A simplified analytical model to predict heating performance of single U-tube underground heat exchangers

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Abstract
In underground U-tube heat exchangers (boreholes) it is important to predict its heating performance to design and select the proper parameters such as length, diameter, material etc. to have an optimized borehole from the point of view of heat capacity and economical aspects. For this reason, having trusty equations is vital to foresee borehole heating performance and applying it in design issues. In this study a single vertical U-tube borehole with constant wall temperature is considered and analytical equations for temperature distribution in the surrounding ground around the borehole is evaluated based on one and two dimensional heat conduction respectively. The analytical equation is compared to experimental data for a borehole with 50 m depth in which warm water of 40 °C is pumped into it a time period of 120 hours and the heat transfer rate per unit length is recorded. The comparison between analytical expression and experimental data shows a good agreement between them. Also the borehole entropy generation number is studied and the optimized parameters are evaluated to minimize it. It is concluded that for the considered borehole, entropy generation number is decreased by increasing its length and by decreasing the borehole radius and pipe outer radius.

Keywords: Borehole, Green's function, Underground heat exchanger, Heat transfer rate, Entropy generation number

1. Introduction
Geothermal energy is one of the important renewable energy which is clean, sustainable and generally available in different area of the world. Due to low variation of ground temperature compared to ambient temperature, ground can absorb heat in hot seasons and inject heat to the considered space in the cold seasons. So boreholes in the form of underground heat exchangers attracted more attention as a media to exchange heat between space and the ground. U-tube heat exchangers are dipped in the ground and then filled with grout. The flowing fluid is the medium that exchange heat load with the surrounding ground. Evaluation of the borehole heating performance is the important part of the borehole design that may help to predict its ability to handle part of required energy. One of the important issues of the borehole design is the economical evaluation as its lifetime would be decreased due to inappropriate size selection. Heat transfer rate evaluation may help to estimate borehole specifications. There are several models in which heating performance may be predicted such as analytical or numerical ones.

The infinite line source model [1] and the cylindrical source model [2] are two analytical solutions that may evaluate long time thermal performance of the borehole heat exchanger. Heat
conduction in vertical direction is neglected in both models and it is assumed that wall is imposed with constant heat flux. Eskilson [3] used a two-dimensional finite difference model to consider the effect of the finite borehole length. In his model, wall temperature was presented in terms of non-dimensional thermal response functions, known as g-function. Zeng et al. [4] suggested a two-dimensional analytical solution for the heat conduction in the ground, which is called finite line-source model. Similar to the previous analytical solutions [1], [2] the heating load was assumed to be constant along the borehole. Aydin et al. [5] used one-dimensional heat transfer model to analyze borehole. They used Laplace transform to predict temperature distribution outside of the borehole and to estimate heat transfer rate per unit length. They validated their calculations with the results that had been found from COMSOL software outputs. Also they stated that analyzing borehole with constant wall temperature is more accurate than borehole with the wall under constant heat flux. Biglarian et al. [6] proposed a numerical model for the simulation of heat transfer in a single borehole heat exchanger. They used 2D finite volume for the outside of the borehole and resistance capacity model for the inside of the borehole. Lee and Lam [7] developed a 3D fully implicit finite difference model in rectangular coordinates. They calculated the wall temperature and heating load along the borehole. Minaei et al. [8] presented an analytical model to predict short term heating performance of the boreholes. Several researchers developed analytical equation based on the Green's function in Cartesian coordinates for infinite boundaries [9], [10] and [11]. They evaluated the heat transfer rate which is carried by the borehole. Lee and Lam [12] used 2×2 ground heat exchanger and the year-round dynamic performance was simulated using TRNSYS. They showed applying displacement ventilation offered energy saving about 23% compared to the mixing ventilation. Saeidi et al. [13] modeled heat pump with a ground heat source and evaluated its thermodynamic performance. They considered horizontal and vertical ground heat exchangers and concluded that using R512a, R600 and R717 may be the most proper substitute refrigerants in the geothermal heat pumps for both economical and environmental viewpoints. Kummert and Bernier [14] presented a simulation study of closed-loop ground coupled heat pump systems for a typical residential building in North America. They compared the results obtained from TRNSYS to models that take dynamic into account in the geothermal fluid loop within borehole. Beier and Holloway [15] determined variations in the ground conductivity and borehole resistance on a set of horizontal boreholes over a two-year period. They found ground thermal conductivity is increased by increasing the depth. Mensah et al. [16] proposed a numerical simulation on the optimal design of boreholes based on the building load and validated the simulation analysis with the experimental data. Cimmino [17] presented an analytical model by using Laplace transform method to calculate fluid temperature in boreholes. Their proposed model had a good agreement with finite difference method. Dehkordi et al. [18] showed that tight boreholes may provide an acceptable heating performance while civil works would be decreased. Tang and Nowamooz [19] analyzed long term performance of shallow borehole heat exchangers. They found that yearly total extracted energy would be decreased annually. Also they showed this reduction becomes less significant after the fourth year. Monzó et al. [20]
employed numerical model for taking into account the effect of thermal resistance between the fluid and the borehole wall. Zhang et al. [21] presented an optimization design methodology with the Hooke-Jeeves pattern search algorithm in order to size and design underground heat exchanger under a given annual cooling and heating load. They found that when inlet fluid temperature is decreased from 5 °C to 0 °C, total borehole length required for the installation of single underground heat exchanger is reduced by 13.3%. Li et al. [22] developed a numerical model considering five strata and the axial temperature profiles of different layers at different distance were presented. They showed that layered subsurface and groundwater flow play non-negligible role on evaluation performance of borehole heat exchangers. Holmberg et al. [23] studied on performance of deep coaxial borehole heat exchangers. They showed that the heat load that can be extracted from boreholes is significantly increased by increasing borehole depth. Also they presented chart as guide when sizing deep coaxial boreholes. Beier et al. [24] applied a borehole heat transfer model that uses a transient weighting factor to calculate an average circulating fluid temperature along the borehole.

<table>
<thead>
<tr>
<th>Nomenclature</th>
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<tr>
<td>$c_p$</td>
<td>heat capacity (J/kg K)</td>
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<td>$E/GN$</td>
<td>entropy generation number</td>
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<tr>
<td>$G$</td>
<td>Green's function</td>
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<td>$h_i$</td>
<td>convective heat transfer coefficient (W/m²K)</td>
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<td>$h$</td>
<td>borehole depth (m)</td>
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<tr>
<td>$\bar{h}$</td>
<td>non-dimensional borehole depth</td>
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<tr>
<td>$J_0$</td>
<td>first kind Bessel’s function of zeroth order</td>
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<tr>
<td>$J_1$</td>
<td>first kind Bessel’s function of 1st order</td>
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<tr>
<td>$k$</td>
<td>thermal conductivity (W/m K)</td>
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<td>$l$</td>
<td>far field distance (m)</td>
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<td>$m$</td>
<td>fluid mass flow rate (kg/s)</td>
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<td>$Nu$</td>
<td>Nusselt number</td>
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<tr>
<td>$q$</td>
<td>heat transfer rate per unit length (W/m)</td>
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<td>$\bar{q}$</td>
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<td>$Q$</td>
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<td>$Re$</td>
<td>Reynolds number</td>
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<td>temperature (°C)</td>
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<td>time</td>
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<thead>
<tr>
<th>Greek Letters</th>
<th>Symbols</th>
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<tr>
<td>$\alpha$</td>
<td>thermal diffusivity (m²/s)</td>
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<tr>
<td>$\beta$</td>
<td>eigenvalue</td>
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<tr>
<td>$\mu$</td>
<td>fluid dynamic viscosity (kg/ms)</td>
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<tr>
<td>$\rho$</td>
<td>density (kg/m³)</td>
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<table>
<thead>
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<th>Subscripts</th>
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<td>averaged value</td>
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<td>borehole</td>
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<td>$eq$</td>
<td>equivalent</td>
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<td>$pi$</td>
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<td>$po$</td>
<td>outer pipe</td>
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<td>$s$</td>
<td>Steady state</td>
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The main goal of this study is to evaluate heating operation of the boreholes to predict its ability to carry heating load. In this research a single U-tube underground heat exchanger is considered. A new analytical solution based on Green's function is proposed to calculate temperature distribution outside of the borehole and to estimate its heating transfer rate capacity. The borehole wall temperature is considered constant. The results have been compared to the experimental data in which a good agreement between them has been concluded.
Also genetic algorithm is applied to optimize borehole from the point of view of entropy generation minimization. Non-dimensional form of entropy generation (entropy generation number (EGN)) is selected as objective function and the optimum technical parameters are calculated to have the minimum EGN.

2. Problem description

Here the heat conduction in the ground is evaluated. Since there is no fluid flow, then there is no heat convection and only heat conduction between the borehole and the ground would be occurred. Heat transfer depends on the ground, grout and the pipe materials and also the type of working fluid. It is assumed that all the materials are homogeneous for the heat transfer modeling.

Neglecting temperature variations in the tangential direction, 2D heat conduction in cylindrical coordinates is used as follows:

\[
\frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha_{\text{ground}}} \frac{\partial T}{\partial t}
\]

To model the borehole, it is assumed that wall temperature is constant due to a better accuracy and shorter time to reach the steady state regime [5, 9, 25, 26].

Temperature distribution in the ground will be evaluated by solving heat conduction equation. In practice boreholes have radius between 0.03m to 0.5m and its length is between 40m to 200m [10]. Since borehole diameter is ignorable compared to its length, so one-dimensional mathematical model can be used in the radial direction [9, 25]. Eskilson [3] suggested that axial heat transfer can be neglected for \( t < \frac{L^2}{90\alpha} \).

Also 2D model is applied to analyze temperature distribution in the surrounding ground around the borehole. In this model it is assumed that in the horizontal and vertical far fields, in which equal to infinity, there is no heat gradient in the \( r \) and \( z \) directions respectively so

\[
\frac{\partial T}{\partial r} = 0 \text{ and } \frac{\partial T}{\partial z} = 0.
\]

In this study, the main goal is to calculate the amount of heat that is exchanged between borehole and the ground. For this purpose, the system and its related boundary conditions may be considered as figure 1:

Figure 1: Schematic of the borehole

As it is mentioned, 1D and 2D heat conduction is used and the Green’s function has been applied to calculate temperature distribution in the ground and also to estimate the amount of heat that is exchanged between the ground and the borehole.
3. Mathematical Model

3.1. Temperature distribution

3.1.1. One-dimensional solution

Considering the previous assumptions, one-dimensional form of heat conduction (ignoring axial heat conduction) has been used. Therefore, one-dimensional form of equation 1 can be written as follows:

$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = \frac{1}{\alpha_{ground}} \frac{\partial T}{\partial t} \quad r_b < r < r_b + l \quad t > 0$  \hspace{1cm} (2)

The related boundary and the initial conditions are given by:

$T \left(r = r_b, t\right) = T_w$  \hspace{1cm} (3)

$\frac{\partial T}{\partial r} \bigg|_{r = r_b + l} = 0$  \hspace{1cm} (4)

$T \left(r, t = 0\right) = T_0$  \hspace{1cm} (5)

In the above equations, $\alpha$ is the ground thermal diffusivity, $T_w$ is the borehole wall temperature, $r_b$ equals to the borehole radius, $T_0$ is initial temperature and $l$ is the horizontal far field distance that temperature variation in radial direction (in that point) is ignorable.

Biglarian et al. [6] suggested $l$ to be equal to $5\sqrt{\alpha t_{\text{max}}}$ in which $t_{\text{max}}$ is the duration of simulation process.

By replacing $T - T_0$ to $\bar{T}$, equations 2 to 5 will be replaced as follows:

$\frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{T}}{\partial \bar{r}} = \frac{1}{\alpha_{ground}} \frac{\partial \bar{T}}{\partial \bar{t}}$  \hspace{1cm} (6)

$\bar{T} \left(r = r_b, t\right) = T_w - T_0 = T_{w1}$  \hspace{1cm} (7)

$\frac{\partial \bar{T}}{\partial \bar{r}} \bigg|_{r = r_b + l} = 0$  \hspace{1cm} (8)

$\bar{T} \left(r, t = 0\right) = 0$  \hspace{1cm} (9)

Dimensionless forms of equations 6 to 9 are given by:

$\frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{T}}{\partial \bar{r}} = \frac{\partial \bar{T}}{\partial \bar{t}} \quad 1 < \bar{r} < \bar{R} \quad \bar{t} > 0$  \hspace{1cm} (10)

$\bar{T} \left(1, \bar{t}\right) = 1$  \hspace{1cm} (11)

$\frac{\partial \bar{T}}{\partial \bar{r}} \left(\bar{R}, \bar{t}\right) = 0$  \hspace{1cm} (12)
The dimensionless parameters are defined as follows:

\[ \bar{T} = \frac{T}{T_w} \]  
\[ \bar{r} = \frac{r}{r_b} \]  
\[ \bar{R} = \frac{r_b + l}{r_b} \]  
\[ \bar{t} = \frac{\alpha_{ground} t}{r_b^2} \]  

The expression of Green’s function for hollow cylinder in which the inner temperature equals to a constant temperature and the outer surface is insulated can be found as follows [27]:

\[
G(r,t,r',\tau) = \frac{\pi}{4} \sum_{m=1}^{\infty} e^{-\beta_m^2(t-\tau)} \left( \frac{\beta_m J_0^2(\beta_m)}{J_0^2(\beta_m) - J_1^2(\beta_m \bar{R})} \right) \times \left[ J_0(\beta_m r) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m r) \right]
\]

\[
\times \left[ J_0(\beta_m r') Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m r') \right]
\]

This function is applied to solve equation 10 to find analytical expression for temperature distribution. In the above equation, \( J_0 \) is the first kind Bessel’s function of zeroth order, \( J_1 \) is the first kind Bessel’s function of 1st order, \( Y_0 \) is the second kind Bessel’s function of zeroth order, \( Y_1 \) is the second kind Bessel’s function of 1st order and \( \beta_m \) is the eigenvalue. Temperature distribution can be found by using Green’s function rules which is given by:

\[
T(r,t) = \int_{t=0}^{t} \left. \frac{\partial G(r,t,r',\tau)}{\partial \tau} \right|_{r'=1} 2 \pi r_i d \tau
\]

Therefore, dimensionless temperature distribution is given by the following expression:

\[
\bar{T} = -\frac{\pi^2}{2} \sum_{m=1}^{\infty} \left[ 1 - e^{-\beta_m^2 \bar{t}} \right] \frac{\beta_m J_0^2(\beta_m)}{J_0^2(\beta_m) - J_1^2(\beta_m \bar{R})} \left( J_0(\beta_m \bar{r}) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m \bar{r}) \right)
\]

\[
\times \left( J_1(\beta_m \bar{r'}) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_1(\beta_m \bar{r'}) \right)
\]

In one hand steady state solution of equation 10 is given by:

\[ T_s = 1 \]

On the other hand steady state of equation 20 can be calculated in the limit as \( \bar{t} \to \infty \) as follows:
\[ T_s = -\frac{\pi^2}{2} \sum_{m=1}^{\infty} \frac{\beta_m J_0^2(\beta_m)}{J_0^2(\beta_m) - J_1^2(\beta_m \bar{R})} \left( J_0(\beta_m \bar{r}) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m \bar{r}) \right) \]
\[ \times \left( J_1(\beta_m) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_1(\beta_m) \right) \]  

Therefore, dimensionless temperature distribution is given by the following expression:
\[ \bar{T} = 1 + \frac{\pi^2}{2} \sum_{m=1}^{\infty} e^{-\beta_m \bar{r}} \frac{\beta_m J_0(\beta_m)}{J_0^2(\beta_m) - J_1^2(\beta_m \bar{R})} \left( J_0(\beta_m \bar{r}) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m \bar{r}) \right) \]
\[ \times \left( J_1(\beta_m) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_1(\beta_m) \right) \]  

\[ (22) \]

\[ \beta_m \text{(eigenvalue)} \text{can be evaluated by applying boundary condition of equation 11 in equation 23. The result may be found by solving the following equation:} \]
\[ J_0(\beta_m) Y_1(\beta_m \bar{R}) - J_1(\beta_m \bar{R}) Y_0(\beta_m) = 0 \]  

\[ (23) \]

Equation 24 may be solved by applying Householder’s iteration perturbation method which is calculated by using modified homotopy perturbation method for nonlinear equations as following equation [28].
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2f'^3(x_n)} \]  

\[ (25) \]

Figure 2 shows temperature variations in surrounding ground around the borehole with increasing the time in several radiuses. It is shown that ground temperature in each location increases with increasing the time due to heat penetration. Also as it is mentioned, temperature variation in far field (\( \bar{r} = 45 \)) is ignorable and the temperature equals to \( T_0 \).

\[ \text{Figure 2: Temperature distribution VS. time in the surrounding ground} \]

Figure 3 shows variation of temperature around the borehole with increasing the non-dimensional radius for several time steps. It is shown that ground temperature would be decreased by increasing the distance from the borehole for each time step due to decreasing the borehole effects in far field points and the far field temperature approximately equals to \( T_0 \). Also as time passes, borehole effects disappear at far enough distances from the borehole and the temperature of those places reaches to \( T_0 \).

\[ \text{Figure 3: Temperature distribution VS. distance from the borehole for different time scales} \]
3.1.2. Two-dimensional solution

In this part, the horizontal and the vertical far fields are assumed equal to infinity and in those points, there is no temperature gradient (equations 42 and 43). Also the axial heat conduction (in the z direction) would be considered so equation 1 may be applied. To evaluate temperature distribution, Green's function in an infinite medium is applied [9], [10], [11] and [29] in which defined as function of temperature raise which is caused by a heat source that is located at point \((x',y',z')\) and activated at instant \(t'\). This function in the Cartesian coordinates is shown as the given equation:

\[
G(x, y, z, t; x', y', z', t') = \frac{1}{8(\pi\alpha(t-t'))^{3/2}} \exp \left\{ -\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4\alpha(t-t')} \right\} \tag{26}
\]

A point \((x,y,z)\) in the Cartesian coordinates may be changed to \((r,\varphi, z)\) in the cylindrical coordinates. The conversion would be done by applying \(x = r\cos\varphi\), \(y = r\sin\varphi\) and \(z = z\). So Green's function in the cylindrical coordinates will be as follows:

\[
G(r, \varphi, z, t; r', \varphi', z', t') = \frac{1}{8(\pi\alpha(t-t'))^{3/2}} \exp \left\{ -\frac{(r\cos\varphi-r'\cos\varphi')^2 + (r\sin\varphi-r'sin\varphi')^2 + (z-z')^2}{4\alpha(t-t')} \right\} \tag{27}
\]

By using the Green's function rules, the temperature rise distribution in the ground as the infinite medium may be as follows [9]:

\[
T(r, \varphi, z, t) - T_0 = \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dz' \int_0^t BG(r, \varphi, z, t; r', \varphi', z', t') dt' \tag{28}
\]

\(B\) is variable that is function of boundary conditions [9]. Various problems may have different \(B\) quantities. In the current problem, it is assumed that borehole wall temperature is constant. Since 2D heat conduction is studied and the temperature distribution is independent of angular coordinate so \(\varphi = 0\) and this temperature rise can be calculated by:

\[
T(r, z, t) - T_0 = \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dz' \int_0^t B \frac{8}{3} \exp \left\{ -\frac{r^2 + r_b^2 - 2rr_b\cos\varphi' + (z-z')^2}{4\alpha(t-t')} \right\} dt', \tag{29}
\]

Integral of angular part is calculated by:

\[
\int_0^{2\pi} \exp \left\{ -\frac{2rr_b\cos\varphi'}{4\alpha(t-t')} \right\} d\varphi' = 2\pi I_0 \left( \frac{r_b}{2\alpha(t-t')} \right) \tag{30}
\]
To analyze borehole, it is assumed that ground surface, keeps a constant temperature as its initial temperature ($T_0$). So a virtual heating sink is assumed on symmetry to the boundary.

The solution of equation 29 may be written as the following equation:

$$ T - T_0 = \frac{B}{8(\pi \alpha(t-t'))^2} \int_0^\infty \frac{r_b}{2\alpha(t-t')} \exp \left( -\frac{r^2 + r_b^2}{4\alpha(t-t')} \right) \left\{ \exp \left( -\frac{(z-z')^2}{4\alpha(t-t')} \right) - \int_0^{z'} \exp \left( -\frac{(z-z')^2}{4\alpha(t-t')} \right) \, dz' \right\} \, dt$$

To solve the above equation, the following relations are applied:

$$ \int \exp \left( -\frac{z^2}{b^2} \right) \, dz = \frac{\sqrt{\pi}}{2} \text{erf} \left( \frac{z}{b} \right) + \text{constant} $$

$$ \text{erf} \left( z \right) = \frac{2}{\sqrt{\pi}} \int_0^z \exp \left( -t^2 \right) \, dt $$

$$ \text{erf} \left( -z \right) = -\text{erf} \left( z \right) $$

So the solution of equation 31 can be written by:

$$ T (r, z, t) - T_0 = \frac{B}{4\alpha(t-t')} \int_0^\infty \frac{r_b}{2\alpha(t-t')} \exp \left( -\frac{r^2 + r_b^2}{4\alpha(t-t')} \right) \left\{ \text{erf} \left( \frac{h-z}{2\sqrt{\alpha(t-t')}} \right) \right\} $$

$$ + 2\text{erf} \left( \frac{z}{2\sqrt{\alpha(t-t')}} \right) - \text{erf} \left( \frac{h+z}{2\sqrt{\alpha(t-t')}} \right) \, dt' $$

Equation 35 is the modified form of the mathematical equation which is proposed by Dehghan [11]. The 2D solution of temperature distribution in the ground may be shown in the non-dimensional form by defining the following parameters:

$$ \tilde{h} = \frac{h}{r_b} $$

$$ \tilde{z} = \frac{z}{h} $$

$$ \tilde{t} = \frac{\alpha t'}{r_b^2} $$

By applying the above relations and the equations 14, 15 and 17, the related non-dimensional initial and boundary conditions are given by:

$$ \tilde{T} (\tilde{r}, \tilde{z}, 0) = 0 $$

$$ \tilde{T} (1, \tilde{z}, \tilde{t}) = 1 $$
\[
T(\vec{r},0,\vec{r}) = 0
\]  \hspace{1cm} (41)

\[
\frac{\partial T(\vec{r},\vec{z},\vec{r})}{\partial \vec{r}} \bigg|_{\vec{r} \to \infty} = 0
\]  \hspace{1cm} (42)

\[
\frac{\partial T(\vec{r},\vec{z},\vec{r})}{\partial \vec{z}} \bigg|_{\vec{z} \to \infty} = 0
\]  \hspace{1cm} (43)

So non-dimensional form of equation 35 will be as follows by considering equations 39-43:

\[
T = \frac{T(r,z,t) - T_0}{T_w - T_0}
\]

\[
T = \frac{\int_0^\infty \frac{1}{(\vec{r} - \vec{r})} \cdot I_0 \left( \frac{\vec{r}}{2(\vec{r} - \vec{r})} \right) \cdot \exp \left( -\frac{1+2^2}{4(\vec{r} - \vec{r})} \right) \cdot \left[ \operatorname{erf} \left( \frac{\vec{h}(1-\vec{z})}{2\sqrt{\vec{r} - \vec{r}}} \right) + 2\operatorname{erf} \left( \frac{\vec{h} \vec{z}}{2\sqrt{\vec{r} - \vec{r}}} \right) - \operatorname{erf} \left( \frac{\vec{h}(1+\vec{z})}{2\sqrt{\vec{r} - \vec{r}}} \right) \right] \, d\vec{r}
\]

The above mathematical relation is other form of temperature distribution equation which is reported by Dehghan [11] due to non-dimensional parameter definition.

To analyze equation 44, a MATLAB code is developed that the results can be seen in figures 4, 5 and 6.

Figure 4 shows temperature variations with increasing the time in several radiuses. It is shown that ground temperature in each location increases with increasing the time due to heat penetration in which is compatible with data in the figure 2.

![Figure 4: Temperature distribution VS. time in the surrounding ground](image)

Figure 5 shows variations of the temperature distribution around the borehole with increasing the distance for several time steps. Similar to the 1D analysis, it is shown that ground temperature would be decreased by increasing distance from the borehole for each time step because borehole effects decrease in the far field points. Also the results of figure 5 are compatible with data in the figure 3.

![Figure 5: Temperature distribution VS. distance from the borehole for different time scales](image)

Figures 6a and 6b show variations of the temperature distribution with increasing the borehole depth after 6 months and 1 year, respectively.
As it is shown, on the ground surface there is $T=T_0$ and by dipping into ground, at the first, the temperature is increased and then it is decreased. Also it is shown that for each time step and each depth, ground temperature would be increased by decreasing the distance to the borehole. For each location, the temperature increases from 6 months to 1 year due to the heat penetration.

3.2. Heat transfer rate per unit length

One of the important parts of this study is to calculate the heating transfer rate per unit length between ground and the borehole. For this purpose, 1D dimensionless heat transfer rate per unit length outside of the borehole can be calculated as follows:

$$\bar{q} = \frac{q_b}{2\pi k_{\text{ground}}T_w} = -\frac{\partial T}{\partial r}|_{r=1}$$ (45)

$$\bar{q} = -\sum_{n=1}^{\infty} e^{-\rho_0 \tau} \frac{\beta_n^2 J_1^2(\beta_n)}{J_0^2(\beta_n) - J_1^2(\beta_n)} \left[ J_1(\beta_nR)Y_1(\beta_n) - J_1(\beta_nR)Y_1(\beta_n) \right] \times \left[ J_1(\beta_nR)Y_1(\beta_n) - J_1(\beta_nR)Y_1(\beta_n) \right]$$ (46)

Also 2D dimensionless heat transfer rate per unit length outside of the borehole can be calculated as follows:

$$\bar{q} = \frac{\partial T}{\partial r}|_{r=1} = \frac{\int_0^\tau \frac{A_1 A_2}{r} d\tau}{\int_0^\tau \frac{A_1}{r} d\tau}$$ (47)

Coefficients $A_1$ and $A_2$ may be shown in equations 48 and 49.

$$A_1 = \frac{1}{2(\tilde{r} - \tilde{r})^2} \exp \left( -\frac{1}{2(\tilde{r} - \tilde{r})^2} \left( J_1 \left( \frac{1}{2(\tilde{r} - \tilde{r})^2} \right) - I_0 \left( \frac{1}{2(\tilde{r} - \tilde{r})^2} \right) \right) \right)$$ (48)

$$A_2 = \text{erf} \left( \frac{h (1 - \tilde{z})}{2\sqrt{(\tilde{r} - \tilde{r})}} \right) + 2\text{erf} \left( \frac{\tilde{h} \tilde{z}}{2\sqrt{(\tilde{r} - \tilde{r})}} \right) - \text{erf} \left( \frac{h (1 + \tilde{z})}{2\sqrt{(\tilde{r} - \tilde{r})}} \right)$$ (49)

In addition, the heat transfer rate per unit length between inside of the borehole and its wall is given by:

$$q_b = \frac{T_{\text{inr}} - T_w}{\frac{\ln \frac{r_b}{r_{eq}}}{2\pi k_{\text{grout}}} + \frac{\ln \frac{r_{po}}{r_{pi}}}{4\pi k_{\text{pipe}}} + \frac{1}{h_i \times 4\pi r_{pi}}}$$ (50)
In the above equation, $T_{\text{avr}}$ equals to the fluid mean temperature $T_{\text{avr}} = (T_{\text{in}} + T_{\text{out}}) / 2$, $T_{\text{w}}$ is the borehole wall temperature, $r_{eq}$ equals to the equivalent radius $r_{eq} = \sqrt{2r_{po}}$ [30], $r_{po}$ is the outer pipe radius, $r_{pi}$ is the inner pipe radius, $r_{b}$ is the borehole radius, $k_{\text{grout}}$ is the grout thermal conductivity coefficient, $k_{\text{pipe}}$ is the pipe thermal conductivity coefficient and $h_{i}$ is convective heat transfer coefficient. It is worth mentioning that equivalent pipe is in the form that its outer surface is equal to sum of outer surfaces of both pipes and thermal resistance of the equivalent pipe (from the fluid to outer surface) is equal to the thermal resistance of two pipes in the parallel form.

Also the heat transfer rate per unit length that fluid transfers to the borehole and the ground can be calculated from the following expression:

$$q_{f} = \frac{\dot{m}c_{p,\text{water}}}{h} (T_{\text{in}} - T_{\text{out}})$$  \hspace{1cm} (51)$$

Here $\dot{m}$ is the fluid mass flow rate, $c_{p,\text{water}}$ is the fluid heat capacity, $h$ is the borehole length, $T_{\text{in}}$ is the inlet fluid temperature that is pumped into the borehole and $T_{\text{out}}$ is the outlet fluid temperature which is exited from the borehole.

In order to calculate convective heat transfer coefficient, $h_{i}$, it may be calculated from the following expression:

$$h_{i} = \frac{Nu \times k_{f}}{d_{pi}}$$  \hspace{1cm} (52)$$

$Nu$ is the Nusselt number that is calculated by the following equation [31]:

$$Nu = 0.023Re^{0.8}Pr^{0.3}$$  \hspace{1cm} (53)$$

Where

$$Re = \frac{2\dot{m}}{\pi r_{pi} \mu_{f}}$$  \hspace{1cm} (54)$$

Equation 53 holds for $Re>2300$. For $Re<2300$ and for internal flows that is faced constant temperature, $Nu$ number is constant and $Nu=3.66$.

As it is seen grout, pipe and ground specification and also the working fluid type is included in $q$ calculation.

4. Results:

4.1. Validation of Analytical Solution

In order to validate the analytical results, experimental data of a single U-tube borehole [9] is applied. Test specification is shown in table 1. For recording test result, water has been heated to 40 C through the electrical resistances and related valves are opened and water enters into the borehole at constant temperature. Then heat transfer rate per unit length is calculated with equation 51.

Table 1: Test conditions of single U-tube borehole
Figure 7 shows analytical solution of heat transfer rate per unit length. As it is seen, analytical solutions have a good trend with the experimental data.

Figure 7: Heat transfer rate per unit length VS. time

Also borehole heat transfer rate per unit length decreases due to increase of temperature around the borehole.

4.2. Long Term Performance Prediction
Heat transfer rate per unit length is calculated analytically and is compared with experimental data for 120 hours. The difference between analytical and experimental data is in a good range. It is beneficial to predict long time continuous operation of the borehole. Figure 8 shows 6 months nonstop operation of the borehole. In real case, borehole works at on/off mode and it does not work continuously. So this result is in a worse case and it can be used to predict borehole performance to select appropriate borehole specification.

Figure 8: Heat transfer rate per unit length VS. time for long term performance prediction

As it is seen, for short time 1D and 2D solutions have good agreement and for longer time they are separated due to axial heat conduction.

4.3. Flow Regime Effect
Flow regime is a factor that can affect heat transfer. In turbulent flow, $h_i$ is calculated from equation 53 and in laminar flow, $h_i$ is calculated from $Nu=3.66$. Figure 9 shows heat transfer rate per unit length for several mass flow rates and $Re$ numbers.

Figure 9: Heat transfer rate per unit length VS. time for various mass flow rates

It is shown that for lower $Re$ numbers, $q$ is approximately constant in each time step and $q$ changes with time at higher $Re$ numbers. In addition for $\dot{m}_f > 0.355\, kg/s$, heat transfer rate per unit length varies fairly by increasing both $\dot{m}_f$ and $Re$ number.

4.4. Borehole Length Effect
Figures 10 and 11 show the effect of length on borehole performance. As it is shown in figure 10, heat transfer rate per unit length has been decreased by increasing borehole length. As can be seen from figure 11, heat transfer rate \(Q\) is increased by increasing borehole length.

Figure 10: Heat transfer rate per unit length VS. time for different borehole lengths

Figure 11: Heat transfer rate VS. time for different borehole lengths

4.5. Entropy Generation

Irreversibility source of a borehole, is consist of heat transfer and pressure drop. It is highly desirable to optimize the borehole so entropy generation minimization would be the objective function. Entropy generation due to temperature difference may be written by [32]:

\[
S_{\text{gen,}\Delta T} = \frac{Q \Delta T}{T_{f,m}^2 (1 + \chi)}
\]  

(55)

\(Q\) is the heat transfer rate which is calculated from the 1D solution. \(\Delta T\) is temperature difference between logarithmic average fluid flow temperature and the wall temperature \((T_{f,m} - T_w)\). \(T_{f,m}\) and \(\chi\) may be calculated from equations 56 and 57.

\[
T_{f,m} = \frac{T_{in} - T_{out}}{\ln \frac{T_{in}}{T_{out}}}
\]  

(56)

\[
\chi = \frac{\Delta T}{T_{f,m}}
\]  

(57)

Entropy generation due to the pressure drop is calculated from the following equation [32]:

\[
S_{\text{gen,}\Delta p} = \frac{\dot{m}_f \Delta p}{\rho_j T_{f,m}}
\]  

(58)

\(\Delta p\) is the pressure drop in the pipe that is given by:

\[
\Delta p = \frac{f \dot{m}_f^2 (2h)}{\rho_j \pi^2 r_i^5}
\]  

(59)

\(f\) is the friction factor that is calculated by equations 60 and 61 for laminar and turbulent flows respectively:

\[
f = \frac{16}{Re} \quad \text{Laminar Flow}
\]  

(60)

\[
f = \frac{0.046}{\sqrt{Re}} \quad \text{Turbulent Flow}
\]  

(61)

The total entropy generation may be written by equation 62:
\[ S_{\text{gen, total}} = S_{\text{gen, } \Delta T} + S_{\text{gen, } \Delta p} = \frac{Q \Delta T}{T_{f,m}^2 (1 + \chi)} + \frac{\dot{m}_f \Delta p}{\rho_f T_{f,m}} \]  

(62)

Bejan [32] suggested non-dimensional form of entropy generation (entropy generation number \( EGN \)) as follows:

\[ EGN = \frac{S_{\text{gen, total}} T_{f,m}}{Q} \]  

(63)

In the current study, the effect of changing of borehole length \( (h) \), mass flow rate \( (\dot{m}) \), borehole radius \( (r_b) \) and pipe outlet radius \( (r_{po}) \) on \( EGN \) is studied. The variation ranges of these parameters are determined based on the recommended values of practical engineering projects and summarized in table 2[33]. Figures 12-15 presents the variations of the \( EGN \) with respect to the change of the mentioned parameters to be optimized by the \( EGN \)-based optimization process while keeping other parameters constant, which was based on the values of the parameters for the design data provided in table 1 and the variation ranges of each design parameter presented in table 2.

Table 2: Ranges of the design parameters of vertical U-tube heat exchangers [33]

Figure 12: \( EGN \) VS. borehole lengths

Figure 13: \( EGN \) VS. fluid mass flow rate

Figure 14: \( EGN \) VS. borehole radius

Figure 15: \( EGN \) VS. pipe outer radius

It is seen that \( EGN \) would be decreased by increasing borehole length and it would be increased by increasing borehole radius and the outer pipe radius.

The optimum parameters are evaluated by applying genetic algorithm method. The objective function is \( EGN \) that should be minimum. Comparison between optimum quantities and the primary specifications (Table 1) of the considered borehole are shown in table 3.

Table 3: Comparison between optimum parameters and the primary specifications of the considered borehole
5. Conclusion:
In this study an analytical model based on Green’s function is presented. In this model, one-dimensional and two-dimensional heat transfer with constant borehole wall temperature was assumed. Borehole heat performance prediction is important as small sizing leads to have a borehole with improper heat performance. Also over sizing leads to increase the cost. For validating analytical solution, the results of calculated heat transfer rate per unit length of 1D and 2D are compared with each other and also with the experimental data. This comparison shows a good agreement between analytical and the experimental data. So borehole heat performance may be predicted based on analytical solution. Heat transfer rate per unit length and also temperature distribution around the borehole is evaluated in 1D and 2D models. It is concluded that heat transfer rate per unit length decreases with passing the time due to increasing temperature around the borehole. Temperature distribution around the borehole shows, at each location, temperature increases with operating time and for each time step ground temperature around the borehole decreases to $T_0$ with taking the distance from the borehole. For more time, this distance is bigger due to heat penetration. These results are compatible with Dehghan et al. study [9]. Also after validating analytical solutions, 6-month nonstop performance of borehole is predicted. Flow regime may affect heat performance of the borehole. In this study various fluid mass flow rates are examined to have laminar and turbulent flows. It is deduced that $q$ will be approximately constant in laminar flow and for turbulent flow $q$ is increased by increasing fluid mass flow rate while it is not linearly proportional. Borehole length is one of the most important parameters that can affect its performance. In this study the length effect on $q$ and $Q$ was evaluated. It is seen that $q$ is decreased by increasing borehole length meanwhile $Q$ increases as well. Entropy generation number of the borehole was studied and it was concluded that for the considered borehole, entropy generation number is decreased by increasing its length and by decreasing the borehole radius and pipe outer radius.

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References


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Figure Caption

Figure 1: Schematic of the borehole
Figure 2: Temperature distribution VS. time in the surrounding ground
Figure 3: Temperature distribution VS. distance from the borehole for different time scales
Figure 4: Temperature distribution VS. time in the surrounding ground
Figure 5: Temperature distribution VS. distance from the borehole for different time scales
Figure 6: Temperature distribution VS. ground depth for various distances after (a) 6 months, (b) 1 year
Figure 7: Heat transfer rate per unit length VS. time
Figure 8: Heat transfer rate per unit length VS. time for long term performance prediction
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Figure 10: Heat transfer rate per unit length VS. time for different borehole lengths
Figure 11: Heat transfer rate VS. time for different borehole lengths
Figure 12: EGN VS. borehole lengths
Figure 13: EGN VS. fluid mass flow rate
Figure 14: EGN VS. borehole radius
Figure 15: EGN VS. pipe outer radius
Table Caption

Table 1: Test conditions of single U-tube borehole
Table 2: Ranges of the design parameters of vertical U-tube heat exchangers [33]
Table 3: Comparison between optimum parameters and the primary specifications of the considered borehole
Figure 1
Figure 3
Figure 4
Figure 6(a)

Figure 6 (b)
Figure 7

- Experimental Data [9]
- Analytical Solution 2D
- Analytical Solution 1D
Figure 8
Figure 9
Figure 10
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