Novel Exponential divergence measure of complex intuitionistic fuzzy sets with an application to decision-making process

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Abstract

As a generalization of the intuitionistic fuzzy sets (IFSs), complex IFSs (CIFSs) is a powerful and worthy tool to realize the imprecise information by using complex-valued membership degrees with an extra term, named as phase term. Divergence measure is a valuable tool to determine the degree of discrimination between the two sets. Driven by these fundamental characteristics, it is fascinating to manifest some divergence measures to the CIFSs. In this paper, we explain a method to solve the multi-criteria decision-making (MCDM) problem under CIFS environment. For it, firstly, the divergence measures are introduced between two CIFSs and examined their several properties and relations. Secondly, a novel algorithm is given based on the proposed measures to solve the problems in which weights corresponding to criteria are resolved using maximizing deviation method. Thirdly, a reasonable example is provided to verify the developed approach and to exhibit its practicality and utility with a comparative analysis to show its more manageable and adaptable nature.

Keywords: Divergence measure; Complex IFS; Decision making; maximize deviation method; MCDM approach

1. Introduction

Today's decision making (DM) is one of the most significant ventures in our regular life, whose mission is to select the best alternative out of the finite ones under the several known or unknown criteria. Multicriteria decision making (MCDM) is the division of the DM and is admitted as a cognitive-based human action. Human beings inescapably are met with numerous decision-making problems (DMPs), which involves multiple fields such as supplier selection, supply chain management, emerging problems, medical problems and so on. In practice, with the growing technological advancements and modern treatment based on new techniques as well as tools, several uncertain cases related to DMP arises which results that decision-makers no longer satisfied with the numerical values expressed in terms of crisp numbers. Therefore, to quantify the different information into the analysis and to analyze the information in a more accurate manner, many researchers have developed several kinds of algorithms by using the theories of fuzzy set (FS) [1] , intuitionistic FS (IFS) [2], linguistic interval-valued IFS (LIVIFS) [3], complex FS [4], complex IFS [5], complex interval-valued IFS [6] are widely used by the researchers. In FS and IFS environment, information associated with each object is defined with membership degrees (MDs) and the non-membership degrees (NMDs) such that their sum isn't more than one. However, in LIVIFS, erudition is gathered not related to quantitative form but in the form of qualitative using linguistic variables. Further, in Complex FS and IFS, information is practiced under the complex environment where the domain of IFS has lengthened the domain of MDs and NMDs from real set to the complex-valued set with a unit disc.

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Under these different environments, various endeavors have been given by the scholars to offer different kinds of methods and algorithm to solve MCDM problem in various fields by using either aggregation operators (AO) [7–14] or information measure (IM) [15–22]. For example, under the IFS environment, the weighted average and geometric AOs are explained by the authors in [7, 8]. Further, Garg [9, 23] elongated them by adding the degree of hesitancy between the pairs of the degrees. Some Hamacher AOs for IFSs are detailed by Huang [11], Garg [24]. Apart from the AOs, the IMs play also a notable role in treating the imperfect and uncertain information. Since it can be worth notifying that similarity, entropy, inclusion, etc., are the measures which are used by the researchers to examine the DMPs. In it, similarity measures deal with two objects to compute the degree of similarity while entropy quantifies the degree of fuzziness in the set. On the other hand, the inclusion measures give the extent to which a set is contained in another set. Apart from them, the divergence measures (DvM) are also one of the most well-known tools for holding the uncertainty associated with the set. It depicts the degree of discrimination between two objects. In the literature, all these measures have been greatly searched by many researchers as vital topics. For example, Kullback and Leibler [25] firstly originate the concept of the DvM between the two probability distribution. Later on, Bhandari and Pal [26] continued this measure to the FSs. Vlachos and Sergiadis [27] extended the idea of the DvM from FSs to IFSs. Zeng and Li [17] presented the correlation coefficients for IFSs. Garg [28] exhibited an improved cosine similarity measure for IFSs. Garg and Kumar [29] presented some similarity measure for IFSs based on the connection numbers of set pair analysis theory. Ohlan [30] presented the IF exponential DvM along with its distinct properties and proposed a method for dealing with DMPs. Garg et al. [31] introduced parametric directed DvM under IFS theory to solve the DMPs. Mishra et al. [32] proposed Jensen-exponential DvM and the corresponding DMP under IFS environment. Besides this, many researchers [33–37] worked on various information measures and showed their applications by applying them to DMPs.

From the earlier comprehensive studies and DMPs, it is perceived that their approaches are restricted to some extent by handling only the uncertainty but concurrently fails to deal with its variations at a given phase of time in the data. However, the information concerned from the medical research, database for biometric and facial recognition, etc., regularly changes with the passage of the time. Therefore, there is a demand to add the supplementary parameter into the study with representing this variation and hence handle the data accurately. For it, Ramot et al. [4] explained the concept of complex FS (CFS) by lengthening the domain of MDs from real set to complex-valued set with a unit disc. Further, to improve this theory, several properties such as the complement, intersection, union, etc., are studied by Ramot et al. [4, 38]. The association between the CFS with the Pythagorean fuzzy set [39] are explored by the authors in Dick et al. [40]. A brief survey on the CFSs and logic are presented by Yazdanbakhsh and Dick [41]. After their actuality, it is remarked that the degree of disagreeing is not included in CFSs. So to discuss the data properly, in 2012, Alkouri and Salleh [5] stretched out the idea of CFS to Complex IFS (CIFS) by including the NMDs of the unit disc along the MDs into the analysis. Later on, some relations, projections, and measures for CIFSs are studied by Alkouri and Salleh [42]. Kumar and Bajaj [43] defined entropy as well as distance measures for CIF soft sets. To further expand the CIFSs, Rani and Garg [44] presented measures the degree of dissimilarities between the CIFSs. Further, to measures, the strength between CIFSs, Garg and Rani [45] defined correlation coefficients for them. Garg and Rani [46, 47] presented some averaging and geometric AOs for CIFSs. Quran and Hassan [48] gave the operations for the complex neutrosophic soft sets. Rani and Garg [49] proposed power AOs for group DMPs under CIFS environment. Garg and Rani [50] presented some generalized Bonferroni mean AOs for CIFSs by using Archimedean t-norm operations. Recently, Garg and Rani [51] presented exponential, logarithm and compensative AOs for aggregating the different CIFSs. However, Garg and Rani [52] presented the study on the various information measures of CIFSs.

The CIFS is a generalization of the IFS considering both the MDs and NMDs on the complex argument plane. Under it, the amplitude term gives the extent of belongingness while the phase term represents the periodicity of an object. Clearly, these phase terms distinguish the CIFS from the traditional IFS theory. In IFS theory, the factor of periodicity is completely ignored and hence there is a certain loss of information.

To avoid, a factor of it is added into the analysis. To further illustrate the concept of phase terms, consider a certain company who wants to purchase cars from the carmakers regarding the features such as (i) Models and (ii) Production dates of cars. Since every year, the carmakers produce the same models of cars with slight improvements and differences, therefore, due to the changes made, people's accept their levels and judgments for the new model. Hence, the production date of the car also plays a significant role during the purchasing or decision. Therefore, such a problem considered as a two-dimensional one which can't be modeled simultaneously in the existing FSs or IFSs environment. Furthermore, in order to execute such kinds of the problem under IFS environment, then there is a need to consider two or more IFSs by the decision makers and then execute it, which leads to the results increasing the execution time, and the number of computations during solving the problem. On the other hand, CIFS is a better representation for such problems in which both the dimensions consider as a single set. Thus, CIFS is a better representation of the data than the existing ones. The salient features of CIFSs over the several existing sets are demonstrated in Table 1.

Insert Table 1 here.

Due to the complex DM process day-by-day, it is inevitable to measure the degree of discrimination between the pairs of the sets. For it, the information measures are the most prosperous tools. Amongst the multiple measures such as entropy, similarity, inclusion, etc., the DvMs have the quality to hold the discrimination degree between the sets. Thus, encouraged from the hallmarks of the CIFS model and the quality of DvM, the center of this paper is to develop some exponential based DvMs to quantify the information. For it, by designating the information under the CIFS model, we quantify the data by proposed measure to solve the DMPs. Some axioms and the properties of it are discussed in details. Later, based on the intended study, we elaborate an algorithm to solve the DMPs and illustrate with several numerical examples. To the best of authors' knowledge, the study on DvM and their impact on DMPs are not utilized so far under CIFS study. Thus, there is a necessity to scrutinize it under the environment and control its impact on the DM process. Accordingly, exciting from it and the advantages of the CIFSs model, the chief augmentation of the work is classified into three parts:

1) to propose the exponential DvM to measure the discrimination between the pairs of CIFSs.
2) to establish DM approach by using proposed measures.
3) to demonstrate the developed method with several examples and shows its feasibility.

The remainder of the manuscript is prepared as follows. Section 2 allots with basic concepts on IFSs and CIFSs. In Section 3, we introduce the concept of DvMs for CIFSs and examines their properties. Section 4 explains the maximizing deviation method for determining the weights and then followed by the DM approach for MCDM problem. In Section 5, we illuminate the approach with some useful models. Lastly, a conclusion regarding the work is given in Section 6.

2. Preliminaries

Let $\mathcal{X}$ be the universal set. Then, we review some basic definitions related to IFSs and CIFSs here.

Definition 2.1. [1] A fuzzy set $\mathcal{F}$ on $\mathcal{X}$ is defined as

$$\mathcal{F} = \{\langle x, u_{\mathcal{F}}(x)\rangle \mid x \in \mathcal{X}\} \tag{1}$$

where $u_{\mathcal{F}}(x) \in [0,1]$ represent MD of element x.

Definition 2.2. [25] The degree of discrimination, known as divergence measure, between two discrete distributions $\mathcal{P} = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ is given as

$$D(\mathcal{P}, Q) = \sum_{j=1}^{n} p_j \log\left(\frac{p_j}{q_j}\right) \tag{2}$$

Definition 2.3. [26] For two FSs $\mathcal{F} = \{(x, u_{\mathcal{F}}(x)) \mid x \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, u_{\mathcal{G}}(x)) \mid x \in \mathcal{X}\}$, the fuzzy DvM is defined as

$$D(\mathcal{F}, \mathcal{G}) = \frac{1}{n}\sum_{j=1}^{n}\left[u_{\mathcal{F}}(x_j)\log\left(\frac{u_{\mathcal{F}}(x_j)}{u_{\mathcal{G}}(x_j)}\right) + (1 - u_{\mathcal{F}}(x_j))\log\left(\frac{1 - u_{\mathcal{F}}(x_j)}{1 - u_{\mathcal{G}}(x_j)}\right)\right] \tag{3}$$

Definition 2.4. [2] An IFS $\mathcal{I}$ on $\mathcal{X}$ is defined as

$$\mathcal{I} = \{\langle x, u_{\mathcal{I}}(x), v_{\mathcal{I}}(x)\rangle \mid x \in \mathcal{X}\}, \tag{4}$$

where $u_{\mathcal{I}}, v_{\mathcal{I}} : \mathcal{X} \to [0, 1]$ represent the MD and NMD function of x to $\mathcal{I}$ respectively such that $u_{\mathcal{I}}(x)+v_{\mathcal{I}}(x) \leq 1$ for each x.

Definition 2.5. [27] For two IFSs $\mathcal{I} = \{(x, u_{\mathcal{I}}(x), v_{\mathcal{I}}(x)) \mid x \in \mathcal{X}\}$ and $\mathcal{J} = \{(x, u_{\mathcal{J}}(x), v_{\mathcal{J}}(x)) \mid x \in \mathcal{X}\}$ defined on $\mathcal{X}$, the DvM of $\mathcal{I}$ relative to $\mathcal{J}$ is defined as:

$$D(\mathcal{I}, \mathcal{J}) = \frac{1}{n}\sum_{j=1}^{n}\left[u_{\mathcal{I}}(x_j)\log\left(\frac{2u_{\mathcal{I}}(x_j)}{u_{\mathcal{I}}(x_j) + u_{\mathcal{J}}(x_j)}\right) + v_{\mathcal{I}}(x_j)\log\left(\frac{2v_{\mathcal{I}}(x_j)}{v_{\mathcal{I}}(x_j) + v_{\mathcal{J}}(x_j)}\right)\right] \tag{5}$$

Later on, Garg et al. [31] defined some generalized parametric divergence of order α and degree β under IFS environment as:

$$\begin{aligned} &D(\mathcal{I}, \mathcal{J}) \\ &= \frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{array}{l} u_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{u_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda u_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)u_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) + v_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{v_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda v_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)v_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) \\ \quad + h_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{h_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda h_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)h_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) \end{array}\right] \\ &+ \frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{array}{l} u_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{u_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda u_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)u_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) + v_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{v_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda v_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)v_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) \\ \quad + h_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)\log\left(\dfrac{h_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j)}{\lambda h_{\mathcal{J}}^{\frac{\alpha}{2-\beta}}(x_j) + (1-\lambda)h_{\mathcal{I}}^{\frac{\alpha}{2-\beta}}(x_j)}\right) \end{array}\right] \end{aligned} \tag{6}$$

Definition 2.6. [4] A CFS $\mathcal{K}$ defined on $\mathcal{X}$ is given as

$$\mathcal{K} = \{(x, \mu_{\mathcal{K}}(x)) : x \in \mathcal{X}\} \tag{7}$$

where $\mu_{\mathcal{K}} : \mathcal{X} \to \{a : a \in C, |a| \leq 1\}$ is a complex-valued MD function and defined by $\mu_{\mathcal{K}}(x) = r_{\mathcal{K}}(x)e^{i2\pi w_{r_{\mathcal{K}}}(x)}$ where $i = \sqrt{-1}$, $0 \leq r_{\mathcal{K}}(x), w_{r_{\mathcal{K}}}(x) \leq 1$.

Definition 2.7. [5] A CIFS $\mathcal{K}$ on $\mathcal{X}$ is given as:

$$\mathcal{K} = \{(x, \mu_{\mathcal{K}}(x), \gamma_{\mathcal{K}}(x)) : x \in \mathcal{X}\} \tag{8}$$

where $\mu_{\mathcal{K}}, \gamma_{\mathcal{K}}$ are the complex-valued MD and NMD functions defined as $\mu_{\mathcal{K}}(x) = r_{\mathcal{K}}(x)e^{2\pi i w_{r_{\mathcal{K}}}(x)}$ and $\gamma_{\mathcal{K}}(x) = k_{\mathcal{K}}(x)e^{2\pi i w_{k_{\mathcal{K}}}(x)}$, where $0 \leq r_{\mathcal{K}}(x), k_{\mathcal{K}}(x) \leq 1$; $0 \leq r_{\mathcal{K}}(x)+k_{\mathcal{K}}(x) \leq 1$ and $0 \leq w_{r_{\mathcal{K}}}(x), w_{k_{\mathcal{K}}}(x), w_{r_{\mathcal{K}}}(x)+w_{k_{\mathcal{K}}}(x) \leq 1$. We denote such pair as $\mathcal{K} = ((r_{\mathcal{K}}, w_{r_{\mathcal{K}}}), (k_{\mathcal{K}}, w_{k_{\mathcal{K}}}))$ and called as complex intuitionistic fuzzy number (CIFN).

Definition 2.8. [5] Let $\mathcal{K} = \{((r_{\mathcal{K}}(x), w_{r_{\mathcal{K}}}(x)), (k_{\mathcal{K}}(x), w_{k_{\mathcal{K}}}(x))) : x \in \mathcal{X}\}$ and $\mathcal{M} = \{((r_{\mathcal{M}}(x), w_{r_{\mathcal{M}}}(x)), (k_{\mathcal{M}}(x), w_{k_{\mathcal{M}}}(x))) : x \in \mathcal{X}\}$ be two CIFSs. Then, we define:

(i) $\mathcal{K} \subseteq \mathcal{M}$ if $r_{\mathcal{K}}(x) \leq r_{\mathcal{M}}(x)$, $k_{\mathcal{K}}(x) \geq k_{\mathcal{M}}(x)$ and $w_{r_{\mathcal{K}}}(x) \leq w_{r_{\mathcal{M}}}(x)$, $w_{k_{\mathcal{K}}}(x) \geq w_{k_{\mathcal{M}}}(x)$;

(ii) $\mathcal{K} = \mathcal{M} \Leftrightarrow \mathcal{K} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{K}$;

(iii) $\mathcal{K}^c = \{((k_{\mathcal{K}}(x), w_{k_{\mathcal{K}}}(x)), (r_{\mathcal{K}}(x), w_{r_{\mathcal{K}}}(x))) : x \in \mathcal{X}\}$;

(iv) $\mathcal{K} \cup \mathcal{M} = \left\{\left((r_{\mathcal{K}\cup\mathcal{M}}(x), w_{r_{\mathcal{K}\cup\mathcal{M}}}(x)), (k_{\mathcal{K}\cup\mathcal{M}}(x), w_{k_{\mathcal{K}\cup\mathcal{M}}}(x))\right) : x \in \mathcal{X}\right\}$ where $r_{\mathcal{K}\cup\mathcal{M}}(x) = \max\{r_{\mathcal{K}}(x), r_{\mathcal{M}}(x)\}$, $k_{\mathcal{K}\cup\mathcal{M}}(x) = \min\{k_{\mathcal{K}}(x), k_{\mathcal{M}}(x)\}$, $w_{r_{\mathcal{K}\cup\mathcal{M}}}(x) = \max\{w_{r_{\mathcal{K}}}(x), w_{r_{\mathcal{M}}}(x)\}$ and $w_{k_{\mathcal{K}\cup\mathcal{M}}}(x) = \min\{w_{k_{\mathcal{K}}}(x), w_{k_{\mathcal{M}}}(x)\}$;

(v) $\mathcal{K} \cap \mathcal{M} = \left\{\left((r_{\mathcal{K}\cap\mathcal{M}}(x), w_{r_{\mathcal{K}\cap\mathcal{M}}}(x)), (k_{\mathcal{K}\cap\mathcal{M}}(x), w_{k_{\mathcal{K}\cap\mathcal{M}}}(x))\right) : x \in \mathcal{X}\right\}$ where $r_{\mathcal{K}\cap\mathcal{M}}(x) = \min\{r_{\mathcal{K}}(x), r_{\mathcal{M}}(x)\}$, $k_{\mathcal{K}\cap\mathcal{M}}(x) = \max\{k_{\mathcal{K}}(x), k_{\mathcal{M}}(x)\}$, $w_{r_{\mathcal{K}\cap\mathcal{M}}}(x) = \min\{w_{r_{\mathcal{K}}}(x), w_{r_{\mathcal{M}}}(x)\}$ and $w_{k_{\mathcal{K}\cap\mathcal{M}}}(x) = \max\{w_{k_{\mathcal{K}}}(x), w_{k_{\mathcal{M}}}(x)\}$.

3. Proposed exponential divergence measure

Let $\Phi(\mathcal{X})$ be the class of CIFSs. Then, here, we have defined the exponential DvM for $\Phi(\mathcal{X})$ and studied their properties.

Definition 3.1. For $\mathcal{K}, \mathcal{M} \in \Phi(\mathcal{X})$, a real function $\mathcal{D}v : \Phi(\mathcal{X}) \times \Phi(\mathcal{X}) \to R^+$ is called a DvM, if

(P1) $\mathcal{D}v(\mathcal{K}, \mathcal{M}) \geq 0$.

(P2) $\mathcal{D}v(\mathcal{K}, \mathcal{M}) = \mathcal{D}v(\mathcal{M}, \mathcal{K})$.

(P3) $\mathcal{D}v(\mathcal{K}, \mathcal{M}) = 0$ if $\mathcal{K} = \mathcal{M}$.

(P4) $\mathcal{D}v(\mathcal{K}, \mathcal{M}) = \mathcal{D}v(\mathcal{K}^c, \mathcal{M}^c)$.

satisfies.

Definition 3.2. For two CIFSs $\mathcal{K} = \{((r_{\mathcal{K}}(x), w_{r_{\mathcal{K}}}(x)), (k_{\mathcal{K}}(x), w_{k_{\mathcal{K}}}(x))) : x \in \mathcal{X}\}$ and $\mathcal{M} = \{((r_{\mathcal{M}}(x), w_{r_{\mathcal{M}}}(x)), (k_{\mathcal{M}}(x), w_{k_{\mathcal{M}}}(x))) : x \in \mathcal{X}\}$, the degree of discrimination between them is defined as

$$E(\mathcal{K}, \mathcal{M}) = \frac{1}{4n(1 - e^{-1})} \sum_{j=1}^{n} \left[\begin{array}{l} 2 - \left(\frac{k_{\mathcal{K}}(x_j) + 1 - r_{\mathcal{K}}(x_j)}{2}\right) \exp\left(\frac{(r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j)) - (k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j))}{2}\right) \\ -\left(\frac{r_{\mathcal{K}}(x_j) + 1 - k_{\mathcal{K}}(x_j)}{2}\right) \exp\left(\frac{(r_{\mathcal{M}}(x_j) - r_{\mathcal{K}}(x_j)) - (k_{\mathcal{M}}(x_j) - k_{\mathcal{K}}(x_j))}{2}\right) \\ -\left(\frac{w_{k_{\mathcal{K}}(x_j)} + 1 - w_{r_{\mathcal{K}}(x_j)}}{2}\right) \exp\left(\frac{(w_{r_{\mathcal{K}}(x_j)} - w_{r_{\mathcal{M}}(x_j)}) - (w_{k_{\mathcal{K}}(x_j)} - w_{k_{\mathcal{M}}(x_j)})}{2}\right) \\ -\left(\frac{w_{r_{\mathcal{K}}(x_j)} + 1 - w_{k_{\mathcal{K}}(x_j)}}{2}\right) \exp\left(\frac{\left(w_{r_{\mathcal{M}}(x_j)} - w_{r_{\mathcal{K}}(x_j)}\right) - \left(w_{k_{\mathcal{M}}(x_j)} - w_{k_{\mathcal{K}}(x_j)}\right)}{2}\right) \end{array} \right] \tag{9}$$

where 'exp' refers to exponential function.

Here, it is clearly seen that $E(\mathcal{K}, \mathcal{M}) \neq E(\mathcal{K}^c, \mathcal{M}^c)$. As it is quite obvious that the degree of discrimination of $\mathcal{K}$ from $\mathcal{M}$ and $\mathcal{K}^c$ from $\mathcal{M}^c$ should be same. So, in order to imbue the measure with symmetry, we define symmetric DvM as follows.

Definition 3.3. A symmetric exponential divergence measure for two CIFSs $\mathcal{K}$ and $\mathcal{M}$, denoted as $\mathcal{D}v(\mathcal{K}, \mathcal{M})$, is defined as

$$\begin{aligned} \mathcal{D}v(\mathcal{K}, \mathcal{M}) &= E(\mathcal{K}, \mathcal{M}) + E(\mathcal{K}^c, \mathcal{M}^c) \\ &= \frac{1}{4n(1 - e^{-1})} \sum_{j=1}^{n} \left[\begin{array}{l} 4 - \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) \exp\left(t_j(\mathcal{K}, \mathcal{M})\right) \\ -\left(1 + t_j(\mathcal{K}, \mathcal{M})\right) \exp\left(-t_j(\mathcal{K}, \mathcal{M})\right) \\ -\left(1 - s_j(\mathcal{K}, \mathcal{M})\right) \exp\left(s_j(\mathcal{K}, \mathcal{M})\right) \\ -\left(1 + s_j(\mathcal{K}, \mathcal{M})\right) \exp\left(-s_j(\mathcal{K}, \mathcal{M})\right) \end{array} \right] \end{aligned} \tag{10}$$

where

$$t_j(\mathcal{K},\mathcal{M}) = \frac{\left(r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j)\right) - \left(k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j)\right)}{2}$$

$$\text{and} \quad s_j(\mathcal{K},\mathcal{M}) = \frac{\left(w_{r_{\mathcal{K}}}(x_j) - w_{r_{\mathcal{M}}}(x_j)\right) - \left(w_{k_{\mathcal{K}}}(x_j) - w_{k_{\mathcal{M}}}(x_j)\right)}{2},$$

provided $t_j \neq 0$ and $s_j \neq 0$.

Remark 3.1. It is evident from it that $t_j(\mathcal{K},\mathcal{M}) = -t_j(\mathcal{M},\mathcal{K})$ and $s_j(\mathcal{K},\mathcal{M}) = -s_j(\mathcal{M},\mathcal{K})$.

Before proving Eq. (10) is valid DvM, we stated two lemmas as follows.

Lemma 3.1. Let $f(y) = 2 - (1-y)\exp(y) - (1+y)\exp(-y)$ be a function, where $y \in [-1, 1]$. Then,

$$0 \leq f(y) \leq 2 - 2\exp(-1)$$

Proof. Since $f(y) = 2-(1-y)\exp(y)-(1+y)\exp(-y)$. It gives that $f'(y) = y(\exp(y)+\exp(-y))$ which follows that $f(y)$ is decreasing in $[-1,0]$ and increasing in $[0,1]$. Therefore, when $y \in [-1,0]$, $f(0) \leq f(y) \leq f(-1)$ i.e, $0 \leq f(y) \leq 2 - 2\exp(-1)$ and similarly for $y \in [0,1]$, $f(0) \leq f(y) \leq f(1)$ i.e, $0 \leq f(y) \leq 2 - 2\exp(-1)$. Hence, for $y \in [-1,1]$, we have $0 \leq f(y) \leq 2 - 2\exp(-1)$.

□

Lemma 3.2. For $y \in [-1, 0]$, the functions $f_1(y) = (1-y)\exp(y)$ and $f_2(y) = (1+y)\exp(-y)$ are increasing functions.

Proof. Since $f_1(y) = (1-y)\exp(y)$. For $y \in [-1,0]$, $f_1'(y) = -y\exp(y) \geq 0$ which gives that $f_1(y)$ is an increasing function. Similarly, we can prove that $f_2(y)$ is an increasing function for $y \in [-1,0]$. □

Theorem 3.1. The measure presented in Definition 3.3 is a valid divergence measure.

Proof. Let $\mathcal{K} = \{((r_{\mathcal{K}}(x), w_{r_{\mathcal{K}}}(x)), (k_{\mathcal{K}}(x), w_{k_{\mathcal{K}}}(x))) : x \in \mathcal{X}\}$, $\mathcal{M} = \{((r_{\mathcal{M}}(x), w_{r_{\mathcal{M}}}(x)), (k_{\mathcal{M}}(x), w_{k_{\mathcal{M}}}(x))) : x \in \mathcal{X}\}$ and $\mathcal{N} = \{((r_{\mathcal{N}}(x), w_{r_{\mathcal{N}}}(x)), (k_{\mathcal{N}}(x), w_{k_{\mathcal{N}}}(x))) : x \in \mathcal{X}\}$ be three CIFSs. Then, to prove the results, we need to show that Eq. (10) satisfies the following axioms:

(P1) $0 \leq \mathcal{D}v(\mathcal{K},\mathcal{M}) \leq 1$.
(P2) $\mathcal{D}v(\mathcal{K},\mathcal{M}) = 0$ if $\mathcal{K} = \mathcal{M}$.
(P3) $\mathcal{D}v(\mathcal{K},\mathcal{M}) = \mathcal{D}v(\mathcal{M},\mathcal{K})$.
(P4) If $\mathcal{K} \subseteq \mathcal{M} \subseteq \mathcal{N}$ then, $\mathcal{D}v(\mathcal{K},\mathcal{N}) \geq \mathcal{D}v(\mathcal{K},\mathcal{M})$ and $\mathcal{D}v(\mathcal{K},\mathcal{N}) \geq \mathcal{D}v(\mathcal{M},\mathcal{N})$.

By definition of CIFSs, we have

(P1) Since $0 \leq r_{\mathcal{K}}(x_j), r_{\mathcal{M}}(x_j), k_{\mathcal{K}}(x_j), k_{\mathcal{M}}(x_j) \leq 1$. It implies that $-1 \leq r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j) \leq 1$; $-1 \leq k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j) \leq 1$ which gives that $-2 \leq (r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j)) - (k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j)) \leq 2$ and hence, $-1 \leq t_j(\mathcal{K},\mathcal{M}) \leq 1$. Similarly, we can prove that, $-1 \leq s_j(\mathcal{K},\mathcal{M}) \leq 1$. Then, by using the above Lemma 3.1, we obtain that $0 \leq 2-\left(1-t_j(\mathcal{K},\mathcal{M})\right)\exp\left(t_j(\mathcal{K},\mathcal{M})\right)-\left(1+t_j(\mathcal{K},\mathcal{M})\right)\exp\left(-t_j(\mathcal{K},\mathcal{M})\right) \leq 2-2\exp(-1)$ and $0 \leq 2-\left(1-s_j(\mathcal{K},\mathcal{M})\right)\exp\left(s_j(\mathcal{K},\mathcal{M})\right)-\left(1+s_j(\mathcal{K},\mathcal{M})\right)\exp\left(-s_j(\mathcal{K},\mathcal{M})\right) \leq 2-2\exp(-1)$ which gives that $0 \leq 4-\left(1-t_j(\mathcal{K},\mathcal{M})\right)\exp\left(t_j(\mathcal{K},\mathcal{M})\right)-\left(1+t_j(\mathcal{K},\mathcal{M})\right)\exp\left(-t_j(\mathcal{K},\mathcal{M})\right)-\left(1-s_j(\mathcal{K},\mathcal{M})\right)\exp\left(s_j(\mathcal{K},\mathcal{M})\right)-\left(1+s_j(\mathcal{K},\mathcal{M})\right)\exp\left(-s_j(\mathcal{K},\mathcal{M})\right) \leq 4-4\exp(-1)$. Hence, $0 \leq \mathcal{D}v(\mathcal{K},\mathcal{M}) \leq 1$.
(P2) For $\mathcal{K} = \mathcal{M}$, we have $r_{\mathcal{K}}(x_j) = r_{\mathcal{M}}(x_j)$, $k_{\mathcal{K}}(x_j) = k_{\mathcal{M}}(x_j)$, $w_{r_{\mathcal{K}}}(x_j) = w_{r_{\mathcal{M}}}(x_j)$ and $w_{k_{\mathcal{K}}}(x_j) = w_{k_{\mathcal{M}}}(x_j)$ for all j which gives that $t_j(\mathcal{K},\mathcal{M}) = s_j(\mathcal{K},\mathcal{M}) = 0$. Thus, $\mathcal{D}v(\mathcal{K},\mathcal{M}) = 0$.
(P3) It is obvious.

(P4) Since $\mathcal{K} \subseteq \mathcal{M} \subseteq \mathcal{N}$. It implies that $r_{\mathcal{K}}(x_j) \leq r_{\mathcal{M}}(x_j) \leq r_{\mathcal{N}}(x_j)$ and $k_{\mathcal{K}}(x_j) \geq k_{\mathcal{M}}(x_j) \geq k_{\mathcal{N}}(x_j)$ which gives that $r_{\mathcal{K}}(x_j) - r_{\mathcal{N}}(x_j) \leq r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j) \leq 0$ and $0 \leq k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j) \leq k_{\mathcal{K}}(x_j) - k_{\mathcal{N}}(x_j)$. It follows that $t_j(\mathcal{K}, \mathcal{N}) \leq t_j(\mathcal{K}, \mathcal{M}) \leq 0$. Also, $t_j(\mathcal{K}, \mathcal{N})$, $t_j(\mathcal{K}, \mathcal{M}) \geq -1$. Then, by using Lemma 3.2, we have $\left(1 - t_j(\mathcal{K}, \mathcal{N})\right) \exp\left(t_j(\mathcal{K}, \mathcal{N})\right) \leq \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) \exp\left(t_j(\mathcal{K}, \mathcal{M})\right)$ and $\left(1 + t_j(\mathcal{K}, \mathcal{N})\right) \exp\left(-t_j(\mathcal{K}, \mathcal{N})\right) \leq \left(1+t_j(\mathcal{K}, \mathcal{M})\right) \exp\left(-t_j(\mathcal{K}, \mathcal{M})\right)$. Similarly, we can prove that $\left(1-s_j(\mathcal{K}, \mathcal{N})\right) \exp\left(s_j(\mathcal{K}, \mathcal{N})\right) \leq \left(1-s_j(\mathcal{K}, \mathcal{M})\right) \exp\left(s_j(\mathcal{K}, \mathcal{M})\right)$ and $\left(1+s_j(\mathcal{K}, \mathcal{N})\right) \exp\left(-s_j(\mathcal{K}, \mathcal{N})\right) \leq \left(1+s_j(\mathcal{K}, \mathcal{M})\right) \exp\left(-s_j(\mathcal{K}, \mathcal{M})\right)$. Hence, $\mathcal{D}v(\mathcal{K}, \mathcal{N}) \geq \mathcal{D}v(\mathcal{K}, \mathcal{M})$. Similarly, we can prove that $\mathcal{D}v(\mathcal{K}, \mathcal{N}) \geq \mathcal{D}v(\mathcal{M}, \mathcal{N})$.

□

The working of the proposed measure is given with example as below.

Example 3.1. For $\mathcal{X} = \{x_1, x_2, x_3\}$, let $\mathcal{K} = \{(x_1, (0.4, 0.1), (0.3, 0.5)), (x_2, (0.5, 0.3), (0.1, 0.2)), (x_3, (0.7, 0.3), (0.2, 0.3))\}$, and $\mathcal{M} = \{(x_1, (0.6, 0.3), (0.3, 0.2)), (x_2, (0.4, 0.3), (0.2, 0.1)), (x_3, (0.7, 0.4), (0.1, 0.2))\}$ be two CIFSs. Then,

$$\begin{aligned} t_1(\mathcal{K}, \mathcal{M}) &= \frac{(r_{\mathcal{K}}(x_1) - r_{\mathcal{M}}(x_1)) - (k_{\mathcal{K}}(x_1) - k_{\mathcal{M}}(x_1))}{2} \\ &= \frac{(0.4 - 0.6) - (0.3 - 0.3)}{2} \\ &= -0.1 \end{aligned}$$

Similarly we can obtain, $t_2(\mathcal{K}, \mathcal{M}) = 0.1$, $t_3(\mathcal{K}, \mathcal{M}) = -0.05$, $s_1(\mathcal{K}, \mathcal{M}) = -0.25$, $s_2(\mathcal{K}, \mathcal{M}) = -0.05$ and $s_3(\mathcal{K}, \mathcal{M}) = -0.1$. Hence, from Eq. (10) we get

$$\begin{aligned} \mathcal{D}v(\mathcal{K}, \mathcal{M}) &= \frac{1}{12(1 - e^{-1})} \begin{bmatrix} 4 - (1 + 0.1)\, e^{-0.1} - (1 - 0.1)\, e^{0.1} - (1 + 0.25)\, e^{-0.25} \\ - (1 - 0.25)\, e^{0.25} + 4 - (1 - 0.1)\, e^{0.1} - (1 + 0.1)\, e^{-0.1} \\ - (1 + 0.05)\, e^{-0.05} - (1 - 0.05)\, e^{0.05} + 4 - (1 + 0.05)\, e^{-0.05} \\ - (1 - 0.05)\, e^{0.05} - (1 + 0.1)\, e^{-0.1} - (1 - 0.1)\, e^{0.1} \end{bmatrix} \\ &= 0.0130 \end{aligned}$$

In the following, we prove some propositions for the proposed divergence measure. For it, we divide the universal set $\mathcal{X}$ into two disjoints subsets $\mathcal{X}_1 = \{x_j \mid \mathcal{K}(x_j) \subseteq \mathcal{M}(x_j)\}$ and $\mathcal{X}_2 = \{x_j \mid \mathcal{M}(x_j) \subseteq \mathcal{K}(x_j)\}$. Then, the following propositions are satisfied based on these considerations.

Proposition 3.1. If $\mathcal{K}$ and $\mathcal{M}$ be two CIFSs defined on $\mathcal{X}$ such that they satisfy for any $x_j \in \mathcal{X}$ either $\mathcal{K} \subseteq \mathcal{M}$ or $\mathcal{K} \supseteq \mathcal{M}$, then

(i) $\mathcal{D}v(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M}) = \mathcal{D}v(\mathcal{K}, \mathcal{M})$
(ii) $\mathcal{D}v(\mathcal{K} \cap \mathcal{M}, \mathcal{K} \cup \mathcal{M}) = \mathcal{D}v(\mathcal{K}, \mathcal{M})$.

Proof. Here, we prove (i) part only while the part (ii) can be deduced similarly. From the Definition 3.3, we have

$$\begin{aligned} &\mathcal{D}v(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M}) \\ &= \frac{1}{4n(1 - e^{-1})} \sum_{j=1}^{n} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \end{bmatrix} \end{aligned}$$

$$= \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_1} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_2} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \\ - \left(1 + s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})\right) e^{-s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{K} \cap \mathcal{M})} \end{bmatrix}$$

$$= \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_1} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{M}, \mathcal{K})\right) e^{t_j(\mathcal{M}, \mathcal{K})} - \left(1 + t_j(\mathcal{M}, \mathcal{K})\right) e^{-t_j(\mathcal{M}, \mathcal{K})} \\ - \left(1 - s_j(\mathcal{M}, \mathcal{K})\right) e^{s_j(\mathcal{M}, \mathcal{K})} - \left(1 + s_j(\mathcal{M}, \mathcal{K})\right) e^{-s_j(\mathcal{M}, \mathcal{K})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_2} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) e^{t_j(\mathcal{K}, \mathcal{M})} - \left(1 + t_j(\mathcal{K}, \mathcal{M})\right) e^{-t_j(\mathcal{K}, \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K}, \mathcal{M})\right) e^{s_j(\mathcal{K}, \mathcal{M})} - \left(1 + s_j(\mathcal{K}, \mathcal{M})\right) e^{-s_j(\mathcal{K}, \mathcal{M})} \end{bmatrix}$$

$$= \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_1} \begin{bmatrix} 4 - \left(1 + t_j(\mathcal{K}, \mathcal{M})\right) e^{-t_j(\mathcal{K}, \mathcal{M})} - \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) e^{t_j(\mathcal{K}, \mathcal{M})} \\ - \left(1 + s_j(\mathcal{K}, \mathcal{M})\right) e^{-s_j(\mathcal{K}, \mathcal{M})} - \left(1 - s_j(\mathcal{K}, \mathcal{M})\right) e^{s_j(\mathcal{K}, \mathcal{M})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})} \sum_{x_j \in \mathcal{X}_2} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) e^{t_j(\mathcal{K}, \mathcal{M})} - \left(1 + t_j(\mathcal{K}, \mathcal{M})\right) e^{-t_j(\mathcal{K}, \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K}, \mathcal{M})\right) e^{s_j(\mathcal{K}, \mathcal{M})} - \left(1 + s_j(\mathcal{K}, \mathcal{M})\right) e^{-s_j(\mathcal{K}, \mathcal{M})} \end{bmatrix}$$

$$= \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K}, \mathcal{M})\right) e^{t_j(\mathcal{K}, \mathcal{M})} - \left(1 + t_j(\mathcal{K}, \mathcal{M})\right) e^{-t_j(\mathcal{K}, \mathcal{M})} \\ - \left(1 - s_j(\mathcal{K}, \mathcal{M})\right) e^{s_j(\mathcal{K}, \mathcal{M})} - \left(1 + s_j(\mathcal{K}, \mathcal{M})\right) e^{-s_j(\mathcal{K}, \mathcal{M})} \end{bmatrix}$$

$$= \mathcal{D}v(\mathcal{K}, \mathcal{M})$$

□

Proposition 3.2. For CIFSs $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{N}$ defined on $\mathcal{X}$, we have

$$\mathcal{D}v(\mathcal{K} \cup \mathcal{M}, \mathcal{N}) + \mathcal{D}v(\mathcal{K} \cap \mathcal{M}, \mathcal{N}) = \mathcal{D}v(\mathcal{K}, \mathcal{N}) + \mathcal{D}v(\mathcal{M}, \mathcal{N})$$

Proof. By using Definition 3.3, we have

$$\mathcal{D}v(\mathcal{K} \cup \mathcal{M}, \mathcal{N}) + \mathcal{D}v(\mathcal{K} \cap \mathcal{M}, \mathcal{N})$$

$$= \frac{1}{4n(1-e^{-1})} \sum_{j=1}^{n} \begin{bmatrix} 4 - \left(1 - t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})\right) e^{t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})} \\ - \left(1 + t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})\right) e^{-t_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})} \\ - \left(1 - s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})\right) e^{s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})} \\ - \left(1 + s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})\right) e^{-s_j(\mathcal{K} \cup \mathcal{M}, \mathcal{N})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})}\sum_{j=1}^{n}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$= \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_1}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_2}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{K}\cup\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_1}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_2}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{K}\cap\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$= \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_1}\begin{bmatrix} 4-\left(1-t_j(\mathcal{M},\mathcal{N})\right)e^{t_j(\mathcal{M},\mathcal{N})} \\ -\left(1+t_j(\mathcal{M},\mathcal{N})\right)e^{-t_j(\mathcal{M},\mathcal{N})} \\ -\left(1-s_j(\mathcal{M},\mathcal{N})\right)e^{s_j(\mathcal{M},\mathcal{N})} \\ -\left(1+s_j(\mathcal{M},\mathcal{N})\right)e^{-s_j(\mathcal{M},\mathcal{N})} \end{bmatrix}$$

$$+ \frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_2}\begin{bmatrix} 4-\left(1-t_j(\mathcal{K},\mathcal{N})\right)e^{t_j(\mathcal{K},\mathcal{N})} \\ -\left(1+t_j(\mathcal{K},\mathcal{N})\right)e^{-t_j(\mathcal{K},\mathcal{N})} \\ -\left(1-s_j(\mathcal{K},\mathcal{N})\right)e^{s_j(\mathcal{K},\mathcal{N})} \\ -\left(1+s_j(\mathcal{K},\mathcal{N})\right)e^{-s_j(\mathcal{K},\mathcal{N})} \end{bmatrix}$$

$$+\frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_1}\begin{bmatrix}4-\Big(1-t_j(\mathcal{K},\mathcal{N})\Big)e^{t_j(\mathcal{K},\mathcal{N})}\\ -\Big(1+t_j(\mathcal{K},\mathcal{N})\Big)e^{-t_j(\mathcal{K},\mathcal{N})}\\ -\Big(1-s_j(\mathcal{K},\mathcal{N})\Big)e^{s_j(\mathcal{K},\mathcal{N})}\\ -\Big(1+s_j(\mathcal{K},\mathcal{N})\Big)e^{-s_j(\mathcal{K},\mathcal{N})}\end{bmatrix}$$

$$+\frac{1}{4n(1-e^{-1})}\sum_{x_j\in\mathcal{X}_2}\begin{bmatrix}4-\Big(1-t_j(\mathcal{M},\mathcal{N})\Big)e^{t_j(\mathcal{M},\mathcal{N})}\\ -\Big(1+t_j(\mathcal{M},\mathcal{N})\Big)e^{-t_j(\mathcal{M},\mathcal{N})}\\ -\Big(1-s_j(\mathcal{M},\mathcal{N})\Big)e^{s_j(\mathcal{M},\mathcal{N})}\\ -\Big(1+s_j(\mathcal{M},\mathcal{N})\Big)e^{-s_j(\mathcal{M},\mathcal{N})}\end{bmatrix}$$

$$=\mathcal{D}v(\mathcal{K},\mathcal{N})+\mathcal{D}v(\mathcal{M},\mathcal{N})$$

□

Proposition 3.3. For CIFSs $\mathcal{K}$ and $\mathcal{M}$ defined on $\mathcal{X}$, we have

(i) $\mathcal{D}v(\mathcal{K},\mathcal{K}\cup\mathcal{M})+\mathcal{D}v(\mathcal{K},\mathcal{K}\cap\mathcal{M})=\mathcal{D}v(\mathcal{K},\mathcal{M})$.
(ii) $\mathcal{D}v(\mathcal{M},\mathcal{K}\cup\mathcal{M})+\mathcal{D}v(\mathcal{M},\mathcal{K}\cap\mathcal{M})=\mathcal{D}v(\mathcal{K},\mathcal{M})$.

Proof. By taking $\mathcal{N}=\mathcal{K}$ in Proposition 3.2, we get $\mathcal{D}v(\mathcal{K}\cup\mathcal{M},\mathcal{K})+\mathcal{D}v(\mathcal{K}\cap\mathcal{M},\mathcal{K})=\mathcal{D}v(\mathcal{K},\mathcal{K})+\mathcal{D}v(\mathcal{M},\mathcal{K})=\mathcal{D}v(\mathcal{K},\mathcal{M})$. Hence, the part (i) holds. Similarly, for the part (ii). □

Proposition 3.4. For CIFSs $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{N}$ defined on $\mathcal{X}$, we have

(i) $\mathcal{D}v(\mathcal{K},\mathcal{N})+\mathcal{D}v(\mathcal{M},\mathcal{N})-\mathcal{D}v(\mathcal{K}\cup\mathcal{M},\mathcal{N})\geq 0$.
(ii) $\mathcal{D}v(\mathcal{K},\mathcal{N})+\mathcal{D}v(\mathcal{M},\mathcal{N})-\mathcal{D}v(\mathcal{K}\cap\mathcal{M},\mathcal{N})\geq 0$.

Proof. From Proposition 3.2, we have $\mathcal{D}v(\mathcal{K},\mathcal{N})+\mathcal{D}v(\mathcal{M},\mathcal{N})-\mathcal{D}v(\mathcal{K}\cup\mathcal{M},\mathcal{N})=\mathcal{D}v(\mathcal{K}\cap\mathcal{M},\mathcal{N})$ and by Theorem 3.1, we have $\mathcal{D}v(\mathcal{K}\cap\mathcal{M},\mathcal{N})\geq 0$. Therefore, $\mathcal{D}v(\mathcal{K},\mathcal{N})+\mathcal{D}v(\mathcal{M},\mathcal{N})-\mathcal{D}v(\mathcal{K}\cup\mathcal{M},\mathcal{N})\geq 0$. □

Proposition 3.5. For CIFSs $\mathcal{K}$ and $\mathcal{M}$ defined on $\mathcal{X}$, we have

(i) $\mathcal{D}v(\mathcal{K},\mathcal{M})=\mathcal{D}v(\mathcal{K}^c,\mathcal{M}^c)$.
(ii) $\mathcal{D}v(\mathcal{K}^c,\mathcal{M})=\mathcal{D}v(\mathcal{K},\mathcal{M}^c)$.
(iii) $\mathcal{D}v(\mathcal{K},\mathcal{M})+\mathcal{D}v(\mathcal{K}^c,\mathcal{M})=\mathcal{D}v(\mathcal{K}^c,\mathcal{M}^c)+\mathcal{D}v(\mathcal{K},\mathcal{M}^c)$.

Proof. Their proofs are direct from the Definition 3.3. □

Next, we define the weighted exponential divergence measure between CIFSs. For it, let $\kappa_j>0$ be the weight vector of $x_j\in\mathcal{X}$ with $\sum_{j=1}^{n}\kappa_j=1$.

Definition 3.4. A weighted exponential divergence measure between two CIFSs $\mathcal{K}$ and $\mathcal{M}$ is defined as

$$\mathcal{D}v_\kappa(\mathcal{K},\mathcal{M})=\frac{1}{4(1-e^{-1})}\sum_{j=1}^{n}\kappa_j\begin{pmatrix}4-(1-t_j(\mathcal{K},\mathcal{M}))\exp\Big(t_j(\mathcal{K},\mathcal{M})\Big)\\ -(1+t_j(\mathcal{K},\mathcal{M}))\exp\Big(-t_j(\mathcal{K},\mathcal{M})\Big)\\ -(1-s_j(\mathcal{K},\mathcal{M}))\exp\Big(s_j(\mathcal{K},\mathcal{M})\Big)\\ -(1+s_j(\mathcal{K},\mathcal{M}))\exp\Big(-s_j(\mathcal{K},\mathcal{M})\Big)\end{pmatrix}\tag{11}$$

where

$$t_j(\mathcal{K},\mathcal{M}) = \frac{(r_{\mathcal{K}}(x_j) - r_{\mathcal{M}}(x_j)) - (k_{\mathcal{K}}(x_j) - k_{\mathcal{M}}(x_j))}{2};$$
$$\text{and} \quad s_j(\mathcal{K},\mathcal{M}) = \frac{(w_{r_{\mathcal{K}}}(x_j) - w_{r_{\mathcal{M}}}(x_j)) - (w_{k_{\mathcal{K}}}(x_j) - w_{k_{\mathcal{M}}}(x_j))}{2}$$

provided $t_j, s_j \neq 0$.

If $\kappa = (1/n, 1/n, \ldots, 1/n)^T$ then Eq. (11) becomes Eq. (10). Further, measure defined in Eq. (11) also satisifies the axioms of divergence measures and hence $0 \leq \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M}) \leq 1$.

Proposition 3.6. Let $\mathcal{K}$, $\mathcal{M}$, $\mathcal{N}$ be three CIFSs defined on $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ such that for every $x_j \in \mathcal{X}$ either $\mathcal{K}(x_j) \subseteq \mathcal{M}(x_j)$ or $\mathcal{M}(x_j) \subseteq \mathcal{K}(x_j)$. Then,

(i) $\mathcal{D}v_\kappa(\mathcal{K}\cup\mathcal{M}, \mathcal{K}\cap\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M})$

(ii) $\mathcal{D}v_\kappa(\mathcal{K}\cap\mathcal{M}, \mathcal{K}\cup\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M})$.

(iii) $\mathcal{D}v_\kappa(\mathcal{K}\cup\mathcal{M}, \mathcal{N}) + \mathcal{D}v_\kappa(\mathcal{K}\cap\mathcal{M}, \mathcal{N}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{N}) + \mathcal{D}v_\kappa(\mathcal{M},\mathcal{N})$

(iv) $\mathcal{D}v_\kappa(\mathcal{K}, \mathcal{K}\cup\mathcal{M}) + \mathcal{D}v_\kappa(\mathcal{K}, \mathcal{K}\cap\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M})$

(v) $\mathcal{D}v_\kappa(\mathcal{M}, \mathcal{K}\cup\mathcal{M}) + \mathcal{D}v_\kappa(\mathcal{M}, \mathcal{K}\cap\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M})$

(vi) $\mathcal{D}v_\kappa(\mathcal{K},\mathcal{N}) + \mathcal{D}v_\kappa(\mathcal{M},\mathcal{N}) - \mathcal{D}v_\kappa(\mathcal{K}\cup\mathcal{M}, \mathcal{N}) \geq 0$

(vii) $\mathcal{D}v_\kappa(\mathcal{K},\mathcal{N}) + \mathcal{D}v_\kappa(\mathcal{M},\mathcal{N}) - \mathcal{D}v_\kappa(\mathcal{K}\cap\mathcal{M}, \mathcal{N}) \geq 0$

(viii) $\mathcal{D}v_\kappa(\mathcal{K},\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K}^c,\mathcal{M}^c)$

(ix) $\mathcal{D}v_\kappa(\mathcal{K}^c,\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M}^c)$

(x) $\mathcal{D}v_\kappa(\mathcal{K},\mathcal{M}) + \mathcal{D}v_\kappa(\mathcal{K}^c,\mathcal{M}) = \mathcal{D}v_\kappa(\mathcal{K}^c,\mathcal{M}^c) + \mathcal{D}v_\kappa(\mathcal{K},\mathcal{M}^c)$.

Proof. As similar to the Propositions 3.1-3.5. □

4. Proposed approach based on Divergence measure

Assume that, a set of alternatives $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ under "n" criteria $C_1, C_2, \ldots, C_n$. Let $\kappa_q > 0$ be the weight vector corresponding to criteria C_q with $\sum_{q=1}^{n} \kappa_q = 1$. An expert evaluated these different alternatives under the set of criteria and gave their preferences in terms of the CIFNs α_{pq}. The collective information of such rating is represented as a matrix $\mathcal{D} = (\alpha_{pq})_{m\times n}$.

Yingming [53] recommended a way of determining weights of criteria more subjectively, which he named as a method of maximizing deviations. In its method, the weight vector κ should be chosen in such a way that deviations of all criteria corresponding to alternatives become maximum. For an arbitrary criteria C_q $(q = 1, 2, \ldots, n)$, the deviation of alternative $\mathcal{A}_p$ from other alternatives is given as:

$$\mathscr{D}_{pq}(\kappa_q) = \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q$$

Then, the total deviations of the criteria C_q to all the alternatives are given as:

$$\mathscr{D}_q = \sum_{p=1}^{m}\sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q$$

Further, the deviations of all criteria to all alternatives are obtained as:

$$\mathscr{D} = \sum_{q=1}^{n}\sum_{p=1}^{m}\sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q$$

Let $\triangle$ be the set of weight information known in anyone of the following forms.

Form 1. A weak ranking: $\kappa_i \geq \kappa_j$;
Form 2. A strict ranking: $\kappa_i - \kappa_j \geq \sigma_i$; $(\sigma_i > 0)$.
Form 3. A ranking with multiples: $\kappa_i \geq \sigma_i \kappa_j$, $(0 \leq \sigma_i \leq 1)$;
Form 4. An interval form: $\lambda_i \leq \kappa_i \leq \lambda_i + \delta_i$, $(0 \leq \lambda_i \leq \lambda_i + \delta_i \leq 1)$;
Form 5. A ranking of differences: $\kappa_i - \kappa_j \geq \kappa_k - \kappa_l$, $(j \neq k \neq l)$.

Now, based on $\mathscr{D}$, we construct nonlinear optimization model to find the optimal weights of the criteria, by assuming that the information related to attribute weights are partially known, as follows:

$$\begin{cases} \max \mathscr{D} = \sum_{q=1}^{n} \sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q \\ \text{subject to} \quad \kappa_q \in \triangle \quad , \quad \sum_{q=1}^{n} \kappa_q = 1 \quad , \quad \kappa_q > 0 \end{cases} \tag{12}$$

where $\mathcal{D}v(\alpha_{pq}, \alpha_{uq})$ is determined using Eq. (10).

If attribute weights information are completely unknown, then we establish another nonlinear optimization model as

$$\begin{cases} \max \mathscr{D} = \sum_{q=1}^{n} \sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q \\ \text{subject to} \quad \sum_{q=1}^{n} \kappa_q = 1 \quad , \quad \kappa_q > 0 \end{cases} \tag{13}$$

In order to obtain the solution of the problem, stated in Eq. (13), consider a function:

$$g(\kappa_q, \lambda) = \sum_{q=1}^{n} \sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})\kappa_q + \lambda \left(\sum_{q=1}^{n} \kappa_q - 1 \right) \tag{14}$$

where λ is Lagrange's multiplier. Now,

$$\frac{\partial g}{\partial \kappa_q} = \sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq}) + \lambda \tag{15}$$

$$\frac{\partial g}{\partial \lambda} = \sum_{q=1}^{n} \kappa_q^2 - 1 \tag{16}$$

Now setting the Eqs. (15) and (16) equal to zero and then, solving them, we obtain:

$$\kappa_q = \frac{\sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})}{\sqrt{\sum_{q=1}^{n} \left(\sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq}) \right)^2}} \tag{17}$$

Further, the normalized value of κ_q can be obtained as:

$$\kappa_q^* = \frac{\kappa_q}{\sum_{q=1}^{n} \kappa_q} = \frac{\sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})}{\sum_{q=1}^{n} \sum_{p=1}^{m} \sum_{u=1}^{m} \mathcal{D}v(\alpha_{pq}, \alpha_{uq})} \tag{18}$$

By solving these models, the optimal weights $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)^T$ are obtained.

Based on the collective information and the weight vector κ, the following steps are proposed to compute the finest alternatives from the given ones.

Step 1: The information about the alternatives is represented as a decision matrix $\mathcal{D}$ given as

$$\mathcal{D} = \begin{array}{c} \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} \end{array} \tag{19}$$

Step 2: Normalize the information, if required, and obtained matrix $\mathcal{R} = (\zeta_{pq})$ where, ζ_{pq} is given by using Eq. (20).

$$\zeta_{pq} = \begin{cases} ((r_{pq}, w_{r_{pq}}), (k_{pq}, w_{k_{pq}})) & ; \quad \text{for benefit type criteria} \\ ((k_{pq}, w_{k_{pq}}), (r_{pq}, w_{r_{pq}})) & ; \quad \text{for cost type criteria} \end{cases} \tag{20}$$

Step 3: Formulate the optimization model either by Eq. (12) or Eq. (18) according to the known information of the weight vector and solve them.

Step 4: Construct the ideal alternative denoted by $\mathcal{A}^*$ as

$$\mathcal{A}^* = \left\{ (C_q, (r_q, w_{r_q}), (k_q, w_{k_q})) \mid q = 1, 2, \dots, n \right\}$$

where $r_q = \max\limits_p \left\{ r_{pq} \right\}$; $k_q = \min\limits_p \left\{ k_{pq} \right\}$; $w_{r_q} = \max\limits_p \left\{ w_{r_{pq}} \right\}$ and $w_{k_q} = \min\limits_p \left\{ w_{k_{pq}} \right\}$ $\forall\ q = 1, 2, \dots, n$.

Step 5: Compute the divergence measure for the alternative $\mathcal{A}_p (p = 1, 2, \dots, m)$ from $\mathcal{A}^*$ as

$$\mathcal{D}v_\kappa(\mathcal{A}_p, \mathcal{A}^*) = \frac{1}{4(1 - e^{-1})} \sum_{q=1}^{n} \kappa_q \begin{pmatrix} 4 - (1 - t_q(\mathcal{A}_p, \mathcal{A}^*)) \exp\left(t_q(\mathcal{A}_p, \mathcal{A}^*) \right) \\ - (1 + t_q(\mathcal{A}_p, \mathcal{A}^*)) \exp\left(- t_q(\mathcal{A}_p, \mathcal{A}^*) \right) \\ - (1 - s_q(\mathcal{A}_p, \mathcal{A}^*)) \exp\left(s_q(\mathcal{A}_p, \mathcal{A}^*) \right) \\ - (1 + s_q(\mathcal{A}_p, \mathcal{A}^*)) \exp\left(- s_q(\mathcal{A}_p, \mathcal{A}^*) \right) \end{pmatrix} \tag{21}$$

Step 6: Based on the argument of $\mathcal{D}v_\kappa$ given as $\arg \min\limits_{1 \le p \le m} \{ \mathcal{D}v_\kappa(\mathcal{A}_p, \mathcal{A}^*) \}$, order the alternatives and select the desired one.

5. Illustrative Example

To illustrate the approach, we consider a case study related to entrepreneur to purchase a new machine out of four different models denoted by $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$, with different date of production of each model. The accessibility of such machines are measured under four different criteria namely, C_1: "Reliability", C_2: "Safety", C_3: "Cost" and C_4: "Productivity for selecting machine". It is quite understood that these factors are changes with the change of the production dates. To evaluate each machine under such factors, preferences are taken from the expert in CIFNs. The steps of the proposed method are illustrated as below.

Step 1: The given information is collective in terms of CIFNs and summarized in Table 2. In this matrix, the entry corresponding to machine $\mathcal{A}_1$ represents that, an expert during the evaluation agree that it is reliable up to 70% under C_1 and unreliable at most 10%. Similarly, with respect to production date, he feels that 50% is compatible and 30% incompatible with C_1. In the similar manner, all data of matrix $\mathcal{D}$ can be interpreted.

Insert Table 2 here

Step 2: As C_3 is the cost type, so by using Eq. (20), the normalized data is given in Table 3.

Insert Table 3 here

Step 3: If we initially assume that the partial information about the weight vector corresponding to criteria is partially known and is given as $\triangle = \{0.2 \le \kappa_1 \le 0.4, 0.15 \le \kappa_2 \le 0.25, 0.25 \le \kappa_3 \le 0.3, 0.1 \le \kappa_4 \le 0.25\}$ such that $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$. Then, by using Eq. (12), we formulate an optimization model as

$$\begin{aligned} \max \mathscr{D}(\kappa) &= 0.1610\kappa_1 + 0.0890\kappa_2 + 0.1272\kappa_3 + 0.0711\kappa_4 \\ \text{subject to } & 0.20 \le \kappa_1 \le 0.40, \\ & 0.15 \le \kappa_2 \le 0.25, \\ & 0.25 \le \kappa_3 \le 0.30, \\ & 0.10 \le \kappa_4 \le 0.25 \\ & \sum_{q=1}^{4} \kappa_q = 1 \quad , \quad \kappa_q > 0 \end{aligned}$$

After solving, we get $\kappa = (0.4, 0.2, 0.3, 0.1)^T$. On the other hand, if we assume that the information related to the criteria weight is completely unknown then by using Eq. (18), we obtain the weight vector as $\kappa = (0.3592, 0.1985, 0.2838, 0.1585)^T$.

Step 4: Based on the matrix $\mathcal{R}$, the ideal alternative is taken as

$$\mathcal{A}^* = \left\{ \begin{matrix} (C_1, (0.7, 0.8), (0.1, 0.1)), (C_2, (0.7, 0.9), (0.2, 0.1)), \\ (C_3, (0.7, 0.7), (0.1, 0.1)), (C_4, (0.7, 0.7), (0.1, 0.1)) \end{matrix} \right\}$$

Step 5: With $\kappa = (0.4, 0.2, 0.3, 0.1)^T$, the exponential divergence measure values are obtained by Eq. (21) and get

$$\begin{aligned} \mathcal{D}v_\kappa(\mathcal{A}_1, \mathcal{A}^*) &= 0.0276, \qquad \mathcal{D}v_\kappa(\mathcal{A}_2, \mathcal{A}^*) = 0.0139, \\ \mathcal{D}v_\kappa(\mathcal{A}_3, \mathcal{A}^*) &= 0.1012, \qquad \mathcal{D}v_\kappa(\mathcal{A}_4, \mathcal{A}^*) = 0.0425. \end{aligned} \tag{22}$$

If $\kappa = (0.3592, 0.1985, 0.2838, 0.1585)^T$ is utilized then the measurement values are obtained as

$$\begin{aligned} \mathcal{D}v_\kappa(\mathcal{A}_1, \mathcal{A}^*) &= 0.0281, \qquad \mathcal{D}v_\kappa(\mathcal{A}_2, \mathcal{A}^*) = 0.0145, \\ \mathcal{D}v_\kappa(\mathcal{A}_3, \mathcal{A}^*) &= 0.0936, \qquad \mathcal{D}v_\kappa(\mathcal{A}_4, \mathcal{A}^*) = 0.0425. \end{aligned} \tag{23}$$

Step 6: By using these values, the ordering of the alternative is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ where "$\succ$" refers "preferred to". Hence, $\mathcal{A}_2$ is the finest one.

5.1. Comparative analysis with CIFS studies

To check the consistency of the method with some existing studies [42, 44, 45, 52] under the CIFS environment, an analysis is conducted by their method and the corresponding results are discussed as below:

(i) By applying the Alkouri and Salleh [42] method based on the distance measure 'd_1' on to the given information, we get $d_1(\mathcal{A}_1, \mathcal{A}^*) = 0.1500$, $d_1(\mathcal{A}_2, \mathcal{A}^*) = 0.1325$, $d_1(\mathcal{A}_3, \mathcal{A}^*) = 0.3310$ and $d_1(\mathcal{A}_4, \mathcal{A}^*) = 0.1885$. Thus, ordering is $\mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3$ and $\mathcal{A}_2$ is the best alternative.

(ii) By applying the Rani and Garg [44] method based on Hamming distance measure 'd_2', we get the values as $d_2(\mathcal{A}_1, \mathcal{A}^*) = 0.1450$, $d_2(\mathcal{A}_2, \mathcal{A}^*) = 0.1125$, $d_2(\mathcal{A}_3, \mathcal{A}^*) = 0.3200$ and $d_2(\mathcal{A}_4, \mathcal{A}^*) = 0.1725$. Hence, from it, it is seen that $\mathcal{A}_2$ is again the best alternative.

(iii) By applying the Garg and Rani [45] method based on correlation coefficient 'C', we obtain the indices values as $C(\mathcal{A}_1, \mathcal{A}^*) = 0.9407$, $C(\mathcal{A}_2, \mathcal{A}^*) = 0.9571$, $C(\mathcal{A}_3, \mathcal{A}^*) = 0.7547$ and $C(\mathcal{A}_4, \mathcal{A}^*) = 0.8926$. Clearly, seen that the best alternative is $\mathcal{A}_2$.

(iv) By performing the similarity measure S_1, as proposed by Garg and Rani [52], on to the considered information under the CIFS environment, we get the measurement value of each alternative as $S_1(\mathcal{A}_1, \mathcal{A}^*) = 0.6733$, $S_1(\mathcal{A}_2, \mathcal{A}^*) = 0.7663$, $S_1(\mathcal{A}_3, \mathcal{A}^*) = 0.5930$ and $S_1(\mathcal{A}_4, \mathcal{A}^*) = 0.6378$. Thus, from it, we conclude that the best alternative is $\mathcal{A}_2$.

From it, we conclude that their position of given alternatives coincides with the given ones which validates the feasibility of the method.

5.2. Comparative analysis with IFS studies

As IFS is one of the special cases of the CIFS with zero phase terms in each CIFNs, so in order to see their performance under the IFS environment also, we conduct a comparative study with the several existing approaches [30, 31, 33–37]. The results obtained through them are listed in Table 4 and obtained that $\mathcal{A}_1$ is the best alternative. However, from the proposed approach, we obtain $\mathcal{A}_2$ is the best alternative. This quiet change in the optimal ranking order is quite significant. This is due to the consideration of the computational procedure in the considered environment. For example, in [30, 31, 33–37] approaches, only one grade of MDs and NMDs are taken into account. Related to the considered problem, we can say that the entire focus is done on the model of the machine by neglecting the production date of each model. Thus, from the analysis, we can say that $\mathcal{A}_1$ is the best machine when there are no limits on the production date. However, in the proposed work, we have investigated the theory based on both the model as well as production dates simultaneously and hence conclude that machine $\mathcal{A}_2$ is the best with the production date also.

Insert Table 4 here.

5.3. Verification and Comparative Analysis

To generalize the capability of CIFS with respect to the features of IFS, we present some examples as follows.

Example 5.1. Consider a DMP which consists of five alternatives in the form of regions namely, $\mathcal{A}_1$: Lalitpur, $\mathcal{A}_2$: Kathmandu, $\mathcal{A}_3$: Gorkha, $\mathcal{A}_4$: Bhaktapur and $\mathcal{A}_5$: Makwanpur, that are affected from the earthquakes, racked Nepal on 25 April 2015. The given task is to identify the most damaged region so that a necessary facilities namely C_1(Food), C_2(Shelter), C_3(Clothes) and C_4(Medical requirements) are provided to victims. Let $\kappa = (0.30, 0.25, 0.15, 0.30)^T$ be the priority weight of them. Before allocating, an expert evaluate the given regions based on facilities and a "reference set" $\mathcal{B}$ is designed in terms of CIFS as

$$B = \left\{ \begin{array}{l} (C_1, (0.7, 0.5), (0.1, 0.3)), (C_2, (0.4, 0.6), (0.5, 0.2)), \\ (C_3, (0.5, 0.5), (0.3, 0.1)), (C_4, (0.8, 0.7), (0.2, 0.1)) \end{array} \right\}$$

During visit to each region, a team of experts have investigated them and summarize their information in Table 5. The ranking results corresponding to this problem is listed in Table 6 along with the several existing MCDM methods [42, 44, 45, 52]. From this table, it is seen that the best alternative remains $\mathcal{A}_3$ but the alternative $\mathcal{A}_1$ is preferable over $\mathcal{A}_4$ while by approaches [17–19, 21, 30, 31, 33–37] under IFS environment, it is observed that $\mathcal{A}_4$ is preferable over $\mathcal{A}_1$. This change in ordering is due to the change in the considered environment. Further, the studies under the IFS are very narrow with respect to CIFS studies.

Insert Tables 5 and 6 here.

Example 5.2. [45] Consider a medical diagnosis problem with four diseases Q_1("Viral fever"), Q_2("Malaria"), Q_3("Typhoid"), Q_4("Stomach Problem") and four symptoms s_1("Temperature"), s_2("HeadAche"), s_3("Stomach Pain"), s_4("Cough"). The rating values of each disease under symptoms are given in Table 7. The weight of each symptom is taken as $\kappa = (0.30, 0.20, 0.10, 0.40)^T$. Consider a patient $\mathcal{P}$ approach to the expert regarding their medical diagnosis. An expert treated this patient as a reference set and rate their values towards each symptom in terms of CIFSs summarized as

$$\mathcal{P} = \left\{ \begin{array}{l} (s_1, (0.8, 0.6), (0.1, 0.2)), (s_2, (0.9, 0.7), (0.1, 0.2)), \\ (s_3, (0.7, 0.8), (0.2, 0.1)), (s_4, (0.6, 0.5), (0.2, 0.4)) \end{array} \right\}$$

The aim of it to recognize the patient $\mathcal{P}$ with the sufficient disease. To it, methods are executed over it along with existing [17–19, 21, 30, 31, 33–37, 42, 44, 45, 52] and their results are given in Table 8. From it, we found that Q_1 is infested diseases by all the methods, which shows the feasibility of the approach.

Insert Tables 7 and 8 here.

5.4. Characteristic comparison

To study the features of developed method over the existing [9, 11, 13, 17, 30–37, 44, 45, 49], we analyze the features over them in Table 9. In this table, from '✓' we mean that the corresponding DMP satisfies the criteria such as an ideal alternative required to compute the process, measure the degree of discrimination and ability to handle the wider information, etc., whereas '×' means that the corresponding method fails. Further, from this table, it is clearly seen that the methods presented in [30–34] under the IFS environment fails to deal with time periodicity problems. Also, the methods presented in [17, 35–37, 44, 45, 49] doesn't measure the degree of discrimination between the two sets. Further, the DM approaches proposed in [30–34] fail to model complex problems whereas in the presented method the range of MDs and NMDs is a unit disc in the complex plane. This extension of ranges will enable the proposed approach to deal with one-dimensional problems described in [30–34] also. Therefore, the developed MCDM approach is more generalized.

Insert Table 9 here.

6. Conclusion

This paper aspires to give an exponential divergence measure for CIFSs to measure the degree of discrimination between the two or more CIFSs. IFS is generally used by the researchers to handle the data. However, CIFS is more extensive and manageable ways to express information and can represent a wide range of fuzzy information. In the presented work, the range of MDs and NMDs are extended from real number to complex number with a unit disc. Based on it, we develop a divergence measures and studied their relevant properties. It is remarked from the study that when additional component i.e., phase terms set to be zero then the relevant study becomes IFSs and hence the approaches under IFS study are the special cases of CIFS study. Further, based on the measure, a DM approach is presented to solve the MCDM problems and some practical examples are considered to verify its feasibility with the several existing approaches. The alignment of the proposed approach to the existing studies is shown and its advantages are outlined eliciting the supreme nature of the proposed theory over the existing ones. Based on its advantages, it is concluded that the presented theory can model the uncertainties with more enhancements as compared to the primitive environments. In the future, we shall lengthen the application of the proposed measure to the diverse fuzzy environment such as Pythagorean set [54–56], linguistic environment [57–59], as well as different fields of application such as supply chain management, emerging decision problems, risk evaluation, etc.[60–63] .

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Table 1: Comparison of CIFS model with existing models in literature

Features	Uncertainty	Falsity	Hesitation	Periodicity	Ability to represent two - dimensional information
FS	✓	×	×	×	×
IVFS	✓	×	×	×	×
IFS	✓	✓	✓	×	×
IVIFS	✓	✓	✓	×	×
CFS	✓	×	×	✓	✓
IVCFS	✓	×	×	✓	✓
CIFS	✓	✓	✓	✓	✓

Abbreviation: FS: Fuzzy set; IVFS: interval-valued fuzzy set; IFS: Intuitionistic fuzzy set; IVIFS: interval-valued intuitionistic fuzzy set; CFS: Complex fuzzy set; IVCFS: interval-valued complex fuzzy sets; CIFS: Complex intuitionistic fuzzy set.

Table 2: Input data for the problem in CIFN format

	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	((0.7, 0.5), (0.1, 0.3))	((0.4, 0.5), (0.3, 0.4))	((0.2, 0.1), (0.6, 0.6))	((0.5, 0.4), (0.1, 0.3))
$\mathcal{A}_2$	((0.7, 0.6), (0.3, 0.3))	((0.4, 0.9), (0.2, 0.1))	((0.2, 0.3), (0.7, 0.7))	((0.4, 0.6), (0.3, 0.1))
$\mathcal{A}_3$	((0.3, 0.4), (0.6, 0.4))	((0.6, 0.6), (0.3, 0.4))	((0.5, 0.6), (0.3, 0.4))	((0.7, 0.7), (0.1, 0.1))
$\mathcal{A}_4$	((0.4, 0.8), (0.5, 0.1))	((0.7, 0.3), (0.3, 0.3))	((0.1, 0.3), (0.6, 0.5))	((0.5, 0.5), (0.3, 0.4))

Table 3: Normalized information data

	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	((0.7, 0.5), (0.1, 0.3))	((0.4, 0.5), (0.3, 0.4))	((0.6, 0.6), (0.2, 0.1))	((0.5, 0.4), (0.1, 0.3))
$\mathcal{A}_2$	((0.7, 0.6), (0.3, 0.3))	((0.4, 0.9), (0.2, 0.1))	((0.7, 0.7), (0.2, 0.3))	((0.4, 0.6), (0.3, 0.1))
$\mathcal{A}_3$	((0.3, 0.4), (0.6, 0.4))	((0.6, 0.6), (0.3, 0.4))	((0.3, 0.4), (0.5, 0.6))	((0.7, 0.7), (0.1, 0.1))
$\mathcal{A}_4$	((0.4, 0.8), (0.5, 0.1))	((0.7, 0.3), (0.3, 0.3))	((0.6, 0.5), (0.1, 0.3))	((0.5, 0.5), (0.3, 0.4))

Table 4: Comparative study results under IFS environment

Ref.	Measurement values of				Ranking
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	
Maheshwari and Srivastava [33] method based on divergence measure	0.0161	0.0491	0.0515	0.0371	$\mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
Ohlan [30] method based on divergence measure	0.0605	0.0986	0.3893	0.1717	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Srivastava and Maheshwari [34] method based on divergence measure	0.1125	0.1722	0.1926	0.1520	$\mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_3$
Garg et al. [31] method based on divergence measure	0.0563	0.2105	0.1697	0.1606	$\mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_3 \succ \mathcal{A}_2$
Shen et al. [35] method based on distance measure	0.0857	0.1439	0.3644	0.2102	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Ye [36] method based on similarity measure	0.9816	0.9500	0.8231	0.9126	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$
Song et al. [37] method based on similarity measure	0.9888	0.9637	0.9155	0.9573	$\mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_3$

Table 5: Input preference for Example 5.1

	C_1	C_2	C_3	C_4
$\mathcal{A}_1$	((0.6, 0.7), (0.1, 0.2))	((0.9, 0.8), (0.1, 0.1))	((0.5, 0.4), (0.3, 0.4))	((0.6, 0.4), (0.2, 0.1))
$\mathcal{A}_2$	((0.4, 0.2), (0.3, 0.1))	((0.5, 0.3), (0.1, 0.1))	((0.6, 0.4), (0.2, 0.3))	((0.8, 0.6), (0.1, 0.2))
$\mathcal{A}_3$	((0.7, 0.7), (0.1, 0.2))	((0.4, 0.6), (0.3, 0.1))	((0.7, 0.7), (0.1, 0.1))	((0.6, 0.5), (0.3, 0.4))
$\mathcal{A}_4$	((0.7, 0.6), (0.3, 0.3))	((0.4, 0.9), (0.2, 0.1))	((0.7, 0.7), (0.2, 0.3))	((0.5, 0.3), (0.3, 0.6))
$\mathcal{A}_5$	((0.2, 0.8), (0.5, 0.1))	((0.7, 0.3), (0.3, 0.3))	((0.6, 0.5), (0.1, 0.3))	((0.6, 0.5), (0.3, 0.4))

Table 6: Comparative analysis of Example 5.1 with existing studies

Approach	Measurement value of $\mathcal{B}$ from					Ranking order
	Study under CIFS environment					
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	
Alkouri and Salleh's method [42] based on distance measure	0.1817	0.1917	0.1400	0.2167	0.2600	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
Rani and Garg's method [44] based on Euclidean distance measure	0.1871	0.1803	0.1374	0.2086	0.2225	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
Garg and Rani's method [45] based on Correlation measure	0.8965	0.9087	0.9439	0.8747	0.8351	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
Garg and Rani's method [52] based on Similarity measure	0.6912	0.6231	0.7400	0.6501	0.5762	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
Proposed method based on divergence measure	0.0325	0.0185	0.0171	0.0391	0.0552	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
	Study under IFS environment					
Approach	Measurement value of $\mathcal{B}$ from					Ranking order
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	
Zeng and Li [17] method based on correlation coefficient	0.8740	0.8874	0.9442	0.8822	0.8262	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
Ye [18] method based on correlation coefficient	0.8808	0.9106	0.9551	0.9246	0.8250	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
Liu et al. [19] method based on correlation coefficient	-0.4603	0.0000	0.5198	0.1143	-0.6336	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
Luo and Ren [21] method based on similarity measure	0.8221	0.8527	0.8827	0.8492	0.7562	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
Maheshwari and Srivastava [33] method based on divergence measure	0.0534	0.0454	0.0249	0.0655	0.0621	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
Ohlan [30] method based on divergence measure	0.2254	0.1395	0.0731	0.0957	0.3216	$\mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5$
Srivastava and Maheshwari [34] method based on divergence measure	0.1520	0.1722	0.1125	0.1926	0.2345	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5$
Garg et al. [31] method based on divergence measure	0.2428	0.1922	0.1196	0.2981	0.2685	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_1 \succ \mathcal{A}_5 \succ \mathcal{A}_4$
Shen et al. [35] method based on distance measure	0.1486	0.2104	0.1078	0.1759	0.2969	$\mathcal{A}_3 \succ \mathcal{A}_1 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_5$
Ye [36] method based on similarity measure	0.9084	0.9525	0.9767	0.9360	0.9099	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_5 \succ \mathcal{A}_1$
Song et al. [37] method based on similarity measure	0.9492	0.9590	0.9812	0.9522	0.9346	$\mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4 \succ \mathcal{A}_1 \succ \mathcal{A}_5$

Table 7: Input data for Example 5.2

	s_1	s_2	s_3	s_4
Q_1	((0.8, 0.7), (0.1, 0.2))	((0.9, 0.6), (0.1, 0.2))	((0.7, 0.8), (0.2, 0.1))	((0.8, 0.7), (0.2, 0.1))
Q_2	((0.6, 0.4), (0.1, 0.5))	((0.4, 0.9), (0.5, 0.1))	((0.5, 0.5), (0.3, 0.3))	((0.4, 0.9), (0.5, 0.1))
Q_3	((0.3, 0.8), (0.3, 0.1))	((0.8, 0.3), (0.1, 0.6))	((0.7, 0.6), (0.2, 0.2))	((0.2, 0.7), (0.8, 0.2))
Q_4	((0.5, 0.3), (0.4, 0.6))	((0.3, 0.1), (0.6, 0.3))	((0.8, 0.3), (0.1, 0.5))	((0.1, 0.3), (0.6, 0.5))

Table 8: Comparative analysis of Example 5.2 with existing studies

	Study under CIFS environment				
Approach	Measurement value of $\mathcal{P}$ from				Ranking order
	Q_1	Q_2	Q_3	Q_4	
Alkouri and Salleh's method [42] based on distance measure	0.0967	0.2717	0.2867	0.3550	$Q_1 > Q_2 > Q_3 > Q_4$
Rani and Garg's method [44] based on Euclidean distance measure	0.1194	0.2291	0.2669	0.3004	$Q_1 > Q_2 > Q_3 > Q_4$
Garg and Rani's method [45] based on Correlation measure	0.9696	0.8486	0.8008	0.6980	$Q_1 > Q_2 > Q_3 > Q_4$
Garg and Rani's method [52] based on Similarity measure	0.8896	0.5723	0.6037	0.4287	$Q_1 > Q_3 > Q_2 > Q_4$
Proposed method based on divergence measure	0.0121	0.0608	0.0804	0.1077	$Q_1 > Q_2 > Q_3 > Q_4$
	Study under IFS environment				
Approach	Measurement value of $\mathcal{P}$ from				Ranking order
	Q_1	Q_2	Q_3	Q_4	
Zeng and Li [17] method based on correlation coefficient	0.9856	0.8461	0.7959	0.7258	$Q_1 > Q_2 > Q_3 > Q_4$
Ye [18] method based on correlation coefficient	0.9912	0.8585	0.7265	0.6645	$Q_1 > Q_2 > Q_3 > Q_4$
Liu et al. [19] method based on correlation coefficient	0.8485	0.1907	0.6608	-0.0690	$Q_1 > Q_3 > Q_2 > Q_4$
Luo and Ren [21] method based on similarity measure	0.9642	0.7394	0.7725	0.6538	$Q_1 > Q_3 > Q_2 > Q_4$
Maheshwari and Srivastava [33] method based on divergence measure	0.0195	0.0523	0.0820	0.0887	$Q_1 > Q_2 > Q_3 > Q_4$
Ohlan [30] method based on divergence measure	0.0100	0.3090	0.3946	0.6407	$Q_1 > Q_2 > Q_3 > Q_4$
Srivastava and Maheshwari [34] method based on divergence measure	0.0365	0.2345	0.2345	0.2996	$Q_1 > Q_2 = Q_3 > Q_4$
Garg et al. [31] method based on divergence measure	0.1024	0.2058	0.3318	0.3170	$Q_1 > Q_2 > Q_4 > Q_3$
Shen et al. [35] method based on distance measure	0.0308	0.2428	0.3339	0.4386	$Q_1 > Q_2 > Q_3 > Q_4$
Ye [36] method based on similarity measure	0.9804	0.9023	0.8000	0.8311	$Q_1 > Q_2 > Q_4 > Q_3$
Song et al. [37] method based on similarity measure	0.9825	0.9468	0.8931	0.8861	$Q_1 > Q_2 > Q_3 > Q_4$

Table 9: The characteristic comparison of different approaches

Method	Needs an ideal alternative to compute the process	Measure the degree of discrimination between the sets	No unknown parameter to choose for aggregation	Ability to capture information using complex numbers	Ability to handle two-dimensional information
Garg [9]	×	×	×	×	×
Huang [11]	×	×	×	×	×
Chen and Chang [13]	×	×	×	×	×
Mishra et al. [32]	✓	✓	×	×	×
Garg et al. [31]	✓	✓	×	×	×
Maheshwari and Srivastava [33]	✓	✓	×	×	×
Ohlan [30]	✓	✓	×	×	×
Srivastava and Maheshwari [34]	✓	✓	×	×	×
Shen et al. [35]	✓	×	×	×	×
Ye [36]	✓	×	×	×	×
Song et al. [37]	✓	×	×	×	×
Zeng and Li [17]	✓	×	×	×	×
Rani and Garg [44]	✓	×	✓	✓	✓
Garg and Rani [45]	✓	×	✓	✓	✓
Rani and Garg [49]	×	×	✓	✓	✓
Proposed method	✓	✓	✓	✓	✓