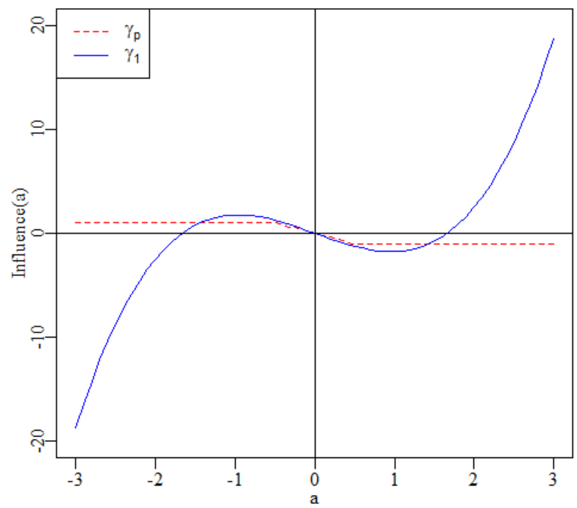

Figure 2: Influence function of γ_1 and γ_p.

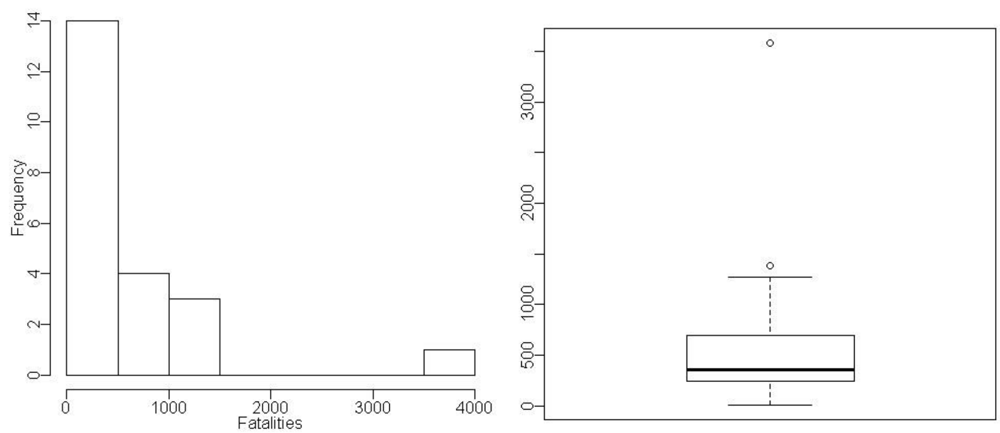

Figure 3: Histogram and box plot depicting the yearly fatality counts for Algeria (1997-2017)

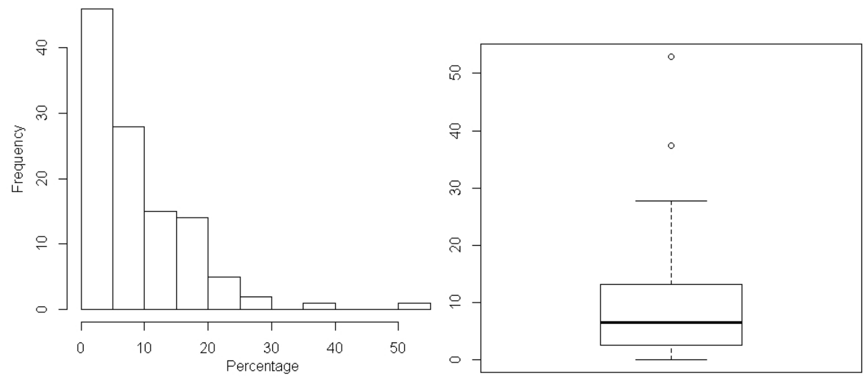

Figure 4: Histogram and box plot depicting the respondents' behaviour form PSLM (2014-15) survey.

Table 1: The simulated results for Chi-Square distribution considering different sample sizes and various combinations of parameter.

	$v=2$					$v=3$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.63212	0.63212	0.63212	0.63212	$F(\mu)$	0.60838	0.60838	0.60838	0.60838
γ_p	0.26426	0.26426	0.26426	0.26426	γ_p	0.21675	0.21675	0.21675	0.21675
γ_1	2.00000	2.00000	2.00000	2.00000	γ_1	2.00000	2.00000	2.00000	2.00000
$\overline{s(\bar{x})}$	0.62739	0.62859	0.62999	0.63196	$\overline{s(\bar{x})}$	0.60340	0.60552	0.60715	0.60834
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.25477 (0.01310)	0.25718 (0.00775)	0.25999 (0.00385)	0.26392 **(0.00078)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.20680 (0.01254)	0.21104 (0.00758)	0.21430 (0.00390)	0.21668 **(0.00077**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.47836 (0.40075)	1.64425 (0.37106)	1.77574 (0.29164)	1.95113 **(0.110120)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.23580 (0.33692)	1.35429 (0.28965)	1.47863 (0.21244)	1.04440 **(0.07322)**
	$v=5$					$v=10$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.58412	0.58412	0.58412	0.58412	$F(\mu)$	0.55951	0.55951	0.55951	0.55951
γ_p	0.16824	0.16824	0.16824	0.16824	γ_p	0.11901	0.11901	0.11901	0.11901
γ_1	1.26491	1.26491	1.26491	1.26491	γ_1	0.89443	0.89443	0.89443	0.89443
$\overline{s(\bar{x})}$	0.58050	0.58187	0.58284	0.58358	$\overline{s(\bar{x})}$	0.55728	0.55739	0.55864	0.55913
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.16100 (0.01228)	0.16375 (0.00773)	0.16567 (0.00378)	0.16715 **(0.00075)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.11456 (0.01224)	0.11478 (0.00738)	0.11728 (0.00360)	0.11827 **(0.00075)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.97527 (0.27722)	1.08051 (0.23096)	1.15893 (0.15366)	1.23948 **(0.04363)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.70700 (0.22945)	0.76368 (0.17113)	0.82892 (0.10472)	0.87929 **(0.00270)**

Table 2: The simulated results for Gamma distribution considering different sample sizes and various combinations of parameters.

$\alpha=1,\ \beta=1$					$\alpha=3,\beta=1$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.63210	0.63210	0.63210	0.63210	$F(\mu)$	0.57681	0.57681	0.57681	0.57681
γ_p	0.26424	0.26424	0.26424	0.26424	γ_p	0.15362	0.15362	0.15362	0.15362
γ_1	2.00000	2.00000	2.00000	2.00000	γ_1	1.15470	1.15470	1.15470	1.15470
$\overline{s(\bar{x})}$	0.62660	0.62811	0.63011	0.63156	$\overline{s(\bar{x})}$	0.57261	0.57565	0.57619	0.57665
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.25327 (0.44856)	0.25623 (0.34128)	0.26023 (0.23889)	0.26313 **(0.10672)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.14521 (0.77165)	0.15130 (0.56716)	0.15239 (0.40098)	0.15329 **(0.17809)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.48760 (0.42324)	1.62557 (0.37028)	1.79768 (0.31140)	1.95260 **(0.17114)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.89422 (0.57222)	0.98710 (0.45717)	1.06955 (0.35081)	1.13657 **(0.17398)**
$\alpha=5\ \beta=1$					$\alpha=10,\beta=1$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.55951	0.55951	0.55951	0.55951	$F(\mu)$	0.54207	0.54207	0.54207	0.54207
γ_p	0.11901	0.11901	0.11901	0.11901	γ_p	0.08414	0.08414	0.08414	0.08414
γ_1	0.89443	0.89443	0.89443	0.89443	γ_1	0.63246	0.63246	0.63246	0.63246
$\overline{s(\bar{x})}$	0.55679	0.55789	0.55856	0.55907	$\overline{s(\bar{x})}$	0.53915	0.54692	0.54086	0.54181
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.11357 (0.01234)	0.11578 (0.00743)	0.11712 (0.00369)	0.11813 **(0.00074)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.07830 (0.01193)	0.08184 (0.00756)	0.08172 (0.00373)	0.08362 **(0.00075)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.70666 (0.22842)	0.77100 (0.16788)	0.82664 (0.10347)	0.87852 **(0.02648)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.49542 (0.18850)	0.55061 (0.14313)	0.58381 (0.07860)	0.61952 **(0.01819)**

Table 3: The simulated results for Weibull distribution considering different sample sizes and various combinations of parameters.

$\alpha=1,\beta=1$					$\alpha=3,\beta=1$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.63212	0.63212	0.63212	0.63212	$F(\mu)$	0.50937	0.50937	0.50937	0.50937
γ_p	0.26424	0.26424	0.26424	0.26424	γ_p	0.01875	0.01875	0.01875	0.01875
γ_1	2.00000	2.00000	2.00000	2.00000	γ_1	0.16810	0.16810	0.16810	0.16810
$\overline{s(\bar{x})}$	0.62509	0.62843	0.62976	0.63181	$\overline{s(\bar{x})}$	0.50834	0.50904	0.50938	0.50934
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.25019 (0.01276)	0.25686 (0.00791)	0.25953 (0.00384)	0.26361 **(0.00077)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.01668 (0.01164)	0.01808 (0.00709)	0.01876 (0.00345)	0.01868 **(0.00070)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.47068 (0.38703)	1.63975 (0.37944)	1.78756 (0.29825)	1.95114 **(0.11215)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.13299 (0.12112)	0.15147 (0.07768)	0.16017 (0.03903)	0.16562 **(0.00822)**
$\alpha=5,\beta=1$					$\alpha=7,\beta=1$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.47928	0.47928	0.47928	0.47928	$F(\mu)$	0.46568	0.46568	0.46568	0.46568
γ_p	-0.04143	-0.04143	-0.04143	-0.04143	γ_p	-0.06864	-0.06864	-0.06864	-0.06864
γ_1	-0.25411	-0.25411	-0.25411	-0.25411	γ_1	-0.46319	-0.46319	-0.46319	-0.46319
$\overline{s(\bar{x})}$	0.48058	0.48000	0.47958	0.47919	$\overline{s(\bar{x})}$	0.46814	0.46583	0.46609	0.46574
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.03885 (0.01185)	-0.03999 (0.00718)	-0.04084 (0.00354)	-0.04160 **(0.00070)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.06373 (0.01235)	-0.06835 (0.00705)	-0.06782 (0.00359)	-0.06852 **(0.00071)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.22019 (0.13947)	-0.23218 (0.08485)	-0.26071 (0.04296)	-0.25205 **(0.0091)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.38519 (0.16125)	-0.41632 (0.10343)	-0.43781 (0.05672)	-0.45739 **(0.01161)**

Table 4: The simulated results for Beta distribution considering different sample sizes and various combinations of parameters.

	$\alpha=1,\ \beta=6$					$\alpha=4,\ \beta=8$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.60343	0.60343	0.60343	0.60343	$F(\mu)$	0.52744	0.52744	0.52744	0.52744
γ_p	0.20686	0.20686	0.20686	0.20686	γ_p	0.05488	0.05488	0.05488	0.05488
γ_1	1.28300	1.28300	1.28300	1.28300	γ_1	0.36422	0.36422	0.36422	0.36422
$\overline{s(\bar{x})}$	0.59994	0.60135	0.60226	0.60304	$\overline{s(\bar{x})}$	0.52522	0.52609	0.52739	0.52743
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.19988 (0.01173)	0.20269 (0.00711)	0.20452 (0.00353)	0.20607 **(0.00069)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.05043 (0.01185)	0.05218 (0.00677)	0.05479 (0.00342)	0.05485 **(0.00068)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.09389 (0.19773)	1.17331 (0.14388)	1.22600 (0.08005)	1.27160 **(0.01750)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.30519 (0.12045)	0.33276 (0.00757)	0.35054 (0.03783)	0.36074 **(0.00769)**
	$\alpha=2,\ \beta=2$					$\alpha=5,\ \beta=5$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.50000	0.50000	0.50000	0.50000	$F(\mu)$	0.50000	0.50000	0.50000	0.50000
γ_p	0.00000	0.00000	0.00000	0.00000	γ_p	0.00000	0.00000	0.00000	0.00000
γ_1	0.00000	0.00000	0.00000	0.00000	γ_1	0.00000	0.00000	0.00000	0.00000
$\overline{s(\bar{x})}$	0.49995	0.49982	0.50018	0.50022	$\overline{s(\bar{x})}$	0.49941	0.50020	0.49986	0.50029
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00010 (0.01067)	-0.00037 (0.00657)	0.00037 (0.00322)	0.000448 **(0.00064)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00118 (0.01143)	0.00040 (0.00678)	-0.00027 (0.00339)	0.00006 **(0.00068)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.00048 (0.07264)	-0.00044 (0.04365)	0.00059 (0.02121)	0.00033 **(0.00419)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.00388 (0.10711)	0.00044 (0.06201)	-0.00084 (0.03098)	-0.00032 **(0.00603)**
	$\alpha=7,\ \beta=3$					$\alpha=9,\ \beta=5$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.46283	0.46283	0.46283	0.46283	$F(\mu)$	0.47860	0.47860	0.47860	0.47860
γ_p	-0.07434	-0.07434	-0.07434	-0.07434	γ_p	-0.04279	-0.04279	-0.04279	-0.04279
γ_1	-0.48249	-0.48249	-0.48249	-0.48249	γ_1	-0.28868	-0.28868	-0.28868	-0.28868
$\overline{s(\bar{x})}$	0.46467	0.46322	0.46329	0.46301	$\overline{s(\bar{x})}$	0.47881	0.47970	0.47875	0.47863
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.07066 (0.01157)	-0.07356 (0.00701)	-0.07341 (0.00339)	-0.07397 **(0.00069)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.04236 (0.01146)	-0.0406 (0.00677)	-0.04060 (0.00677)	-0.4273 **(0.00069)**

$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.41517 (0.12078)	-0.446521 (0.07754)	0.46500 (0.03792)	-0.47825 **(0.00808)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-.25137 (0.12244)	-0.26340 (0.07568)	-0.26340 (0.07568)	-0.28528 **(0.00797)**

Table 5: The simulated results for Student's T-distribution considering different sample sizes and various combinations of parameters.

		$v=2$					$v=3$		
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.50000	0.50000	0.50000	0.50000	$F(\mu)$	0.50000	0.50000	0.50000	0.50000
γ_p	0.00000	0.00000	0.00000	0.00000	γ_p	0.00000	0.00000	0.00000	0.00000
γ_1	----	----	----	----	γ_1	----	----	----	----
$\overline{s(\bar{x})}$	0.50127	0.50093	0.50123	0.49994	$\overline{s(\bar{x})}$	0.49959	0.50017	0.50057	0.49992
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.00253 (0.06263)	0.00187 (0.04547)	0.00246 (0.03114)	-0.00013 **(0.00968)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00081 (0.02645)	0.00034 (0.01765)	0.00113 (0.00934)	-0.00015 **(0.00200)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.01069 (3.67913)	0.04112 (5.3898)	0.06651 (9.99695)	-0.02304 **(36.5364)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.00351 (1.62034)	-0.00785 (2.14600)	0.00568 (2.88789)	-0.04103 **(6.22120)**
		$v=4$					$v=5$		
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.50000	0.50000	0.50000	0.50000	$F(\mu)$	0.50000	0.50000	0.50000	0.50000
γ_p	0.00000	0.00000	0.00000	0.00000	γ_p	0.00000	0.00000	0.00000	0.00000
γ_1	0.00000	0.00000	0.00000	0.00000	γ_1	0.00000	0.00000	0.00000	0.00000
$\overline{s(\bar{x})}$	0.49966	0.49958	0.50021	0.50024	$\overline{s(\bar{x})}$	0.49888	0.50073	0.49966	0.49977
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00067 (0.01990)	-0.00084 (0.01223)	0.00042 (0.00625)	0.00048 **(0.00122)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00224 (0.01689)	0.00145 (0.01029)	-0.00069 (0.00503)	-0.00046 **(0.00104)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.00129 (1.00591)	0.00212 (1.10064)	-0.00404 (1.15580)	0.00158 **(1.39235)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.00892 (0.67060)	0.00685 (0.68144)	-0.00975 (0.62416)	-0.01682 **(0.44776)**

Table 6: The simulated results for F-distribution considering different sample sizes and various combinations of parameters.

	$v_1=1,\ v_2=3$					$v_1=3, v_2=5$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.81831	0.81831	0.81831	0.81831	$F(\mu)$	0.71221	0.71221	0.71221	0.71221
γ_p	0.63662	0.63662	0.63662	0.63662	γ_p	0.42441	0.42441	0.42441	0.42441
γ_1	---	---	---	---	γ_1	---	---	---	---
$\overline{s(\bar{x})}$	0.67264	0.77583	0.78896	0.80606	$\overline{s(\bar{x})}$	0.69118	0.69854	0.70512	0.71062
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.52527 (0.02458)	0.55166 (0.01887)	0.57793 (0.01252)	0.61213 **(0.00535)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.38237 (0.02064)	0.39708 (0.01464)	0.41024 (0.00852)	0.42122 **(0.00215)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	3.11141 (1.22462)	3.94584 (2.07043)	5.40987 (4.18508)	10.97980 **(22.22940)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	2.38910 (1.06552)	2.93949 (1.63729)	3.80013 (2.94790)	6.42382 **(11.93600)**
	$v_1=7, v_2=7$					$v_1=9, v_2=9$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.66588	0.66588	0.66588	0.66588	$F(\mu)$	0.64289	0.64289	0.64289	0.64289
γ_p	**0.33177**	0.33177	0.33177	0.33177	γ_p	0.28578	0.28578	0.28578	0.28578
γ_1	**10.15590**	10.15590	10.15590	10.15590	γ_1	4.39205	4.39205	4.39205	4.39205
$\overline{s(\bar{x})}$	0.65367	0.65825	0.66175	0.66494	$\overline{s(\bar{x})}$	0.63340	0.63738	0.63941	0.64244
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.30733 (0.01899)	0.31650 (0.01164)	0.32351 (0.00634)	**0.32989** **(0.00134)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.26681 (0.01701)	0.27476 (0.01034)	0.27882 (0.00547)	0.28488 **(0.00112)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.97263 (0.92258)	2.36775 (1.23576)	2.90375 (1.90147)	**4.25472** **(5.08084)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.72144 (0.79281)	2.02557 (0.98828)	2.44363 (1.34279)	3.25094 **(12.60029)**
	$v_1=3, v_2=7$					$v_1=7, v_2=9$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.67962	0.67962	0.67962	0.67962	$F(\mu)$	0.64551	0.64551	0.64551	0.64551
γ_p	**0.35923**	0.35923	0.35923	0.35923	γ_p	0.29103	0.29103	0.29103	0.29103
γ_1	**11.00000**	11.00000	11.00000	11.00000	γ_1	4.47214	4.47214	4.47214	4.47214
$\overline{s(\bar{x})}$	0.66570	0.67163	0.67554	0.67876	$\overline{s(\bar{x})}$	0.63427	0.63970	0.64233	0.64473
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.33140 (0.01825)	0.34327 (0.01188)	0.35107 (0.00618)	**0.35752** **(0.00132)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.26853 (0.01718)	0.27941 (0.01024)	0.28468 (0.00537)	0.28946 **(0.00114)**

$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	2.07855 (0.89883)	2.50569 (1.26351)	3.10938 (1.98945)	**4.53298** **(5.31268)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.72786 (0.75854)	2.06022 (0.98613)	2.47838 (1.39227)	3.31528 **(2.50372)**

$v_1 = 9, v_2 = 7$					$v_1 = 11, v_2 = 9$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.66394	0.66394	0.66394	0.66394	$F(\mu)$	0.64134	0.64134	0.64134	0.64134
γ_p	**0.32790**	0.32790	0.32790	0.32790	γ_p	0.28269	0.28269	0.28269	0.28269
γ_1	**10.03800**	10.03800	10.03800	10.03800	γ_1	4.34484	4.34484	4.34484	4.34484
$\overline{s(\bar{x})}$	0.65055	0.65631	0.66045	0.66290	$\overline{s(\bar{x})}$	0.63189	0.63525	0.63859	0.64076
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.30110 (0.01871)	0.31262 (0.01179)	0.32089 (0.00641)	**0.32581** **(0.00138)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.26378 (0.01677)	0.27049 (0.01051)	0.277184 (0.00543)	0.28152 **(0.00112)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.92299 (0.89460)	2.34468 (1.23039)	2.92540 (1.99575)	**4.20557** **(4.77499)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	1.6967 (0.76439)	2.00512 (0.96459)	2.41802 (1.32659)	3.2203 **(2.4167)**

Table 7: The simulated results for Rayleigh distribution considering different sample sizes and various combinations of parameter.

$\alpha = 0.5$					$\alpha = 4$				
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.54406	0.54406	0.54406	0.54406	$F(\mu)$	0.54406	0.54406	0.54406	0.54406
γ_p	0.08812	0.08812	0.08812	0.08812	γ_p	0.08812	0.08812	0.08812	0.08812
γ_1	0.63111	0.63111	0.63111	0.63111	γ_1	0.63111	0.63111	0.63111	0.63111
$\overline{s(\bar{x})}$	0.54221	0.54298	0.54341	0.54391	$\overline{s(\bar{x})}$	0.54235	0.54292	0.54331	0.54394
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.084417 (0.01169)	0.08596 (0.00693)	0.08683 (0.00353)	0.08782 **(0.00069)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.08471 (0.01169)	0.08584 (0.00696)	0.08663 (0.00349)	0.08787 **(0.00069)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.52539 (0.15383)	0.56564 (0.10240)	0.59762 (0.05665)	0.62406 **(0.05665)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.52309 (0.15317)	0.56224 (0.10187)	0.59639 (0.05649)	0.62592 **(0.01251)**

Table 8: The simulated results for Cauchy distribution considering different sample sizes and various combinations of parameters.

	$\alpha=-1,\beta=1$					$\alpha=-1,\beta=3$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.5	0.5	0.5	0.5	$F(\mu)$	0.5	0.5	0.5	0.5
γ_p	0	0	0	0	γ_p	0	0	0	0
γ_1	---	---	---	---	γ_1	---	---	---	---
$\overline{s(\bar{x})}$	0.50711	0.51216	0.49956	0.50136	$\overline{s(\bar{x})}$	0.50199	0.50086	0.49752	0.50279
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.00227 (0.27918)	0.00243 (0.30180)	-0.00088 (0.31092)	0.00272 **(0.32522)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	0.00387 (0.27905)	0.00172 (0.30271)	-0.00496 (0.31073)	0.00558 **(0.32908)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.01604 (10.082)	0.00821 (17.6950)	0.01821 (36.1892)	0.09924 **(184.6340)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	0.02335 (10.06320)	0.02081 (17.7665)	-0.03202 (36.1368)	0.08241 **(186.71300)**
	$\alpha=0,\beta=1$					$\alpha=0,\beta=3$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.5	0.5	0.5	0.5	$F(\mu)$	0.5	0.5	0.5	0.5
γ_p	0	0	0	0	γ_p	0	0	0	0
γ_1	---	---	---	---	γ_1	---	---	---	---
$\overline{s(\bar{x})}$	0.49684	0.50005	0.49864	0.50400	$\overline{s(\bar{x})}$	0.49855	0.50261	0.49649	0.50612
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00632 (0.27746)	0.00010 (0.29562)	-0.002728 (0.31109)	0.00800 **(0.32602)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00289 (0.27769)	0.00522 (0.29590)	-0.00701 (0.31254)	0.012239 **(0.328688)**
$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.041012 (10.01940)	0.02998 (17.50450)	-0.037145 (35.98690)	0.18268 **(184.9340)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.01164 (9.974747)	0.04368 (17.4140)	-0.05769 (36.0447)	0.163290 **(187.395)**
	$\alpha=5,\beta=1$					$\alpha=5,\ \beta=3$			
n	30	50	100	500	n	30	50	100	500
$F(\mu)$	0.5	0.5	0.5	0.5	$F(\mu)$	0.5	0.5	0.5	0.5
γ_p	0	0	0	0	γ_p	0	0	0	0
γ_1	---	---	---	---	γ_1	---	---	---	---
$\overline{s(\bar{x})}$	0.49789	0.40452	0.50187	0.49299	$\overline{s(\bar{x})}$	0.49689	0.50066	0.49658	0.50047
$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00421 (0.28464)	0.00904 (0.29411)	0.00373 (0.30972)	-0.01401 **(0.33416)**	$\bar{\hat{\gamma}}_p$ $V(\hat{\gamma}_p)$	-0.00623 (0.28457)	0.00131 (0.29544)	-0.00685 (0.31489)	-0.000094 **(0.32468)**

$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.02821 (10.18350)	0.06320 (17.28080)	-0.00185 (36.06790)	-0.34661 **(90.82100)**	$\bar{\hat{\gamma}}_1$ $V(\hat{\gamma}_1)$	-0.02065 (10.27350)	0.02882 (17.5818)	-0.07722 (36.5457)	-0.05429 **(186.9300)**

Table 9: Performance comparison of $\hat{\gamma}_p$ and $\hat{\gamma}_1$ in the presence of outliers - Algeria data.

	Full dataset	Reduced dataset	RC
$\hat{\gamma}_p$	0.3636	0.3000	17.49%
$\hat{\gamma}_1$	2.8932	1.1453	60.41%

Table 10: Performance comparison of $\hat{\gamma}_p$ and $\hat{\gamma}_1$ in the presence of outliers - PSLM data.

	Full dataset	Reduced dataset	RC
$\hat{\gamma}_p$	0.2613	0.2294	12.21%
$\hat{\gamma}_1$	1.9405	0.8616	55.60%

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