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Variable sample size of exponentially weighted moving average chart with measurement errors

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Abstract. Several researchers have investigated the effect of measurement errors on adaptive Shewhart charts. However, the effect of measurement errors on the performance of Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control charts has not been evaluated yet. In this regard, the present study aims to investigate the performance of the VSS EWMA chart in the presence of measurement errors using a linear covariate error model and Markov chain method. The results indicated that the presence of measurement errors could significantly affect the performance of the VSS EWMA chart. In addition, the effect of taking multiple measurements for each item in a subgroup on the performance of the VSS EWMA chart was evaluated. Moreover, the performance of the VSS EWMA control chart was compared with those of several other control charts in the presence of measurement errors. Finally, an illustrative example was presented to demonstrate the application of the VSS EWMA control chart with measurement errors.

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1. Introduction

Although the Shewhart $\bar{X}$ control chart is effectively used for detecting large process mean shifts, Exponentially Weighted Moving Average (EWMA) control chart has a better performance in detecting small and moderate shifts. Many developments have been made to improve the performance of control charts in detecting the process mean shifts. One of the main approaches to improving the performance of EWMA charts is using adaptive procedures and designing adaptive control charts. An adaptive control chart is a chart where at least one of its parameters (sample size, sampling interval, and control limit coefficient) is regarded as variable throughout the process.

A number of researches have been conducted to develop adaptive control charts and according to the findings, they outperformed their Fixed Sampling Rate (FSR) control chart counterparts in detecting small and moderate shifts. In this section, the literature concerning adaptive Shewhart $\bar{X}$ control charts is reviewed. Prabhu et al. [1] and costa [2] presented the Variable Sample Size (VSS) $\bar{X}$ control chart. Reynolds [3] investigated the Variable Sampling Interval (VSI) $\bar{X}$ control chart. Amiri et al. [4] investigated the economic-statistical design of VSI and VSS $\bar{X}$ control charts through Taguchi's loss function approach. Prabhu et al. [5] and Costa [6] studied the combined Variable Sample Size and Sampling Interval (VSSI) $\bar{X}$ control schemes. Costa [7] examined the $\bar{X}$ chart with Variable Parameters (VP). The next section briefly introduces the studies conducted on the Adaptive EWMA (AEWMA) charts.

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In addition to the studies on adaptive Shewhart $\bar{X}$ control chart, further researches have also been conducted on the AEWMA control chart. For instance, Saccucci et al. [8] investigated the VSI EWMA control chart. Reynolds Jr and Arnold [9] considered the VSS and VSI EWMA control charts and developed integral equation and Markov chain methods to evaluate the statistical performance of these charts. Castagliola et al. [10,11] presented the VSI R-EWMA and VSI S2-EWMA control charts to monitor the process variance. Kazemzadeh et al. [12] designed a VSI EWMA control chart based on the t distribution. In addition, Tran et al. [13] evaluated the efficiency of the VSI EWMA-$\tilde{X}$ (median) control chart. Ugaz et al. [14] proposed an AEWMA control chart with the time-varying smoothing parameter. Tang et al. [15] introduced an optimal design of the AEWMA chart for the mean based on both median run and expected median run lengths. Amiri et al. [16] designed a new adaptive Variable Sample Size EWMA (VSS EWMA) control chart. Capizzi and Masarotto [17] proposed an AEWMA-$\bar{X}$ control chart. Ong et al. [18] studied the VSI EWMA chart when the process parameters were unknown. Tran et al. [19] presented an EWMA-type control chart to monitor the ratio of two normal quality characteristics. Teoh et al. [20] investigated the VSI EWMA $\bar{X}$ chart when the process parameters were unknown. Haq and Khoo [21] introduced an Adaptive Multivariate EWMA (AMEWMA) chart and calculated the run length characteristics of this chart using the Monte Carlo simulation runs.

The above-mentioned studies did not take into account the measurement errors. However, in real world, there are measurement errors in many processes. Some researchers have studied the effect of measurement errors on the performance of Shewhart control charts. For instance, Bennet [22] introduced the effect of measurement errors using the model $X = Y + \varepsilon$, where X is the observed value, Y the true value of the quality characteristic, and ε a random measurement error. Linna and Woodall [23] suggested using the linearly covariate model $X = A + BY + \varepsilon$, where A and B were two constants.

Here, a brief summary of the studies on the effect of measurement errors on the adaptive Shewhart charts is given. Linna and Woodall [23] and Kanazuka [24] evaluated the effect of measurement errors on the performance of $\bar{X}$-R control charts. Costa and Castagliola [25] investigated the effect of measurement errors on the $\bar{X}$ control chart with autocorrelated data, and Hu et al. [26] evaluated this effect on the synthetic $\bar{X}$ control chart. Linna et al. [27] analyzed the performance of the FSR Hotelling's T^2 control chart in the presence of measurement errors. Hu et al. [28,29] considered the effect of measurement errors on the performance of the VSS $\bar{X}$ and VSI $\bar{X}$ control charts, respectively. Sabahno and Amiri [30] investigated the effect of measurement errors on the performance of the VSSI $\bar{X}$ control chart. Sabahno et al. [31–33] investigated the effect of measurement errors on the performance of VP $\bar{X}$, VSS-T^2 and VSI-T^2 control charts, respectively. Ghashghaei et al. [34] delved into the effect of measurement errors on the joint monitoring of the process mean and variability under Ranked Set Sampling (RSS) scheme. Amiri et al. [35] investigated the effect of measurement errors on the performance of Exponentially-weighted Likelihood Ratio (ELR) control chart for simultaneous monitoring of the mean vector and covariance matrix in the multivariate normal processes. Maleki et al. [36] considered the ELR control chart for simultaneous monitoring of the mean vector and covariance matrix and evaluated the effect of measurement errors with linearly increasing error variance on the performance of this control chart. Nguyen et al. [37] assessed the performance of a VSI Shewhart control chart monitoring the coefficient of variation in the presence of measurement errors. Finally, Tran et al. [38] evaluated the performance of coefficient of variation charts in the presence of measurement errors.

Stemann and Weihs [39] took into account the AEWMA-type charts in the presence of measurement errors to introduce the EWMA-$\bar{X}$-S control chart and evaluate its performance. Maravelakis et al. [40] considered the EWMA control chart with measurement errors. Yang et al. [41] evaluated the effect of measurement errors on the performance of an EWMA control chart for monitoring two dependent process steps. Haq et al. [42] investigated the effect of measurement errors on the EWMA control chart under RSS scheme. Abbasi [43,44] studied the effect of measurement errors on the EWMA control chart with two-component measurement errors (i.e., additive as well as multiplicative ones). Saghaei et al. [45] investigated the economic design of an EWMA control chart with measurement errors using a genetic algorithm. Cheng and Wang [46] explored the effect of measurement errors on the EWMA-$\tilde{X}$ and CUSUM-$\tilde{X}$ control charts. Most of studies on the AEWMA control charts have been carried out without considering the measurement errors. Tang et al. [47] examined the effect of measurement errors on the AEWMA-$\bar{X}$ chart.

Maleki et al. [48] conducted a review of the effect of measurement errors on control charts. To the best of the authors' knowledge and based on this review, the effect of measurement errors on the VSS EWMA-$\bar{X}$ control chart has not been investigated yet. Hence, this study aims to evaluate the effect of measurement errors on the performance of the VSS EWMA chart. To this end, the linear covariate error model proposed by Linna and Woodall [23] was employed and both fixed and linearly increasing error variances were taken into account. Then, the effects of the measurement errors, multiple measurements, and constant B on the performance of the adaptive VSS EWMA control chart were evaluated. A modified Markov chain method (inspired by the one proposed by Lucas and Saccucci [49]) was also used for calculating the run length properties of the VSS EWMA control chart. Further, this study compared the performance of the proposed VSS EWMA with that of FSR EWMA and also that of the Shewhart VSS $\bar{X}$ with VSI $\bar{X}$ control charts proposed by Hu et al. [28,29] in detecting step shifts in the process mean in terms of Average Run Length (ARL).

This study is organized as follows. The next section introduces the linearly covariate error model. Section 3 discusses the VSS EWMA control chart with measurement errors in detail. Section 4 evaluates the effect of the measurement errors and multiple measurements on the performance of the VSS EWMA in terms of the ARL and Average Number of Observations to Signal (ANOS) using the Markov chain. Section 5 compares the performances of both VSS EWMA and FSR EWMA control charts. Section 6 presents an illustrative example based on a real case. Finally, Section 7 concludes the study and provides further suggestions for future researches.

2. Linearly covariate error model

The linearly covariate measurement error model was first introduced by Linna and Woodall [23]. Assume that Y_{ij}, the jth observation of the quality characteristic at time i ($i = 1, 2, ...,$ and $j = 1, 2, ..., n_s$) with $n_s = \{n_1, n_2\}$, where n_1 and n_2 are respectively the small and large sample sizes, follow a normal $N\left(\mu_0 + \delta\sigma_0, \sigma_0^2\right)$ distribution, where μ_0 and σ_0 are the in-control mean and standard deviation of the process, respectively, both considered known. Here, δ is the magnitude of the mean shift in unit σ_0 (note that when the process is in-control, $\delta = 0$). In practice, the true value of the quality characteristic Y_{ij} is not directly observed, but it can be estimated from $k = 1, ..., m$ measurements X_{ijk} related to the true value Y_{ij} using the linearly covariate model:

$$X_{ijk} = A + BY_{ij} + \varepsilon_{ijk}, \tag{1}$$

where A and B are constants, and ε_{ijk} is the random error term that is assumed to be normally distributed with mean 0 and variance σ_M^2. Assume that we have N standard parts with known but different quality characteristic values. The quality characteristic of these standard parts is measured through some measurement systems. Due to the presence of measurement errors, the observed values are different from the real ones. The parameters A, B, and σ_M^2 are estimated using the true and observed values for the quality characteristics and fitting a simple linear regression. In addition, μ_X and σ_X^2 can be easily estimated using the sample mean μ_X and sample standard deviation σ_X of the measured values for the quality characteristic. Then, the values μ_Y and σ_Y^2 are estimated through the following equations:

$$\mu_Y = \frac{\mu_X - A}{B} \quad \text{and} \quad \sigma_Y^2 = \frac{\sigma_X^2 - \sigma_M^2}{B^2}.$$

According to Montgomery and Runger [50] and Linna and Woodall [23], there are two possible scenarios for σ_M^2: first, σ_M^2 is considered as a constant and in the second one, $\sigma_M^2 = C + D\mu_Y$ where C and D are two extra constants. Similar to parameters A and B, parameters C and D can be estimated through a dedicated simple linear regression model.

At time $i = 1, 2, ...$, the sample mean $\bar{X}_i$ of the subgroup $\{X_{i11}, ..., X_{i1m}, ..., X_{in1}, ..., X_{inm}\}$ is equal to:

$$\bar{X}_i = \frac{1}{mn_s} \sum_{j=1}^{n_s} \sum_{k=1}^{m} X_{ijk}. \tag{2}$$

By substituting Eq. (1) into Eq. (2), we have:

$$\bar{X}_i = A + \frac{1}{n_s} \left(B \sum_{j=1}^{n_s} Y_{ij} + \frac{1}{m} \sum_{j=1}^{n_s} \sum_{k=1}^{m} \varepsilon_{ijk} \right). \tag{3}$$

It can be easily shown that the expected value and the variance of $\bar{X}_i$, are respectively equal to:

$$\mathrm{E}\left(\bar{X}_i\right) = A + B\left(\mu_0 + \delta\sigma_0\right), \tag{4}$$

$$\mathrm{Var}\left(\bar{X}_i\right) = \frac{1}{n_s} \left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m} \right). \tag{5}$$

3. VSS EWMA control chart with measurement errors

As mentioned earlier, in case the sample sizes are variable, the control chart for this scheme is called VSS control chart. As mentioned in the previous section, in this study, we assumed that there were two types of sample sizes ($n_1 < n_2$). It can be clearly stated that based on Eq. (1) related to Y and X, X is normally distributed with the mean of $A + B\left(\mu_0 + \delta\sigma_0\right)$ and variance of $B^2\sigma_0^2 + \sigma_M^2$. With the assumption that $n_s = \{n_1, n_2\}$ at each sampling point, the mean of these observations can be calculated as $\bar{X}_i$ standardized below:

$$U_i = \frac{\bar{X}_i - \mu_{0_{\bar{X}_i}}}{\sigma_{\bar{X}_i}}, \tag{6}$$

where $\sigma_{\bar{X}_i}$ is the standard deviation of $\bar{X}_i$ obtained from taking the square root of Eq. (5) and $\mu_{0_{\bar{X}_i}}$ the in-control expected value of $\bar{X}_i$ obtained from Eq. (4).

Then, $U_i \sim N(0,1)$ when the process is in-control. Further, U_i is used in the EWMA statistic, as shown in the following:

$$Z_i = \lambda U_i + (1-\lambda) Z_{i-1}, \tag{7}$$

where Z_i follows a normal distribution with the following mean and asymptotic variance $\mu_{Z_i} = 0, \sigma^2_{Z_i} = \frac{\lambda}{2-\lambda}$, respectively. Hence, the steady state upper and lower control limits of the VSS EWMA chart are obtained by Eqs. (8) and (9), respectively:

$$UCL = L\sqrt{\frac{\lambda}{2-\lambda}}, \tag{8}$$

$$LCL = -L\sqrt{\frac{\lambda}{2-\lambda}}. \tag{9}$$

Similarly, the steady state upper and lower warning limits are respectively as follows:

$$UWL = W\sqrt{\frac{\lambda}{2-\lambda}}, \tag{10}$$

$$LWL = -W\sqrt{\frac{\lambda}{2-\lambda}}, \tag{11}$$

where $0 < W < L$. The VSS strategy is as follows:

- If $Z_i \in \Omega_1 = [LWL, UWL]$, the process is identified as "in-control", and the small sample size n_1 is taken for the next sampling period;
- If $Z_i \in \Omega_2 = [LCL, LWL) \cup (UWL, UCL]$, the process is also identified as "in-control", but the large sample size n_2 is chosen for the next sampling period;
- If $Z_i \notin \Omega_3 = [LCL, UCL]$, the process is declared as "out-of-control" and the corrective actions must be undertaken.

For transient states, we have:

$$Pr(Z_i \in \Omega_1) = \Phi(W) - \Phi(-W) = 2\Phi(W) - 1, \tag{12}$$

$$Pr(Z_i \in \Omega_2) = \Phi(K) - \Phi(W) + \Phi(-W) - \Phi(-K)$$
$$= 2(\Phi(K) - \Phi(W)), \tag{13}$$

$$Pr(Z_i \in \Omega_3) = \Phi(K) - \Phi(-K) = 2\Phi(K) - 1, \tag{14}$$

where $\Phi(.)$ denotes the cumulative distribution function (c.d.f) of the $N(0,1)$ distribution. To obtain the warning limit coefficient, the balanced equation $n_0 = n_1P_1 + n_2P_2$, is employed where n_0 is the average sample size (the weighted mean of n_1 and n_2), and P_1 and P_2 are the conditional probabilities of Z_i falling in Ω_1 and Ω_2, respectively, provided that Z_i is "in-control"(Ω_3). Hence, we have:

$$n_0 = n_1\frac{Pr(Z_i \in \Omega_1)}{Pr(Z_i \in \Omega_3)} + n_2\frac{Pr(Z_i \in \Omega_2)}{Pr(Z_i \in \Omega_3)}. \tag{15}$$

By substituting Eqs. (12)–(14) into Eq. (15) and determining W, the following equation is obtained:

$$W = \Phi^{-1}\left(\frac{2\Phi(L)(n_0 - n_2) - n_0 + n_1}{2(n_1 - n_2)}\right), \tag{16}$$

where $\Phi^{-1}(.)$ denotes the inverse c.d.f. of the $N(0,1)$ distribution.

4. The effect of measurement errors on the VSS EWMA control chart

In this section, the main focus is given to the performance of VSS EWMA control chart with a linearly covariate error model over a range of shift sizes. Following the calculation of W and consideration of the other parameters $(n_1, n_2, n_0, m, A, B, L)$, we compare the performance of the VSS EWMA control chart in terms of ARL and ANOS. In addition, Markov chain approach is employed to compute the ARL and ANOS for EWMA control chart, according to the method presented by Lucas and Saccucci [49]. The distance between the control limits is divided into $2p+1$ states. Each state has a width of $2d$, where $2d = \frac{UCL-LCL}{2p+1}$. The midpoint of each state is $f_t, t \in \{-p, ..., p\}$. The statistic Z_i is in the state t if $f_t - d \leq Z_i \leq f_t + d$. In case Z_i is within the control limits, it is in a transient state; and in case Z_i is outside the control limits, it enters the absorbing state. Let $\mathbf{Q}$ be the $(2p+1, 2p+1)$ matrix of transition probabilities Q_{gh} of from state g to state h:

$$Q_{gh} = Pr(f_h - d < Z_i < f_h + d \,|Z_{i-1} = f_g), \tag{17}$$

$$Q_{gh} = Pr(Z_i < f_h + d \,|Z_{i-1} = f_g)$$
$$-Pr(Z_i < f_h - d \,|Z_{i-1} = f_g). \tag{18}$$

By replacing Z_i with U_i based on Eq. (7), we have:

$$Q_{gh} = Pr\left(U_i < \frac{f_h + d - (1-\lambda) f_j}{\lambda}\right)$$
$$-Pr\left(U_i < \frac{f_h - d - (1-\lambda) f_g}{\lambda}\right). \tag{19}$$

Substituting U_i with $\bar{X}_i$ based on Eq. (6), Q_{gh} is obtained as follows:

$$Q_{gh} = Pr\left(\bar{X}_i < \left(\frac{f_h + d - (1-\lambda) f_g}{\lambda}\right) \sigma_{\bar{X}_i} + \mu_{0_{\bar{X}_i}}\right)$$
$$-Pr\left(\bar{X}_i < \left(\frac{f_h - d - (1-\lambda) f_g}{\lambda}\right) \sigma_{\bar{X}_i} + \mu_{0_{\bar{X}_i}}\right). \tag{20}$$

For computing this probability, Eq. (20) should be standardized; hence, by using Eqs. (4) and (5), we have:

$$Q_{gh} = Pr\left(\frac{\bar{X}_i - \mu_{\bar{X}_i}}{\sigma_{\bar{X}_i}} < \frac{\left[\left(\frac{f_h + d - (1-\lambda) f_g}{\lambda}\right)\sigma_{\bar{X}_i} + \mu_{0_{\bar{X}_i}} - \mu_{\bar{X}_i}\right]}{\sigma_{\bar{X}_i}}\right) - Pr\left(\frac{\bar{X}_i - \mu_{\bar{X}_i}}{\sigma_{\bar{X}_i}} < \frac{\left[\left(\frac{f_h - d - (1-\lambda) f_g}{\lambda}\right)\sigma_{\bar{X}_i} + \mu_{0_{\bar{X}_i}} - \mu_{\bar{X}_i}\right]}{\sigma_{\bar{X}_i}}\right). \tag{21}$$

Considering $\mu_{0_{\bar{X}_i}} = A + B\mu_0$ (when $\delta = 0$), $\mu_{\bar{X}_i} = A + B\left(\mu_0 + \delta\sigma_0\right)$, and $\sigma_{\bar{X}_i} = \sqrt{\frac{1}{n}\left(B^2\sigma_0^2 + \frac{\sigma_M^2}{m}\right)}$ and some simplifications in Eq. (21), we have:

$$Q_{gh} = \Phi\left(\left(\frac{f_h + d - (1-\lambda) f_g}{\lambda}\right) - \frac{B\delta\sigma_0}{\sigma_{\bar{X}_i}}\right) - \Phi\left(\left(\frac{f_h - d - (1-\lambda) f_g}{\lambda}\right) - \frac{B\delta\sigma_0}{\sigma_{\bar{X}_i}}\right). \tag{22}$$

Let $\mathbf{q}$ and $\mathbf{n}$ be the $(2p+1,1)$ vector of the initial probabilities and $(2p+1,1)$ vector of the sample sizes, respectively, associated with each transient state, i.e.:

$$\mathbf{q_t} = \begin{cases} 0 & \text{if } \ Z_i \notin (f_t - d, f_t + d] \\ 1 & \text{if } \ Z_i \in (f_t - d, f_t + d] \end{cases} \tag{23}$$

$$\mathbf{n_t} = \begin{cases} n_S & \text{if } \ LWL \le f_t \le UWL \\ n_L & \text{otherwise} \end{cases}$$

Finally, the ARL and ANOS of the VSS EWMA control chart can be obtained using:

$$ARL_1 = \mathbf{q}^T\left(\mathbf{I} - \mathbf{Q}\right)^{-1}\mathbf{1}, \tag{24}$$

$$ANOS_1 = \mathbf{q}^T\left(\mathbf{I} - \mathbf{Q}\right)^{-1}\mathbf{n}, \tag{25}$$

where $\mathbf{I}$ is the $(2p+1, 2p+1)$ identity matrix and $\mathbf{1}$ is the $(2p+1,1)$ unit column vector.

To evaluate the effect of measurement errors on the performance of the VSS EWMA chart, $\lambda = 0.2$ and $L = 2.962$ were employed to achieve an in-control ARL of 500. In addition, the $2p+1$ states were used for the implementation of the Markov chain approach where $p = 105$ based on Maravelakis et al. [40]. Moreover, in the present study, the variances of both constant measurement errors as well as those of the linearly increasing measurement errors were taken into account.

4.1. A constant measurement error variance

The ARL_1 and ANOS_1 for the VSS EWMA chart with a linearly covariate error model are listed in Table 1 for different combinations of measurement errors $\left(\sigma_M^2 = 0, 0.3, 0.7, 1\right)$ and shift magnitudes in process mean $(\delta = 0.1, 0.5, 1, 2)$, $n_1 = 1, 3, 5$, and $n_2 = 6, 7, 10$ when $m = 1$ and $B = 1$. According to this table, in case the variance of measurement errors (σ_M^2) increases, the ARL_1 and ANOS_1 also increase, thus increasing the negative effect of measurement errors. For example, if $\delta = 0.1, n_1 = 1, n_2 = 6$, and $\sigma_M^2 = 0$, we have $\text{ARL}_1 = 184.8$ and $\text{ANOS}_1 = 691.6$; and if $\sigma_M^2 = 1$, we have $\text{ARL}_1 = 276.43$ and $\text{ANOS}_1 = 1004.11$. Table 2 shows the effect of the number of measurements (m) when $\sigma_M^1 = 1$ and $B = 1$ for different shifts in the process mean $(\delta = 0.1, 0.5, 1, 2)$. According to Table 2, using multiple measurements can attenuate the negative effect of measurement errors. For example, if we consider the combination $(m = 1, \delta = 0.1, n_1 = 1, n_2 = 6)$, we have $\text{ARL}_1 = 276.43$ and $\text{ANOS}_1 = 1004.11$; and if $m = 2$ we have $\text{ARL}_1 = 238.2$ and $\text{ANOS}_1 = 874.5$.

Table 3 presents the results of sensitivity analysis of the parameter B. According to this table, in case $\sigma_M^2 = 1$ and $m = 1$ for different shifts in the process mean, the positive effect of parameter B can significantly reduce the negative effects of measurement errors. For example, considering the combination $(\delta = 0.1, n_1 = 1, n_2 = 6)$ for $B = 1$ leads to $\text{ARL}_1 = 276.43$ and $\text{ANOS}_1 = 1004.11$ while, for the same combination and $B = 2$, we have $\text{ARL}_1 = 213.9$ and $\text{ANOS}_1 = 791.6$. Table 4 shows the ARL and ANOS of the VSS EWMA chart at different values of σ_M^2 when $m = 5$ and $B = 1$. A comparison between the results of Tables 1 and 4 shows that increasing m at different values of σ_M^2 reduces the negative effect of the measurement errors. For example, considering the combination $(\delta = 0.1, n_1 = 5, n_2 = 10, \sigma_M^2 = 0.3)$ and $m = 1$ and $m = 5$ lead to $\text{ARL}_1 = 139.2$ and $\text{ARL}_1 = 117.4$, respectively. Table 5 reports ARL_1 and ANOS_1 for different values of B, $\sigma_M^1 = 1$, and $m = 5$. A comparison between Tables 5 and 3 shows that when $m = 5$, the effect of measurement errors is greater than that when $m = 1$. For example, considering the combination $(\delta = 0.1, n_1 = 5, n_2 = 10)$ when $B = 3$ and $m = 1$, ARL_1 is equal to 122.3 while when $m = 5$, the ARL_1 decreases to 113.8.

4.2. Linearly increasing measurement errors variance

In the case of linearly increasing measurement errors variance, we have $\sigma_M^2 = C + D\mu_Y$. We investigate the effect of parameter D on the performance of the VSS EWMA chart. Table 6 presents the ARL_1 and ANOS_1 performances of the VSS EWMA chart with linearly increasing variance for different values of D, $B = 1$, $C = 0$, and $m = 1$. From Table 6, increasing

Table 1. ARL_1, and $ANOS_1$ when $ARL_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $B = 1$, $m = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
$ANOS_0$	1753.6	3751.7	2501.8	3254.2
		$\sigma_M^2 = 0$		
$\delta = 0.1$	(184.8, 691.6)	(111.6, 892.6)	(152.4, 805.27)	(118.07, 838.01)
$\delta = 0.5$	(9.54, 40.07)	(5.68, 47.55)	(7.47, 43.00)	(6.16, 45.96)
$\delta = 1$	(4.13, 15.16)	(2.59, 20.31)	(3.18, 17.05)	(2.86, 19.47)
$\delta = 2$	(2.25, 7.51)	(1.67, 11.78)	(1.94, 9.56)	(1.93, 12.30)
		$\sigma_M^2 = 0.3$		
$\delta = 0.1$	(219.1, 809.4)	(139.2, 1099.7)	(184.5, 964.4)	(146.7, 1024.5)
$\delta = 0.5$	(11.65, 50.06)	(6.82, 57.73)	(9.14, 53.19)	(7.36, 55.91)
$\delta = 1$	(4.74, 17.79)	(2.95, 23.42)	(3.66, 19.88)	(3.25, 22.51)
$\delta = 2$	(7.51, 8.26)	(1.84, 13.46)	(2.03, 10.12)	(2.00, 12.82)
		$\sigma_M^2 = 0.7$		
$\delta = 0.1$	(254.99, 931.39)	(170.7, 1335.2)	(219.3, 1136.01)	(179.2, 1234.02)
$\delta = 0.5$	(14.57, 63.67)	(8.35, 71.37)	(11.42, 67.03)	(8.95, 69.24)
$\delta = 1$	(5.49, 21.18)	(3.39, 27.31)	(4.26, 23.47)	(3.73, 26.27)
$\delta = 2$	(2.75, 9.42)	(1.95, 14.53)	(2.18, 11.02)	(2.08, 13.41)
		$\sigma_M^2 = 1$		
$\delta = 0.1$	(276.43, 1004.11)	(191.16, 1486.8)	(240.8, 1241.7)	(200.0, 1367.5)
$\delta = 0.5$	(16.85, 74.06)	(9.51, 81.72)	(13.17, 77.63)	(10.16, 79.35)
$\delta = 1$	(6.04, 23.67)	(3.71, 30.10)	(4.69, 26.08)	(4.07, 28.97)
$\delta = 2$	(2.96, 10.28)	(2.02, 15.14)	(2.31, 11.81)	(2.16, 14.02)

D significantly increases ARL_1 and $ANOS_1$, indicating the negative effect of the linearly increasing variance on the performance of the VSS EWMA chart. Table 7 shows ARL_1 and $ANOS_1$ based on different values of D for $m = 5$. Comparison between Tables 6 and 7 shows that when $m = 5$, the effect of parameter D on increasing ARL_1 and $ANOS_1$ is less than that in the case of $m = 1$.

5. Performance validation of the VSS EWMA

In this section, we provide a comparison between the VSS EWMA, FSR EWMA, adaptive Shewhart control charts (VSS $\bar{X}$ and VSI $\bar{X}$) for a wide range of mean shifts ($\delta = [0, 0.9]$) with and without measurement errors in terms of the ARL criterion. For this aim, we considered the following parameters ($\lambda = 0.2, n_1 = 3, n_2 = 7, n_0 = 5, B = 1$) and an in-control ARL equal to 500 for all control charts. As a first case, it is assumed that there are no measurement errors ($\sigma_M^2 = 0$) and the results are shown in Figure 1. As illustrated in this figure, the VSS EWMA control chart outperforms the FSR EWMA, VSS, and VSI $\bar{X}$ charts for detecting shifts in the process mean. For example, when $\delta = 0.2$, ARL_1 for the VSS EWMA is equal to 41.28, while this value is equal to 52.48, 224.8, and 231.8 for the FSR EWMA, VSS $\bar{X}$, and VSI $\bar{X}$ control charts, respectively.

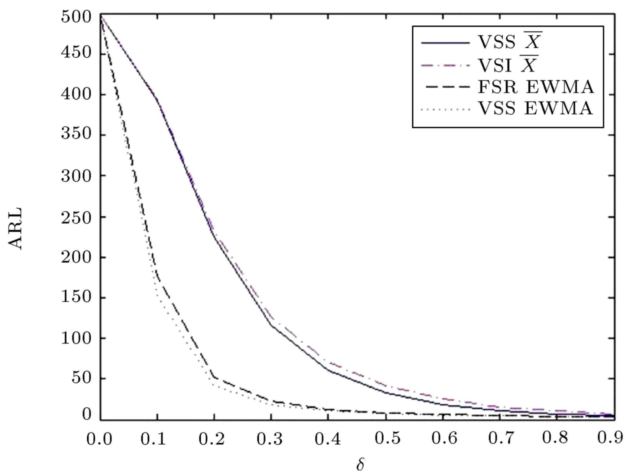

Figure 1. The comparison of the performance of Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control chart with those of FSR EWMA, Variable Sample Size (VSS) $\bar{X}$, and Variable Sampling Interval (VSI) $\bar{X}$ control charts with no measurement errors.

Table 2. ARL_1 and ANOS_1 when $\mathrm{ARL}_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $B = 1$, $\sigma_M^2 = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
ANOS_0	1753.6	3751.7	2501.8	3254.2
		$m = 1$		
$\delta = 0.1$	(276.43, 1004.11)	(191.16, 1486.8)	(240.8, 1241.7)	(200.0, 1367.5)
$\delta = 0.5$	(16.85, 74.06)	(9.51, 81.72)	(13.17, 77.63)	(10.16, 79.35)
$\delta = 1$	(6.04, 23.67)	(3.71, 30.10)	(4.69, 26.08)	(4.07, 28.97)
$\delta = 2$	(2.96, 10.28)	(2.02, 15.14)	(2.31, 11.81)	(2.16, 14.02)
		$m = 2$		
$\delta = 0.1$	(238.2, 874.5)	(155.6, 1222.7)	(202.9, 1055.19)	(163.7, 1134.2)
$\delta = 0.5$	(13.10, 56.82)	(7.58, 64.53)	(10.27, 60.07)	(8.15, 62.56)
$\delta = 1$	(5.12, 19.50)	(3.17, 25.40)	(3.96, 21.69)	(3.498, 24.423)
$\delta = 2$	(2.60, 8.84)	(1.91, 14.07)	(2.10, 10.54)	(2.03, 13.09)
		$m = 3$		
$\delta = 0.1$	(222.5, 820.9)	(142.06, 1121.06)	(187.7, 980.3)	(149.6, 1043.5)
$\delta = 0.5$	(11.89, 51.18)	(6.95, 58.86)	(9.33, 54.33)	(7.49, 57.02)
$\delta = 1$	(4.80, 18.08)	(2.99, 23.76)	(3.71, 20.19)	(3.29, 22.83)
$\delta = 2$	(2.48, 8.35)	(1.85, 13.58)	(2.043, 10.19)	(2.00, 12.86)
		$m = 4$		
$\delta = 0.1$	(213.9, 791.6)	(134.8, 1067.2)	(179.6, 939.9)	(142.2, 995.37)
$\delta = 0.5$	(11.30, 48.38)	(6.63, 56.04)	(8.86, 51.48)	(7.16, 54.25)
$\delta = 1$	(4.64, 17.36)	(2.89, 22.9)	(3.58, 19.42)	(3.19, 22.02)
$\delta = 2$	(2.42, 8.12)	(1.82, 13.26)	(2.01, 10.03)	(1.99, 12.75)
		$m = 5$		
$\delta = 0.1$	(208.5, 773.1)	(130.4, 1033.9)	(174.50, 914.6)	(137.6, 965.4)
$\delta = 0.5$	(10.94, 46.71)	(6.44, 54.3)	(8.58, 49.78)	(6.96, 52.60)
$\delta = 1$	(4.54, 16.93)	(2.83, 22.41)	(3.5, 18.95)	(3.12, 21.52)
$\delta = 2$	(2.38, 7.99)	(1.80, 13.04)	(2.00, 9.93)	(1.98, 12.68)

Figure 2 shows the ARL curves for the considered charts in the presence of measurement errors ($\sigma_M^2 = 1$). In fact, a comparison was made between the performance of the proposed control chart and those of the FSR EWMA and adaptive Shewhart control charts (VSS $\bar{X}$ and VSI $\bar{X}$), suggested by Hu et al. [28,29]. The curves were plotted by taking into account $m = 1$ as well as the parameters mentioned in the previous comparison. According to Figure 2, the proposed VSS EWMA chart outperforms its counterparts, i.e., FSR EWMA, VSS $\bar{X}$, VSI $\bar{X}$. To be specific, when $\delta = 0.2$, the ARL_1 values for the VSS EWMA and FSR EWMA as well as VSS $\bar{X}$ and VSI $\bar{X}$ charts are equal to 83.49 and 101.9 as well as 318.9 and 322.5, respectively. In Figure 3, we again considered the parameters used in Figure 2. However, the number of measurements increased to $m = 5$. The results showed high sensitivity for all control charts since the out-of-control ARL values obtained for $m = 5$ were smaller

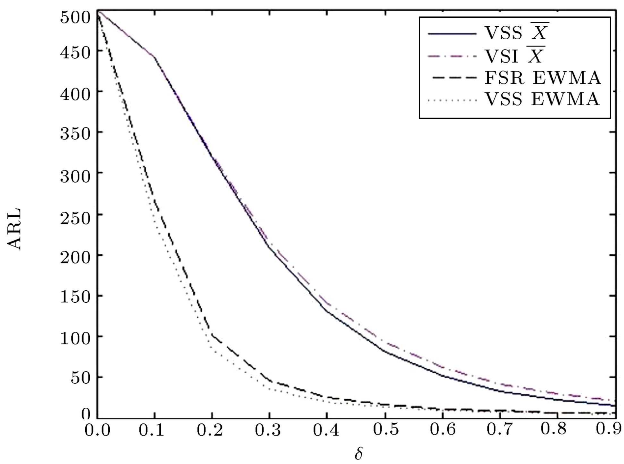

Figure 2. The comparison of the performance of Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control chart with those of FSR EWMA, Variable Sample Size (VSS) $\bar{X}$, and Variable Sampling Interval (VSI) $\bar{X}$ control charts when $\sigma_M^2 = 1$ and $m = 1$.

Table 3. ARL_1 and ANOS_1 when $\mathrm{ARL}_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $m = 1$, $\sigma_M^2 = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
ANOS_0	1753.6	3751.7	2501.8	3254.2
		$B = 1$		
$\delta = 0.1$	(276.43, 1004.11)	(191.16, 1486.8)	(240.8, 1241.7)	(200.0, 1367.5)
$\delta = 0.5$	(16.85, 74.06)	(9.51, 81.72)	(13.17, 77.63)	(10.16, 79.35)
$\delta = 1$	(6.04, 23.67)	(3.71, 30.10)	(4.69, 26.08)	(4.07, 28.97)
$\delta = 2$	(2.96, 10.28)	(2.02, 15.14)	(2.31, 11.81)	(2.16, 14.02)
		$B = 2$		
$\delta = 0.1$	(213.9, 791.6)	(134.8, 1067.2)	(179.6, 939.9)	(142.2, 995.37)
$\delta = 0.5$	(11.30, 48.38)	(6.63, 56.04)	(8.86, 51.48)	(7.16, 54.25)
$\delta = 1$	(4.64, 17.36)	(2.89, 22.92)	(3.58, 19.42)	(3.19, 22.02)
$\delta = 2$	(2.42, 8.12)	(1.82, 13.26)	(2.01, 10.03)	(1.99, 12.75)
		$B = 3$		
$\delta = 0.1$	(198.4, 738.5)	(122.3, 972.7)	(165.04, 867.7)	(129.1, 910.4)
$\delta = 0.5$	(10.31, 43.75)	(6.10, 51.32)	(8.09, 46.76)	(6.60, 49.65)
$\delta = 1$	(4.36, 16.15)	(2.73, 21.49)	(3.36, 18.12)	(3.01, 20.62)
$\delta = 2$	(2.32, 7.76)	(1.75, 12.56)	(1.97, 9.77)	(1.96, 12.53)
		$B = 4$		
$\delta = 0.1$	(192.6, 718.5)	(117.7, 938.2)	(159.6, 841.00)	(124.3, 879.2)
$\delta = 0.5$	(9.97, 42.14)	(5.92, 49.67)	(7.82, 45.11)	(6.41, 48.04)
$\delta = 1$	(4.26, 15.72)	(2.67, 20.98)	(3.28, 17.66)	(2.95, 20.12)
$\delta = 2$	(2.29, 7.65)	(1.72, 12.25)	(1.96, 9.68)	(1.95, 12.44)

Table 4. ARL_1 and ANOS_1 when $\mathrm{ARL}_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $B = 1$, $m = 5$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
ANOS_0	1753.6	3751.7	2501.8	3254.2
		$\sigma_M^2 = 0$		
$\delta = 0.1$	(184.8, 691.6)	(111.6, 892.6)	(152.4, 805.27)	(118.07, 838.01)
$\delta = 0.5$	(9.54, 40.07)	(5.68, 47.55)	(7.47, 43.00)	(6.16, 45.96)
$\delta = 1$	(4.13, 15.16)	(2.59, 20.31)	(3.18, 17.05)	(2.86, 19.47)
$\delta = 2$	(2.25, 7.51)	(1.67, 11.78)	(1.94, 9.56)	(1.93, 12.30)
		$\sigma_M^2 = 0.3$		
$\delta = 0.1$	(192.3, 717.4)	(117.4, 936.4)	(159.3, 839.6)	(124.1, 877.6)
$\delta = 0.5$	(9.96, 42.05)	(5.91, 49.59)	(7.81, 45.03)	(6.40, 47.95)
$\delta = 1$	(4.26, 15.69)	(2.67, 20.95)	(3.28, 17.63)	(2.94, 20.10)
$\delta = 2$	(2.29, 7.64)	(1.72, 12.23)	(1.96, 9.68)	(1.95, 12.44)
		$\sigma_M^2 = 0.7$		
$\delta = 0.1$	(201.8, 750.02)	(124.9, 992.9)	(168.1, 883.3)	(131.9, 928.5)
$\delta = 0.5$	(10.52, 44.71)	(6.21, 52.30)	(8.25, 47.74)	(6.72, 50.61)
$\delta = 1$	(4.42, 16.40)	(2.76, 21.79)	(3.41, 18.39)	(3.05, 20.92)
$\delta = 2$	(2.34, 7.83)	(1.77, 12.73)	(1.98, 9.82)	(1.97, 12.58)
		$\sigma_M^2 = 1$		
$\delta = 0.1$	(208.5, 773.1)	(130.4, 1033.9)	(174.50, 914.6)	(137.6, 965.4)
$\delta = 0.5$	(10.94, 46.71)	(6.44, 54.3)	(8.58, 49.78)	(6.96, 52.60)
$\delta = 1$	(4.54, 16.93)	(2.83, 22.41)	(3.5, 18.95)	(3.12, 21.52)
$\delta = 2$	(2.38, 7.99)	(1.80, 13.04)	(2.00, 9.93)	(1.98, 12.68)

Table 5. ARL_1 and $ANOS_1$ when $ARL_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $m = 5$, $\sigma_M^2 = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
$ANOS_0$	1753.6	3751.7	2501.8	3254.2
		$B = 1$		
$\delta = 0.1$	(208.5, 773.1)	(130.4, 1033.9)	(174.5, 914.6)	(137.6, 965.4)
$\delta = 0.5$	(10.94, 46.71)	(6.44, 54.34)	(8.58, 49.78)	(6.96, 52.60)
$\delta = 1$	(4.54, 16.93)	(2.83, 22.41)	(3.50, 18.95)	(3.12, 21.52)
$\delta = 2$	(2.38, 7.99)	(1.8, 13.04)	(2.00, 9.93)	(1.98, 12.68)
		$B = 2$		
$\delta = 0.1$	(191.09, 713.2)	(116.5, 929.2)	(158.2, 833.9)	(123.1, 871.1)
$\delta = 0.5$	(9.89, 41.72)	(5.87, 49.25)	(7.75, 44.69)	(6.36, 47.62)
$\delta = 1$	(4.23, 15.61)	(2.65, 20.84)	(3.26, 17.54)	(2.93, 20.00)
$\delta = 2$	(2.28, 7.62)	(1.71, 12.16)	(1.95, 9.66)	(1.950, 12.42)
		$B = 3$		
$\delta = 0.1$	(187.6, 701.3)	(113.8, 909.03)	(155.04, 818.14)	(120.3, 852.8)
$\delta = 0.5$	(9.69, 40.80)	(5.76, 48.30)	(7.60, 43.75)	(6.25, 46.70)
$\delta = 1$	(4.181, 15.36)	(2.62, 20.55)	(3.22, 17.27)	(2.89, 19.70)
$\delta = 2$	(2.269, 7.56)	(1.69, 11.96)	(1.949, 9.60)	(1.943, 12.36)
		$B = 4$		
$\delta = 0.1$	(186.4, 697.1)	(112.9, 901.9)	(153.9, 812.54)	(119.3, 846.3)
$\delta = 0.5$	(9.62, 40.48)	(5.73, 47.97)	(7.54, 43.42)	(6.21, 46.38)
$\delta = 1$	(4.161, 15.27)	(2.61, 20.45)	(3.20, 17.17)	(2.88, 19.60)
$\delta = 2$	(2.263, 7.542)	(1.68, 11.88)	(1.946, 9.58)	(1.940, 12.33)

Table 6. ARL_1 and $ANOS_1$ when $ARL_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $m = 1$, $B = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
$ANOS_0$	1753.6	3751.7	2501.8	3254.2
		$D = 1$		
$\delta = 0.1$	(438.8, 1550.4)	(393.5, 2976.1)	(422.2, 2126.8)	(399.8, 2629.4)
$\delta = 0.5$	(98.12, 388.2)	(52.78, 440.02)	(77.02, 424.6)	(55.98, 421.5)
$\delta = 1$	(24.91, 109.4)	(13.51, 117.1)	(19.33, 114.04)	(14.36, 113.8)
$\delta = 2$	(8.23, 33.91)	(4.96, 41.13)	(6.44, 36.68)	(5.40, 39.70)
		$D = 2$		
$\delta = 0.1$	(465.9, 1641.3)	(438.2, 3303.6)	(456.1, 2291.1)	(442.2, 2895.8)
$\delta = 0.5$	(168.6, 636.05)	(99.63, 801.3)	(137.8, 732.3)	(105.4, 755.01)
$\delta = 1$	(49.06, 206.8)	(25.57, 220.2)	(37.81, 217.8)	(27.08, 213.4)
$\delta = 2$	(13.56, 58.95)	(7.82, 66.67)	(10.63, 62.24)	(8.40, 64.64)
		$D = 3$		
$\delta = 0.1$	(476.3, 1676.05)	(456.4, 3436.7)	(469.3, 2355.3)	(459.3, 3003.02)
$\delta = 0.5$	(219.1, 809.4)	(139.2, 1099.7)	(184.5, 964.4)	(146.7, 1024.5)
$\delta = 1$	(73.29, 298.04)	(38.49, 326.1)	(56.87, 319.7)	(40.80, 314.3)
$\delta = 2$	(19.30, 85.01)	(10.74, 92.63)	(15.05, 88.84)	(11.45, 90.00)
		$D = 4$		
$\delta = 0.1$	(481.8, 1694.3)	(466.3, 3508.8)	(476.3, 2389.5)	(468.6, 3060.8)
$\delta = 0.5$	(256.5, 936.6)	(172.2, 1345.9)	(220.9, 1143.6)	(180.7, 1243.4)
$\delta = 1$	(96.13, 381.0)	(51.60, 430.6)	(75.38, 416.2)	(54.72, 412.7)
$\delta = 2$	(25.43, 111.6)	(13.77, 1.194)	(19.73, 116.3)	(14.63, 116.09)

Table 7. ARL_1 and $ANOS_1$ when $ARL_0 = 500$, $k = 2.962$, $\lambda = 0.2$, $m = 5$, $B = 1$.

n_1, n_2	$n_1 = 1, n_2 = 6$	$n_1 = 5, n_2 = 10$	$n_1 = 3, n_2 = 7$	$n_1 = 3, n_2 = 10$
W	0.672	0.672	0.672	0.672
$ANOS_0$	1753.6	3751.7	2501.8	3254.2
		$D = 1$		
$\delta = 0.1$	(327.7, 1177.5)	(245.1, 1886.3)	(294.6, 1504.8)	(254.6, 1714.6)
$\delta = 0.5$	(25.75, 113.01)	(13.93, 120.7)	(19.97, 117.75)	(14.79, 117.41)
$\delta = 1$	(8.15, 33.50)	(4.91, 40.70)	(6.37, 36.26)	(5.34, 39.28)
$\delta = 2$	(3.81, 13.79)	(2.41, 18.68)	(2.93, 15.57)	(2.65, 17.85)
		$D = 2$		
$\delta = 0.1$	(381.6, 1358.7)	(310.9, 2370.7)	(354.4, 1796.9)	(319.8, 2126.8)
$\delta = 0.5$	(44.14, 187.6)	(23.07, 199.1)	(34.02, 197.00)	(24.42, 193.2)
$\delta = 1$	(12.01, 51.75)	(7.01, 59.43)	(9.42, 54.90)	(7.55, 57.57)
$\delta = 2$	(5.02, 19.07)	(3.12, 24.91)	(3.89, 21.24)	(3.43, 23.95)
		$D = 3$		
$\delta = 0.1$	(409.7, 1453.2)	(349.8, 2655.7)	(387.3, 1956.7)	(357.7, 2365.1)
$\delta = 0.5$	(62.84, 259.2)	(32.80, 279.9)	(48.58, 275.8)	(34.75, 270.4)
$\delta = 1$	(16.08, 70.58)	(9.12, 78.26)	(12.58, 74.08)	(9.76, 75.97)
$\delta = 2$	(6.13, 24.08)	(3.76, 30.55)	(4.76, 26.51)	(4.12, 29.41)
		$D = 4$		
$\delta = 0.1$	(427.09, 1511.2)	(375.3, 2843.3)	(408.02, 2057.4)	(382.4, 2520.3)
$\delta = 0.5$	(80.95, 326.1)	(42.79, 360.6)	(63.02, 352.02)	(45.36, 346.9)
$\delta = 1$	(20.39, 89.87)	(11.28, 97.48)	(15.89, 93.83)	(12.02, 94.73)
$\delta = 2$	(7.19, 29.00)	(4.37, 35.91)	(5.61, 31.63)	(4.77, 34.62)

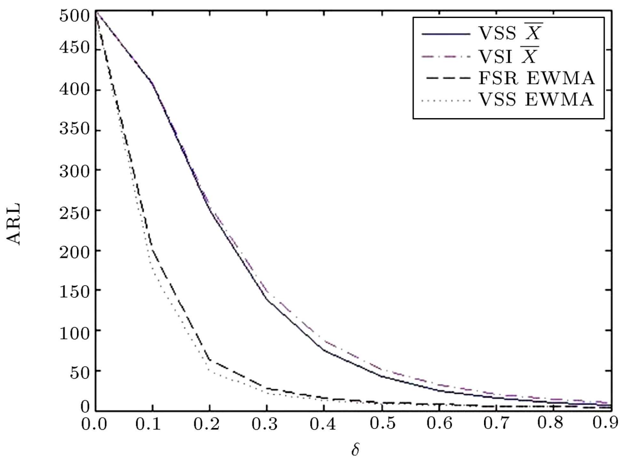

Figure 3. The comparison of the performance of Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control chart with those of FSR EWMA, Variable Sample Size (VSS) $\bar{X}$, and Variable Sampling Interval (VSI) $\bar{X}$ control charts when $\sigma_M^2 = 1$ and $m = 5$.

than those obtained for $m = 1$. Moreover, VSS EWMA control chart still outperformed the FSR EWMA, VSS $\bar{X}$, and VSI $\bar{X}$ control charts in this case, too.

6. An illustrative example

This section illustrates the application of the VSS EWMA control chart with measurement errors. To this end, a Costa and Castagliola [25] case is used corresponding to a production line of yogurt. The quality characteristic to be monitored is the weight of each yogurt cup. From a historical perspective, we know that $\mu_0 = 124.9$, $\sigma_0^2 = 0.578$, and $\sigma_M^2 = 0.058$. We also assume that $\lambda = 0.2$, $L = 2.962$, $A = 0$, $B = 1$, $m = 2$, $n_1 = 2$, and $n_2 = 5$ for monitoring purposes based on Eqs. (8)–(11); then, we obtained UCL= 0.987, LCL= −0.987, UWL = 0.224, and LWL = −0.224. Then, based on the mentioned parameters, we generated new data, computed the statistics $\bar{X}_i$, U_i, and Z_i until the VSS EWMA control chart would signal the out-of-control status. The results are reported in Table 8 and graphically presented in Figure 4. According to the results, the statistic Z_i is located below LCL in sample 20, signaling an 'out-of-control' situation. Hence, the process needs corrective actions.

7. Conclusion and suggestions for future research

The main objective of the present study was to evaluate the effect of measurement errors on the Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control chart using covariate error model. The performance of the proposed VSS EWMA chart was evaluated in terms of the Average Run Length

Table 8. The sample size n and statistics $\bar{X}_i$, U_i, Z_i and process status for simulated data generated based on the parameters reported by Costa and Castagliola [25].

Time (i)	n_i	$\bar{X}_i$	U_i	Z_i	Status
1	2	124.86	−1.12	−0.12	In-control
2	2	124.69	−1.43	−0.38	In-control
3	5	125.49	0.02	−0.30	In-control
4	5	125.11	−1.06	−0.45	In-control
5	5	125.58	0.28	−0.30	In-control
6	5	125.53	0.14	−0.21	In-control
7	2	125.59	0.19	−0.13	In-control
8	2	125.69	0.38	−0.03	In-control
9	2	124.88	−1.08	−0.24	In-control
10	5	125.45	−0.08	−0.21	In-control
11	2	125.54	0.10	−0.14	In-control
12	2	124.79	−1.24	−0.36	In-control
13	5	125.82	0.97	−0.09	In-control
14	2	126.68	2.16	0.35	In-control
15	5	125.64	0.45	0.37	In-control
16	5	125.93	1.28	0.55	In-control
17	5	125.99	1.44	0.73	In-control
18	5	126.15	1.91	0.97	In-control
19	5	125.61	0.37	0.85	In-control
20	5	126.06	1.64	1.01	Out-of-control

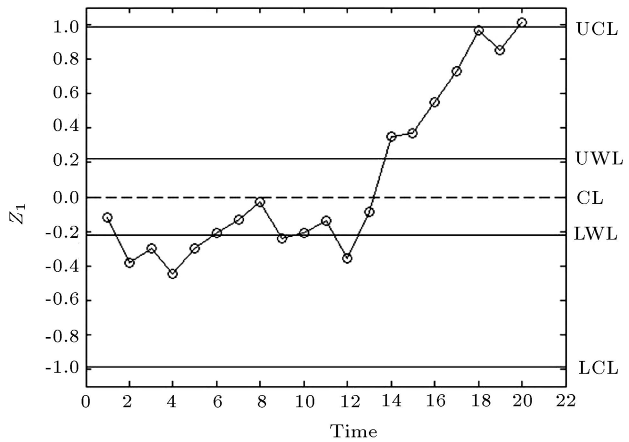

Figure 4. Variable Sample Size Exponentially Weighted Moving Average (VSS EWMA) control chart for a real-case example.

(ARL) and Average Number of Observations to Signal (ANOS) performance measures through the Markov chain approach. The results pointed to the negative effect of the measurement errors on the performance of the VSS EWMA control chart. In addition, upon increasing multiple measurements as well as the parameter B, the negative effect of the measurement errors on the performance of the VSS EWMA control chart would be attenuated. Moreover, the performance of the proposed VSS EWMA control chart was compared with those of FSR EWMA, Variable Sample Size (VSS) $\bar{X}$, and Variable Sampling Interval (VSI) $\bar{X}$ control charts in terms of ARL criterion in the presence of measurement errors, the results of which indicated that the VSS EWMA chart outperformed its counterparts in detecting shifts in the process mean in the presence of measurement errors.

Statistical and economic-statistical designing of other adaptive procedures such as VSI, VSSI, and Variable Parameters (VP) for the Exponentially Weighted Moving Average (EWMA) control chart was suggested for future study. In this study, the application of multiple measurements, which is the most common solution for reducing the effect of measurement errors, was taken into account. Evaluating other methods for reducing the effect of measurement errors on the performance the VSS EWMA control chart as well as for other adaptive EWMA control charts can be considered for future studies (see Ghashghaei et al. [34] and Amiri et al. [35]).

References

1. Prabhu, S.S., Runger, G.C., and Keats, J.B. "$\bar{X}$ chart with adaptive sample sizes", *International Journal of Production Research,* **31**(12), pp. 2895–2909 (1993).

2. Costa, A.F.B. "$\bar{X}$ charts with variable sampling size", *Journal of Quality Technology*, **26**(3), pp. 155–163 (1994).

3. Reynolds, M.R. "Variable sampling interval control charts with sampling at fixed times", *IIE Transactions*, **28**(6), pp. 497–510 (1996).

4. Amiri, F., Noghondarian, K., and Noorossana, R. "Economic-statistical design of adaptive $\bar{X}$ control chart: a Taguchi loss function approach", *Scientia Iranica. Transaction E: Industrial Engineering*, **21**(3), pp. 1096–1104 (2014).

5. Prabhu, S.S., Montgomery, D.C., and Runger, G.C. "A combined adaptive sample size and sampling interval $\bar{X}$ control scheme", *Journal of Quality Technology*, **26**(3), pp. 164–176 (1994).

6. Costa, A.F.B. "$\bar{X}$ chart with variable sample size and sampling intervals", *Journal of Quality Technology*, **29**(2), pp. 197–204 (1997).

7. Costa, A.F.B. "$\bar{X}$ charts with variable parameters", *Journal of Quality Technology*, **31**(4), pp. 408–416 (1999).

8. Saccucci, M.S., Amin, R.W., and Lucas, J.M. "Exponentially weighted moving average control schemes with variable sampling intervals", *Communications in Statistics-Simulation and Computation*, **21**(3), pp. 627–657 (1992).

9. Reynolds Jr, M.R. and Arnold, J.C. "EWMA control charts with variable sample sizes and variable sampling intervals", *IIE Transactions*, **33**(6), pp. 511–530 (2001).

10. Castagliola, P., Celano, G., and Fichera, S. "Evaluation of the statistical performance of a variable sampling interval R EWMA control chart", *Quality Technology & Quantitative Management*, **3**(3), pp. 307–323 (2006).

11. Castagliola, P., Celano, G., Fichera, S., and Giuffrida, F. "A variable sampling interval S_2-EWMA control chart for monitoring the process variance", *International Journal of Technology Management*, **37**(1–2), pp. 125–146 (2006).

12. Kazemzadeh, R.B., Karbasian, M., and Babakhani, M.A. "An EWMA t chart with variable sampling intervals for monitoring the process mean", *The International Journal of Advanced Manufacturing Technology*, **66**(1–4), pp. 125–139 (2013).

13. Tran, K.P, Castagliola, P., Nguyen, T.H., and Cuzol, A. "The efficiency of the VSI exponentially weighted moving average median control chart", *Proceeding of 24th ISSAT International Conference on Reliability and Quality in Design*, Toronto, Canada (2018).

14. Ugaz, W., Sánchez, I., and Alonso, A.M. "Adaptive EWMA control charts with time-varying smoothing parameter", *The International Journal of Advanced Manufacturing Technology*, **93**(9–12), pp. 3847–3858 (2017).

15. Tang, A., Castagliola, P., Sun, J., and Hu, X. "Optimal design of the adaptive EWMA chart for the mean based on median run length and expected median run length", *Quality Technology & Quantitative Management*, **16**(4), pp. 439–45 (2018).

16. Amir,i A., Nedaie, A., and Alikhani, M. "A new adaptive variable sample size approach in EWMA control chart", *Communications in Statistics-Simulation and Computation*, **43**(4), pp. 804–812 (2014).

17. Capizzi, G. and Masarotto, G. "An adaptive exponentially weighted moving average control chart", *Technometrics*, **45**(3), pp. 199–207 (2003).

18. Ong, L.V., Teoh, W.L., Khoo, M.B.C., Chong, Z.L., and Yeong, W.C. "A study on the variable sampling interval EWMA chart when the process parameters are unknown", In *Journal of Physics: Conference Series*, **1132**(1), p. 012084, IOP Publishing (2018).

19. Tran, K.P., Castagliola, P., and Celano, G. "Monitoring the ratio of two normal variables using EWMA type control charts", *Quality and Reliability Engineering International*, **32**(2), pp. 1853–1869 (2015).

20. Teoh, W., Ong, L., Khoo, M., Castagliola, P., and Chong, Z. "The variable sampling interval EWMA $\bar{\mathrm{X}}$ chart with estimated process parameters", *Journal of Testing and Evaluation*, **49**(2), pp. 1237–1265 (2021).

21. Haq, A. and Khoo, M.B.C. "An adaptive multivariate EWMA chart", *Computers and Industrial Engineering*, **131**, p. 459 (2019).

22. Bennet, C.A. "Effect of measurement error on chemical process control", *Industrial Quality Control*, **10**(4), pp. 17–20 (1954).

23. Linna, K.W. and Woodall, W.H. "Effect of measurement error on Shewhart control charts", *Journal of Quality Technology*, **33**(2), pp. 213–222 (2001).

24. Kanazuka, T. "The effect of measurement error on the power of $\bar{X}$-R charts", *Journal of Quality Technology*, **18**(2), pp. 91–95 (1986).

25. Costa, A.F. and Castagliola, P. "Effect of measurement error and autocorrelation on the chart", *Journal of Applied Statistics*, **38**(4), pp. 661–673 (2011).

26. Hu, X., Castagliola, P., Sun, J., and Khoo, M.B. "The effect of measurement errors on the synthetic chart", *Quality and Reliability Engineering International*, **31**(8), pp. 1769–1778 (2014).

27. Linna, K.W., Woodall, W.H., and Busby, K.L. "The performance of multivariate control charts in the presence of measurement error", *Journal of Quality Technology*, **33**(3), pp. 349–355 (2018).

28. Hu, X., Castagliola, P., Sun, J., and Khoo, M.B. "The performance of variable sample size chart with measurement errors", *Quality and Reliability Engineering International*, **32**(3), pp. 969–983 (2015).

29. Hu, X., Castagliola, P., Sun, J., and Khoo, M.B. "Effect of measurement errors on the VSI $\bar{X}$ chart", *European J. Industrial Engineering*, **10**(2), pp. 224–242 (2016).

30. Sabahno, H. and Amiri, A. "The effect measurement errors on the performance of variable sample size and sampling interval control chart", *International Journal of Engineering-Transactions A: Basics*, **30**(7), pp. 995–1004 (2017).

31. Sabahno, H, Amiri, A., and Castagliola, P. "Performance of the variable parameters control chart in presence of measurement errors", *Journal of Testing and Evaluation*, **47**(1), pp. 480–497 (2019).

32. Sabahno, H., Amiri, A., and Castagliola, P. "Optimal performance of the variable sample sizes Hotelling's T^2 control chart in the presence of measurement errors", *Quality Technology & Quantitative Management*, **16**(5), pp. 588–612 (2019).

33. Sabahno, H., Amiri, A., and Castagliola, P. "Evaluating the effect of measurement errors on the performance of the variable sampling intervals Hotelling's T^2 control charts", *Quality and Reliability Engineering International*, **34**(8), pp. 1785–1799 (2018).

34. Ghashghaei, R., Bashiri, M., Amiri, A., and Maleki, M.R. "Effect of measurement error on joint monitoring

of process mean and variability under ranked set sampling", *Quality and Reliability Engineering International*, **32**(8), pp. 3035–3050 (2016).

35. Amiri, A., Ghashghaei, R., and Maleki, M.R. "On the effect of measurement errors in simultaneous monitoring of mean vector and covariance matrix of multivariate processes", *Transactions of the Institute of Measurement and Control*, **40**(1), pp. 318–330 (2018).

36. Maleki, M.R., Amiri, A., and Ghashghaei, R. "Simultaneous monitoring of multivariate process mean and variability in the presence of measurement error with linearly increasing variance under additive covariate model", *International Journal of Engineering-Transactions A: Basics*, **29**(4), pp. 471–480 (2016).

37. Nguyen, H.D., Nguyen, Q.T., Tran, K.P., and Ho, D.P. "On the performance of VSI Shewhart control chart for monitoring the coefficient of variation in the presence of measurement errors", *The International Journal of Advanced Manufacturing Technology*, **104**(1–4), pp. 211–243 (2019).

38. Tran, K.P., Heuchenne, C., and Balakrishnan, N. "On the performance of coefficient of variation charts in the presence of measurement errors", *Quality and Reliability Engineering International*, **35**(1), pp. 329–350 (2019).

39. Stemann, D. and Weihs, C. "The EWMA-X-S-control chart and its performance in the case of precise and imprecise data", *Statistical Papers*, **42**(2), pp. 207–223 (2001).

40. Maravelakis, P, Panaretos, J., and Psarakis, S. "EWMA chart and measurement error", *Journal of Applied Statistics*, **31**(4), pp. 445–455 (2004).

41. Yang, S.F., Ho, H.W., and Rahim, M.A. "Effects of measurement error on controlling two dependent process steps", *Economic Quality Control*, **22**(1), pp. 127–139 (2007).

42. Haq, A, Brown, J., Moltchanova, E., and Al-Omari, A.I. "Effect of measurement error on exponentially weighted moving average control charts under ranked set sampling schemes", *Journal of Statistical Computation and Simulation*, **85**(6), pp. 1224–1246 (2014).

43. Abbasi, S.A. "On the performance of EWMA chart in the presence of two-component measurement error", *Quality Engineering*, **22**(3), pp. 199–213 (2010).

44. Abbasi, S.A. "Exponentially weighted moving average chart and two-component measurement error", *Quality and Reliability Engineering International*, **32**(2), pp. 499–504 (2015).

45. Saghaei, A., Fatemi Ghomi, S.M.T., and Jaberi, S. "Economic design of exponentially weighted moving average control chart based on measurement error using genetic algorithm", *Quality and Reliability Engineering International*, **30**(8), pp. 1153–1163 (2014).

46. Cheng, X.B. and Wang, F.K. "The performance of EWMA median and CUSUM median control charts for a normal process with measurement errors", *Quality and Reliability Engineering International*, **34**(2), pp. 203–213 (2017).

47. Tang, A., Castagliola, P., Sun, J., and Hu, X. "The effect of measurement errors on the adaptive EWMA chart", *Quality and Reliability Engineering International*, **34**(4), pp. 609–630 (2018).

48. Maleki, M.R., Amiri, A., and Castagliola, P. "Measurement errors in statistical process monitoring: a literature review", *Computers & Industrial Engineering*, **103**, pp. 316–329 (2016).

49. Lucas, J.M. and Saccucci, M.S. "Exponentially weighted moving average control schemes: properties and enhancements", *Technometrics*, **32**, pp. 1–12 (1990).

50. Montgomery, D.C. and Runger, G.C. "Gauge capability and designed experiments. Part I: basic methods", *Quality Engineering*, **6**(1), pp. 115–135 (1994).

Appendix A

The proof of Z_i distribution

Based on Eqs. (4) and (5), we have:

$$\bar{X}_i \sim N(A + B\left(\mu_0 + \delta\sigma_0\right), \frac{1}{n}\left(B^2\sigma_0^2 + \sigma_M^2\right)).$$

Let U_i be the standardized version of the statistic $\bar{X}_i$:

$$U_i = \frac{\bar{X}_i - \mu_{0_{\bar{X}_i}}}{\sigma_{\bar{X}_i}}. \tag{A.1}$$

Since $\bar{X}_i$ follows the normal distribution with the mean of $\mu_{0_{\bar{X}_i}}$ and variance of $\sigma^2_{\bar{X}_i}$, we have $U_i \sim N(0,1)$. Here, U_i is used in the EWMA statistic, as shown in the following:

$$Z_i = \lambda U_i + (1-\lambda) Z_{i-1}. \tag{A.2}$$

The expansion of is provided as shown in Eqs. (A.3)–(A.5):

$$Z_i = \lambda U_i + (1-\lambda)\Big(\lambda U_{i-1} + (1-\lambda) Z_{i-2}\Big), \tag{A.3}$$

where:

$$Z_{i-2} = \lambda U_{i-2} + (1-\lambda) Z_{i-3},$$

and in the same way:

$$Z_i = \lambda U_i + (1-\lambda)\Big(\lambda U_{i-1} + (1-\lambda)\left(\lambda U_{i-2} + (1-\lambda) Z_{i-3}\right)\Big). \tag{A.4}$$

Finally, Z_i is expanded as presented below:

$$Z_i = \sum_{e=0}^{i-1} \lambda (1-\lambda)^e U_{i-e} + (1-\lambda)^i Z_0, \tag{A.5}$$

where $Z_0 = \mu_{U_i}$. In addition, the expected value and variance of Z_i are obtained as follows:

$$\begin{aligned}\mathrm{E}(Z_i) &= \mathrm{E}\left(\sum_{e=0}^{i-1} \lambda (1-\lambda)^e U_{i-e} + (1-\lambda)^i \mu_{U_i}\right)\\ &= \mu_{Ui}\left(1-(1-\lambda)^i\right) + \mu_{Ui}(1-\lambda)^i\\ &= \mu_{Ui}. \end{aligned}\tag{A.6}$$

$$\begin{aligned}\mathrm{Var}(Z_i) &= \mathrm{Var}\left(\sum_{e=0}^{i-1} \lambda (1-\lambda)^e U_{i-e} + (1-\lambda)^i \mu_{U_i}\right)\\ &= \sigma_{U_i}^2 \sum_{e=0}^{i-1} \lambda^2 (1-\lambda)^{2e} + 0\\ &= \sigma_{U_i}^2 \frac{\lambda^2\left(1-(1-\lambda)^{2i}\right)}{1-(1-\lambda)^2}\\ &= \sigma_{U_i}^2 \frac{\lambda\left(1-(1-\lambda)^{2i}\right)}{2-\lambda}. \end{aligned}\tag{A.7}$$

Thus:

$$Z_i \sim N\left(\mu_{Ui}, \sigma_{U_i}^2 \frac{\lambda\left(1-(1-\lambda)^{2i}\right)}{2-\lambda}\right),$$

and for asymptotic variance, we have $\sigma_{U_i}^2 \frac{\lambda}{2-\lambda}$. Since $U_i \sim N(0,1)$, $Z_i \sim N\left(0, \frac{\lambda}{2-\lambda}\right)$.

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