On the Bayesian Analysis of Two-Component Mixture of Transmuted Weibull Distribution

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Abstract

Transmuted distributions are skewed distributions and recently attracted a great attention of researchers due to their applications in reliability and statistics. In this article, our main focus is on the Bayesian estimation of two-component mixture of the Transmuted Weibull Distribution (TWD) under type-I right censored sampling scheme. In order to estimate the unknown parameters, non-informative and informative priors under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF) and Quadratic Loss Function (QLF) are assumed when computing the posterior estimations. In addition, the Bayesian credible intervals (BCI) are also constructed. A Markov Chain Monte Carlo (MCMC) technique is adopted to generate samples from the posterior distributions and in turn computing different posterior summaries, including Bayes estimates (BEs), posterior risks (PRs) and Bayesian credible intervals (BCI). As an illustration, comparison of these Bayes estimators is made through simulated under different loss functions in terms of their respective posterior risks assuming different sample sizes and censoring rates. Two real-life examples, the first being the survival times of hepatitis B & C patients while the second being the hole diameter of 12 mm and the sheet thickness is 3.15 mm, are also discussed to illustrate the potential application of the proposed methodology.

Keywords: Transmuted Weibull Distribution, Mixture Model, Loss functions, Bayes Estimators, Posterior risks, Uniform prior, Informative prior, Bayesian intervals, MCMC, and Type-I right Censoring.

1. Introduction

Shaw and Buckley [1] introduced a technique to construct skewed distributions by using the quadratic rank transmutation map (QRTM) to generate a flexible family of probability distributions. The authors considered the extreme value distribution as the base line distribution and used a transmuted parameter to enhance the flexibility of the base line distribution [2]. Currently, transmuted distributions are applied in many diverse fields such as reliability studies, lifetime analysis, engineering, economics, medicine, insurance and environmental sciences [3].

A random variable X is said to follow a transmuted distribution if its probability density function (PDF) and cumulative distribution function (CDF) can be written as:

$$f(x) = g(x)\{1 + \lambda - 2\lambda G(x)\} \tag{1}$$

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x) \tag{2}$$

where $x > 0$ and $|\lambda| \leq 1$ is the transmuted parameter, and g(x) and G(x) are the PDF and CDF of the baseline distribution, respectively.

The Weibull distribution is a popular continuous probability distribution introduced by Swedish physicist Waloddi Weibull in 1939 in order to describe the behavior of the breaking strength of materials [4]. Since its inception, the Weibull distribution has proved to be a versatile lifetime distribution and many other distributions are its special cases based on different values of the shape parameter. The Weibull distribution has extensively used for analyzing lifetime data in reliability engineering, astronomy, medicine, psychology, botany, zoology, agriculture, fisheries, aerospace, and electronics [5]. However, to generalize the Weibull distribution for non-monotonic hazard rate, several distributions have been introduced in the literature. The first generalization of the Weibull distribution with application to survival data is given by Mudholkar et al. [6]. Later, Aryal and Tsokos [7] introduced the transmuted Weibull distribution which is a generalization of Weibull probability distribution. Khan and King [8] introduced the transmuted modified Weibull distribution which is more flexible than the transmuted Weibull distribution. The new modified distribution reduces to the transmuted Weibull distribution by setting the scale parameter equal to zero. Merovci et al. [9] proposed the transmuted generalized inverse Weibull distribution and discussed some of its mathematical properties. The authors also estimated the parameters using the method of maximum likelihood. Khan et al. [5] introduced transmuted generalized Weibull distribution using the QRTM technique and explored its mathematical properties including the expressions for the quantile function, moments, entropies, mean deviation, Bonferroni and Lorenz curves and moments of the order statistics. The authors also estimated the model parameters using the method of the maximum likelihood. This distribution is an important competitive model that contains twenty three lifetime distributions as special cases. More recently, Abdurrahman [10] used the method of the maximum likelihood and method of moment estimators to estimate its parameters. Similarly, Nofal et al. [11] proposed a generalized transmuted Weibull distribution based on generalized transmuted-G family and derived its properties. The main feature of this distribution is that two additional parameters are inducted in the PDF and CDF of Weibull distribution to provide greater flexibility for the generated distribution and contains eleven lifetime distributions as special cases. The generalized transmuted Weibull distribution can be reduced into transmuted class (TC) studied by Shaw and Buckley [1] by equating two additional parameters equal to one.

Finite Mixture models provide a flexible framework to handle heterogeneous data with a finite number of unobserved subpopulations and also have been widely applied to classification, clustering, and pattern identification problems. The mixture models have received great attention in the recent era due to their flexibility. A finite mixture of probability distributions is suitable to study a population categorized in a number of subpopulations mixed in an unknown proportion. The concept of the finite mixture distribution was pioneered by Newcomb [12] for modeling outliers. The mixture models can be used even when available data are generated from a mixture of two or more distributions. This motivated us to mix two or more statistical models to get a new mixture model. The analysis of mixture models under the Bayesian framework has gained a significant interest among the statisticians. Most of the researchers worked on two-component mixture models using both classical and Bayesian analysis, e.g., Feroze and Aslam [13] presented the Bayesian analysis of doubly censored lifetime data using two-component mixture of the Weibull distribution. Sindhu et al. [14] studied the two-component mixture of inverse Weibull distributions under doubly censored sample using various loss functions. Aslam et al. [15,16] and Tahir et al. [17] studied the properties of three-component mixture of Rayleigh distributions while Ateya [18] discussed the mixture of generalized exponential distribution. Similarly, Benaicha and Chaker [19] also studied the mixture of the Weibull distribution. These contributions to the mixture models are great motivations for the recent studies.

Motivated by the popularity of the mixture models, this study considers the Bayesian parameter estimation of a two-component mixture of the transmuted Weibull distribution. Bayesian estimation is done assuming different noninformative (uniform) and informative (gamma and inverse gamma) priors and three loss functions are used to obtain the posterior summaries. Censoring is an important aspect of the lifetime data because most of the times it is not possible to carry on the experiment until the last observation in order to obtain a complete data set. A censored data set contains at least one observation about which only partial information on the exact failure time is available. There are three types of censored observations, the left, the interval and the right censored observations. Due to nonavlibilty of true lifetime of certain objects, type-I right censoring is used by taking some pre-specified test termination time [20-22], e.g., if a patient survives until the end of a study, the patient's time of death is right-censored. Since the marginal posterior distributions are not in closed forms, we proposed a new Markov Chain Monte Carlo (MCMC) algorithm in order to obtain Bayes estimates, posterior risks and 95% credible intervals.

The rest of the article is structured as follows: The transmuted Weibull mixture model, sampling scheme, likelihood function, the expressions of posterior distributions using noninformative and informative priors, and marginal posterior densities for censored data are discussed in Section 2. The expression for the Bayes Estimators (BEs) and their respective Posterior Risks (PRs) under different loss functions are presented in Section 3. To obtain the posterior summaries, an MCMC algorithm is presented in Section 3. The results of BEs and their PRs based on a simulation study are tabulated in Section 4. In Section 5, the Bayesian Credible Intervals (BCI) are discussed mathematically and numerically. Two real-life data sets are presented in Sections 6 while some concluding remarks are given in Section 7.

2. The Mixture of Transmuted Weibull Model

In this section, we introduce the likelihood function for a two-component mixture of transmuted Weibull distribution.

A random variable X is said to follow a finite mixture distribution with m-components and unknown mixing proportion (p) if the probability density function of X can be written as:

$$f(x \mid \theta) = \sum_{i=1}^{m} p_i \, f(x \mid \theta_i) \text{ where } p_i > 0 \; ; \; i = 1,2,...m \text{ and } \sum_{i=1}^{m} p_i = 1$$

The parameter p_i is the mixing proportion of i^{th} component, whereas f_i(x) denotes the density of i^{th} component parameterized by θ_i.

A finite two-component mixture model of transmuted Weibull distribution with unknown mixing proportions p_1 and p_2 has the following form.

$$f(x;\Omega) = p_1 f_1(x;\Omega_1) + (1-p_1) f_2(x;\Omega_2) \;\&\; p_2 \le 1-p_1 \quad 1$$

where $\Omega = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, p_1)$, $\Omega_i = (\alpha_i, \beta_i, \lambda_i)$, $i = 1,2$, and $f_i(x;\Omega_i)$ denote the PDF of the i^{th} component that can be written as:

$$f_i(x;\Omega_i) = p_i \frac{\alpha_i}{\beta_i} x_i^{\alpha_i - 1} \exp\left(-\frac{x_i^{\alpha_i}}{\beta_i}\right)\left\{1-\lambda_i + 2\lambda_i \exp\left(-\frac{x_i^{\alpha_i}}{\beta_i}\right)\right\}, \quad \mathrm{x_i} \ge 0, \alpha_i, \beta_i > 0 \text{ and } |\lambda_i| \le 1 \; i = 1,2 \qquad (3)$$

Further, the CDF of a two-component mixture of Transmuted Weibull Distribution is:

$$F(x;\Omega) = p_1 F_1(x;\Omega_1) + (1-p_1) F_2(x;\Omega_2)$$

where the CDF of the i^{th} component is given by

$$F_i(x;\Omega_i) = p_i \exp\left(-\frac{x^{\alpha_i}}{\beta_i}\right)\left\{1-\lambda_i + \lambda_i \exp\left(-\frac{x^{\alpha_i}}{\beta_i}\right)\right\} \; \mathrm{x_i} \ge 0, \alpha_i, \beta_i > 0, |\lambda_i| \le 1 \text{ and } i = 1,2 \qquad (4)$$

Special Cases

- If λ=0 and p_1=1 then the two-component mixture of the transmuted Weibull distribution reduced to the Weibull distribution.
- If $\alpha = 1$ and p_1=1then we have transmuted exponential distribution. In addition, we obtained the exponential distribution by assuming $\lambda = 0$.
- If $\alpha = \beta = 1$ and p_1=1 then the resulting distribution is the transmuted standard exponential distribution.
- If α =2 and p_1=1 then the transmuted Rayleigh distribution is obtained. Further, if $\lambda = 0$, then the ordinary Rayleigh distribution is obtained.

2.1 Sampling Scheme

Suppose that *n* units are put on a life testing experiment with a fixed termination time *t*. At the end of the experiment, it is found that *r* units out of *n* objects have failed and *n-r* objects are still functioning. As noted by Mendenhall and Hader [23], in many real-life situations only the failed units can easily be categorized into either a member of subpopulation-I or subpopulation-II. For example, an engineer can identify whether a failed electronic object is a member of the first or the second subpopulation based on the cause of its failure. Thus, out of *r* failures, r_1 and r_2 failures belong to subpopulation-I and subpopulation-II, respectively. However, $r = r_1 + r_2$ denotes the number of observed observations while *n-r* observations are the censored. Now, we define x_{lj}, $0 < x_{lj} < t$ be the failure time of the j^{th} object belonging to the l^{th} sub-population, where l =1,2 and j =1,2,…r_l.

2.2 The Likelihood Function

For a 2-component mixture model, the likelihood function is

$$L(\mathbf{x};\Omega) = \left\{\prod_{j=1}^{r_1} p_1 f_1(x_{1j})\right\}\left\{\prod_{j=1}^{r_2} p_2 f_2(x_{2j})\right\}\{1 - F(T)\}^{n-r}$$

where $\Omega = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, p_1)$ $\mathbf{x} = (x_{11}, x_{12}, \ldots x_{1r_1}, x_{21}, x_{22}, \ldots x_{2r_2})$

and F(T) denote the CDF at time(T).

For the transmuted Weibull distribution, the likelihood expression can be written as:

$$L(\mathbf{x};\Omega) = \left[\prod_{j=1}^{r_1} p_1 \frac{\alpha_1}{\beta_1} x_{1j}^{\alpha_1 - 1} \exp\left(-\frac{x_{1j}^{\alpha_1}}{\beta_1}\right)\left\{1-\lambda_1 + 2\lambda_1 exp\left(-\frac{x_{1j}^{\alpha_1}}{\beta_1}\right)\right\}\right]\left[\prod_{j=1}^{r_{21}} (1-p_1)\frac{\alpha_2}{\beta_2} x_{2j}^{\alpha_2-1} \exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right.$$
$$\left.\left\{1-\lambda_2 + 2\lambda_2 exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right\}\right]\left[1 - p_1 exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\left\{1-\lambda_1 + 2\lambda_1 exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\right\} - (1-p_1) exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\right.$$
$$\left.\left\{1-\lambda_{2-} + 2\lambda_2 exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\right\}\right]^{n-r}$$

which can be simplified as

$$\mathrm{L}(x;\Omega) \propto p_1^{E_{11}-1}(1-p_1)^{E_{12}-1}\alpha_1^{E_{11}-1}\exp(-\alpha_1 F_{11})\alpha_2^{E_{12}-1}\exp(-\alpha_2 F_{12})\frac{1}{\beta_1^{E_{21}+1}}\exp\left(-\frac{G_{11}}{\beta_1}\right)\frac{1}{\beta_2^{E_{22}+1}}\exp\left(-\frac{G_{12}}{\beta_2}\right)\exp(H_{11})$$
$$\exp(H_{12}) I^{n-r} \qquad (5)$$

where $E_{11} = r_1 + 1, E_{12} = r_2 + 1$, $F_{11} = \sum_{j=1}^{r_1} log\left(\frac{1}{x_{1j}}\right)$, $F_{12} = \sum_{j=1}^{r_2} log\left(\frac{1}{x_{2j}}\right)$,

$E_{21} = r_1 - 1, E_{22} = r_2 - 1$, $G_{11} = \sum_{j=1}^{r_1} x_{1j}^{\alpha_2}$, $G_{12} = \sum_{j=1}^{r_1} x_{2j}^{\alpha_2}$,

$H_{11} = \sum_{j=i}^{r_1} \log\left\{1 - \lambda_1 + 2\lambda_1 \exp\left(-\frac{x_{1j}^{\alpha_1}}{\beta_1}\right)\right\}$ $H_{12} = \exp\left[\sum_{j=i}^{r_2} \log\left\{1 - \lambda_2 + 2\lambda_2 \exp\left(-\frac{x_{2j}^{\alpha_2}}{\theta_2}\right)\right\}\right]$,

$$I = 1 - p_1 \exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\left\{1-\lambda_1+2\lambda_1\exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\right\} - (1-p_1)\exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\left\{1-\lambda_2+\lambda_2 \exp p\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\right\}.$$

In the next section, we discuss the posterior distribution assuming different priors.

2.3 The Posterior Distribution using UP

The main difference between the Bayesian and classical inference is the prior information. The prior distribution uses the past information about a phenomenon under investigation. However, selecting an appropriate prior is definitely the most important task of the Bayesian analysis [24-25], because posterior distribution depends heavily on the prior information. There are situations where the sufficient prior information regarding the parameter of interest is available. An important noninformative prior, proposed by Laplace [26], is the uniform prior, which has been applied to many problems, and mostly provides satisfactory results. We define the noninformative uniform prior for $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1 as:

$\alpha_1 \sim Uniform(0,\infty)$, $\alpha_2 \sim Uniform(0,\infty)$, $\beta_1 \sim Uniform(0,\infty)$ and $\beta_2 \sim Uniform(0,\infty)$.

Furthermore, the UP over the interval (0, 1) is assumed for mixing proportion p_1 , i.e., $p_1 \sim Uniform(0,1)$ while the priors for transmuted parameters λ_1 and λ_2 are *UP (-1, 1)*. In our study, it is assumed that the prior distributions of $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1 are independent [27-28]. The joint prior distribution of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1 is:

$$\pi_{UP}(\Omega) \propto 1;\ \alpha_1, \alpha_2 > 0, \beta_1, \beta_2 > 0, -1 < \lambda_1, \lambda_2 < 1 \ \& \ 0 < p_1 < 1$$

By the Bayes theorem $g(\Omega|\mathbf{x}) = \dfrac{L(x;\Omega)\pi(\Omega)}{\int_{\Omega}^{\infty} L(x;\Omega)\pi(\Omega d\Omega}$, where $\pi(\Omega)$ denotes the joint prior distribution of $\Omega = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, p_1)$, $\mathbf{x} = (x_{11}, x_{12},...x_{1r_1}, x_{21}, x_{22},...x_{2r_2})$, $L(\mathbf{x};\Omega)$ denotes the likelihood function and $g(\Omega|\mathbf{x})$ is the joint posterior distribution; we have the joint posterior distribution as follows:

$$g_{UP}(\Omega \mid \mathbf{x}) = \frac{\begin{array}{r} p_1^{E_{11}}(1-p_1)^{E_{12}-1}\alpha_1^{E_{11}-1}\exp(-\alpha_1 F_{11})\alpha_2^{E_{12}-1}\exp(-\alpha_2 F_{12})\frac{1}{\beta_1^{E_{21}+1}}\exp\left(-\frac{G_{11}}{\beta_1}\right) \\ \frac{1}{\beta_2^{E_{22}+1}}\exp(-\beta_2 G_{12})\exp(H_{11})\exp(H_{12})(K)^{n-r} \end{array}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1 \begin{array}{l} p_1^{E_{11}}(1-p_1)^{E_{12}-1}\alpha_1^{E_{11}-1}\exp(-\alpha_1 F_{11})\alpha_2^{E_{12}-1}\exp(-\alpha_2 F_{12})\frac{1}{\beta_1^{E_{21}+1}}\exp\left(-\frac{G_{11}}{\beta_1}\right) \\ \frac{1}{\beta_2^{E_{22}+1}}\exp(-\beta_2 G_{12})\exp(H_{11})\exp(H_{12})(K)^{n-r}\,dp_1 d\lambda_1 d\lambda_2 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \end{array}} \quad (6)$$

The marginal posterior density of each parameter can be obtained by integrating the joint posterior distribution with respect to nuisance parameter.

Forexample, $g_{UP}(\alpha_1 \mid \mathbf{x}) = \int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1\int_0^1 g_{UP}(\Omega|x)\,dp_1 d\lambda_1 d\lambda_2 d\beta_1 d\beta_2 d\alpha_2$.

2.4 The Posterior Distribution using IP

If some specific and definite information about the values of the parameters is available, it is quantified as an informative prior and must be included in the analysis. For the mixture of transmuted Weibull distribution, we assume different prior distributions such as gamma, inverse gamma, uniform, and beta for $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1. To this end, let $\alpha_1 \sim gamma(a_1, b_1)$ $\alpha_2 \sim gamma(a_2, b_2)$, $\beta_1 \sim inverse-gamma(c_1, d_1)$, $\beta_2 \sim inverse-gamma(c_2, d_2)$ $\lambda_1 \sim Uniform(l_1, l_2)$, $\lambda_2 \sim Uniform(m_1, m_2)$ and $p_1 \sim beta(e_1, e_2)$. These priors are selected by keeping in mind the range of the parameters [29]. Assuming independence, we have the joint prior distribution of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1 :

$$\pi_{IP}(\Omega) \propto \alpha_1^{a_1-1}e^{-b_1\alpha_1}\alpha_2^{a_2-1}e^{-b_2\alpha_2}\frac{1}{\beta_1^{c_1+1}}e^{-\frac{d_1}{\beta_1}}\frac{1}{\beta_2^{c_2+1}}e^{-\frac{d_2}{\beta_2}}p_1^{e_1-1}(1-p_1)^{e_2-1}$$

The joint posterior distribution of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$ and p_1 given data $\mathbf{x}$ is:

$$g_{IP}(\Omega \mid \mathbf{x}) = \frac{p_1^{E_{21}}(1-p_1)^{E_{22}-1}\alpha_1^{F_{21}-1}\exp(-\alpha_1 F_{22})\alpha_2^{G_{21}-1}\exp(-\alpha_2 G_{22})\frac{1}{\beta_1^{H_{21}+1}}\exp\left(-\frac{H_{22}}{\beta_1}\right)\frac{1}{\beta_2^{I_{21}+1}}\exp(-\beta_2 I_{22})\exp(J_{21})\exp(J_{22})(K)^{n-r}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1 p_1^{E_{21}}(1-p_1)^{E_{22}-1}\alpha_1^{F_{21}-1}\exp(-\alpha_1 F_{22})\alpha_2^{G_{21}-1}\exp(-\alpha_2 G_{22})\frac{1}{\beta_1^{H_{21}+1}}\exp\left(-\frac{H_{22}}{\beta_1}\right)\frac{1}{\beta_2^{I_{21}+1}}\exp(-\beta_2 I_{22})\exp(J_{21})\exp(J_{22})(K)^{n-r}\,dp_1 d\lambda_1 d\lambda_2 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2} \tag{7}$$

where $E_{21}=e_1+r_1, E_{22}=e_2+r_2,\ F_{21}=a_1+r_1,\ F_{22}=b_1+\sum_{j=1}^{r_1}\log\left(\frac{1}{x_{1j}}\right)$,

$G_{21}=a_2+r_2,\ G_{22}=b_2+\sum_{j=1}^{r_2}\log\left(\frac{1}{x_{2j}}\right),\ H_{21}=c_1+r_1,\ H_{21}=d_1+\sum_{j=1}^{r_1}x_{1j}^{\alpha_2}$,

$I_{21}=c_2+r_2,\ I_{22}=d_2+\sum_{j=1}^{r_1}x_{2j}^{\alpha_2},\ J_{21}=\sum_{j=i}^{r_1}\log\left\{1-\lambda_1+2\lambda_1\exp\left(-\frac{x_{1j}^{\alpha_1}}{\beta_1}\right)\right\}$,

$J_{22}=\exp\left[\sum_{j=i}^{r_2}\log\left\{1-\lambda_2+2\lambda_2\exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right\}\right]$,

$K=1-p_1\exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\left[1-\lambda_1+2\lambda_1\exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\right]-(1-p_1)\exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\left[1-\lambda_2+\lambda_2\exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\right]$.

The marginal posterior densities can be obtained by integrating out the nuisance parameters, for example, to derive the marginal density of the parameter α_1, we proceed as follows:

$g_2(\alpha_1|\mathbf{x})=\int_0^\infty\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1\int_0^1 g_2(\Omega|\mathbf{x})dp_1 d\lambda_1 d\lambda_2 d\beta_1 d\beta_2 d\alpha_2$ and vice versa.

3. Bayes Estimators and Posterior Risks under Different Loss Functions

In this section, we focus on the derivation of the Bayes estimators and posterior risks under different loss functions. As the choice of a suitable loss function, i.e., symmetric or asymmetric, depends on the problem at hand, there is no rule to select an appropriate loss function [30]. To estimate the unknown parameter in Bayesian, one must specify a loss function. A loss function $L(\theta,d)\geq 0$ is a function which enables us to estimate the unknown parameter by an estimator $\hat{d}$. The Bayes estimator is a posterior estimator which minimizing the posterior risk.

The worth of a decision is measured by the expected loss, which is known as the posterior risk. If $\hat{d}$ is a Bayes estimator, then $\rho(\hat{d})$ is called the posterior risk [31], defined as

$$\rho(\hat{d}) = E_{\theta|\mathbf{x}}\left\{L(\theta,\hat{d})\right\} = \int L(\theta,\hat{d})\,p(\theta\,|\,\mathbf{x})\,d\theta \tag{8}$$

One of the aims of this study is to suggest a suitable loss function for the parameter estimation of the transmuted Weibull distribution. To this end, three different loss functions, namely the squared error loss function (SELF), precautionary loss function (PLF) and quadratic loss function (QLF) are used in this study. A loss function that yields the minimum posterior risk will be selected as the appropriate loss function. A brief discussion of these loss functions is given below.

The first loss function is the square error loss function which is a symmetric loss function proposed by Legendre [32] and used by Gauss [33] to develop the least square theory. Further, it is also used in estimation problems. In fact, it is a quadratic deviation from the true parameter value. Mathematically, it is defined as

$$L(\beta,d) = (\beta-d)^2 \tag{9}$$

where d is a decision that a statistician has to take in order to approximate an unknown β, the so called parameter, is often used to summarize a posterior distribution.
The Bayes estimator and the posterior risk under SELF are:

$$\hat{\mathrm{d}} = \mathrm{E}_{\beta|\mathrm{x}}(\beta) \;\&\; \rho(\hat{d}) = \mathrm{E}(\beta^2|\mathbf{x}) - \left\{\mathrm{E}(\beta|\mathbf{x})\right\}^2 = \mathrm{Var}(\beta|\mathbf{x})$$

The precautionary loss function (PLF) suggested by Norstrom [34] is a asymmetric loss function and defined as:

$$L(\beta,d) = \frac{(\beta-d)^2}{d}$$

The Bayes estimator and the posterior risk under PLF are:

$$\hat{\mathrm{d}} = \left\{\mathrm{E}(\beta^2|\mathrm{x}\right\}^{\frac{1}{2}} \;\&\; \rho(\hat{d}) = 2\left\{\mathrm{E}(\beta^2|\mathrm{x}\right\}^{\frac{1}{2}} - 2\mathrm{E}(\beta|\mathrm{x})$$

The third loss function is the quadratic loss function (QLF), which is another asymmetric loss function and defined as:

$$L(\beta,d) = \left(\frac{\beta-d}{\beta}\right)^2.$$

The Bayes estimator and posterior risk for this loss function are:

$$\hat{\mathrm{d}} = \frac{\mathrm{E}(\beta^{-1}|\mathrm{x})}{\mathrm{E}(\beta^{-2}|\mathrm{x})} \;\&\; \rho(\hat{d}) = 1 - \frac{\left\{\mathrm{E}(\beta^{-1}|\mathrm{x})\right\}^2}{\mathrm{E}(\beta^{-2}|\mathrm{x})}$$

The details about these loss functions can be seen in Ali [35-36].

3.1 Posterior Summaries by Markov Chain Monte Carlo Techniue

In the previous section, we observed that the expressions of the posterior densities are in intractable form and cannot solve directly. Thus, we adopt a numerical technique in order to find the posterior summaries, such as the mean and the quantiles. To this end, we use a Markov Chain Monte Carlo (MCMC) technique similar to Ali [36]. In particular, we implement the Gibbs sampling [37] with Metropolis Hasting step [38-39]. To obtain the posterior summaries using the MCMC approach, the posterior densities assuming the uniform and the informative priors can be written as:

$$g_{UP}(\Omega|\mathbf{x}) \propto f_{\alpha_1}\left(r_1+1,\sum_{j=1}^{r_1}log\left(\frac{1}{x_{1j}}\right)\right)\times f_{\alpha_2}\left(r_2+1,\sum_{j=1}^{r_2}log\left(\frac{1}{x_{2j}}\right)\right)\times f_{\beta_1|\alpha_1}\left(r_1-1,\sum_{i=1}^{r_1}(x_{1i}^{\alpha_{1i}}+(n-r)T^{\alpha_{1i}}\right)\times$$
$$f_{\beta_2|\alpha_2}\left(r_2-1,\sum_{i=1}^{r_2}(x_{2i}^{\alpha_2}+(n-r)T^{\alpha_{2i}}\right)\times f_{\lambda_1}\left(exp\left[\sum_{i=1}^{r_1}log\left\{1-\lambda_1+2\lambda_1 exp\left(-\frac{x^{\alpha_1}{}_{1j}}{\beta_1}\right)\right\}\right]\right)\times$$
$$f_{\lambda_2}\left(exp\left[\sum_{i=1i}^{r_2}log\left\{1-\lambda_2+2\lambda_2 exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right\}\right]\right)$$

$$g_{IP}(\Omega|x) \propto f_{\alpha_1}\left(a_1+r_1+1,b_1+\sum_{j=1}^{r_1}log\left(\frac{1}{x_{1j}}\right)\right)\times f_{\alpha_2}\left(a_2+r_2+1,b_2+\sum_{j=1}^{r_2}log\left(\frac{1}{x_{2j}}\right)\right)\times$$
$$f_{\beta_1|\alpha_1}\left(c_1+r_1,d_1+\sum_{i=1}^{r_1}(x_{1i}^{\alpha_1}+(n-r)T^{\alpha_1}\right)\times f_{\beta_2|\alpha_2}\left(c_2+r_2 d_2+\sum_{i=1}^{r_2}(x_{2i}^{\alpha_2}+(n-r)T^{\alpha_{2i}}\right)\times$$
$$f_{\lambda_1}\left(exp\left[\sum_{j=i}^{r_1}log\left\{1-\lambda_1+2\lambda_1 exp\left(-\frac{x_{1i}^{\alpha_1}}{\beta_1}\right)\right\}\right]\right)\times f_{\lambda_2}\left(exp\left[\sum_{j=i}^{r_2}log\left\{1-\lambda_2+2\lambda_2 exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right\}\right]\right)$$

where f_{α_1} and $f_{\beta_1|\alpha_1}$ denote the probability density functions of gamma and inverse gamma distributions while $f_{\lambda 1}$ is the probability density functions of λ for the first component. Similarly f_{α_2} , $f_{\beta_2|\alpha_2}$ and $f_{\lambda 2}$ represent the probability density function of a gamma, inverse gamma and λ for the second component. In order to obtain the Bayes estimates and their respective posterior risks, we proceed as follows:

Generate random numbers from the transmutedWeibull distribution by using the inverse integral transformation, i.e., $u_i=\left(1-e^{-\frac{x^{\alpha_i}}{\beta_i}}\right)\left(1-\lambda_i+\lambda_i e^{-\frac{x^{\alpha_i}}{\beta_i}}\right)$, where i=1, 2 and $u_i \sim$ uniform (0, 1). After simplification, we obtain $x_i=\left\{-\beta_i \ln\left(\frac{1+\lambda_i-\sqrt{(1+\lambda_i)^2-4\lambda_i u_i}}{2\lambda_i}\right)\right\}^{\frac{1}{\alpha_i}}$. Thus, one can obtain the desired random sample by providing the required parameters. To generate the right censored data, one needs to fix T and record units that are less than equal to the censoring time. The number of units that are greater than censoring time would be considered censored observations. To generate data from the mixture model, fix p and generate uniform (0, 1). If the generated uniform random number is smaller than p, generate X_1 from $TWD(\alpha_1, \beta_1, \lambda_1)$ otherwise from $TWD(\alpha_2, \beta_2, \lambda_2)$.

Next, to implement the MCMC for finding the posterior summaries, we propose the following steps:

Algorithm 1.

At the i^{th} step, repeat the following steps:

1. Generate $\alpha_{1i} \sim f_{\alpha_1} = Gamma\left(r_1 + 1, \sum_{j=1}^{r_1} log\left(\frac{1}{x_{1j}} \right) \right)$,

   $$\beta_{1i} \sim f_{\beta_1|\alpha_{1(i-1)}} = Inverse-Gamma\left(r_1 - 1, \sum_{i=1}^{r_1} (x_{1i}^{\alpha_{1i}} + (n-r)T^{\alpha_{1i}} \right),$$

   $$\alpha_{2i} \sim f_{\alpha_2} = Gamma\left(r_2 + 1, \sum_{j=1}^{r_2} log\left(\frac{1}{x_{2j}} \right) \right) \text{ and}$$

   $$\beta_{2i} \sim f_{\beta_2|\alpha_{1(i-1)}} = Inverse-Gamma(r_2 - 1, \sum_{i=1}^{r_2} (x_{2i}^{\alpha_2} + (n-r)T^{\alpha_{2i}}).$$
2. Take $\mathrm{Y}_{1i} \sim Uniform(0,1)$ and $\mathrm{Y}_{2\mathrm{i}} \sim Uniform(0,1)$.
3. Generate $U \sim Uniform(0,1)$ and, λ_{1i}^* and λ_{2i}^* from Uniform (-1, 1).
4. Let $f_1^{(i)} = \exp\left[\sum_{i=1}^{r_1} \left\{ \log\left(1 - \lambda_1^{*(i-1)} + 2\lambda_1^{*(i-1)} \exp\left(-x_1^{\alpha_1^{(i)}} / \beta_1^{(i)}\right)\right)\right\} \right]$

   $$f_2^{(i)} = \exp\left[\sum_{i=1}^{r_1} \left\{ \log\left(1 - Y_1^{(i)} + 2Y_1^{(i)} \exp\left(-x_1^{\alpha_1^{(i)}} / \beta_1^{(i)}\right)\right)\right\} \right]$$

   $$f_3^{(i)} = \exp\left[\sum_{i=1}^{r_2} \left\{ \log\left(1 - \lambda_2^{*(i-1)} + 2\lambda_2^{*(i-1)} \exp\left(-x_2^{\alpha_2^{(i)}} / \right)\beta_2^{(i)} \right)\right\} \right]$$

   $$f_4^{(i)} = \exp\left[\sum_{i=1}^{r_2} \left\{ \log\left(1 + Y_2^{(i)} - 2Y_2^{(i)} \exp\left(-x_2^{\alpha_2^{(i)}} / \beta_2^{(i)}\right)\right)\right\} \right]$$

   $$\rho_1^{(i)} = \min\left[1, \frac{f_1^{(i)} \left\{ dunif\left(\lambda_1^{*(i-1)}, 0,1\right)\right\}}{f_2^{(i)} \left\{ dunif\left(\lambda_1^{*(i-1)}, 0,1\right)\right\}} \right] \text{and } \rho_2^{(i)} = \min\left[1, \frac{f_3^{(i)} \left\{ dunif\left(\lambda_2^{*(i-1)}, 0,1\right)\right\}}{f_4^{(i)} \left\{ dunif\left(\lambda_2^{*(i-1)}, 0,1\right)\right\}} \right]$$
5. Select,

   $$\lambda_1^{(i)} = \lambda_1^{*(i)} \begin{cases} \lambda_1^{*(\mathrm{i})} = \lambda_1^{*(i-1)} & \text{if } \rho_1^{(i)} > U^{(i)} \\ \lambda_1^{*(\mathrm{i})} = \mathrm{Y}_1^{*(i-1)} & \text{otherwise} \end{cases}$$

$$\lambda_2^{(i)}=\lambda_2^{*(i)}\begin{cases}\lambda_2^{*(i)}=\lambda_2^{*(i-1)} & \text{if } \rho_2^{(i)}>U^{(i)}\\ \lambda_2^{*(i)}=Y_2^{*(i-1)} & \text{otherwise}\end{cases}$$

6. Repeat Steps 1-6, N-times to obtain $(\alpha_1,\beta_1,\lambda_1),(\alpha_2,\beta_2,\lambda_2),...(\alpha_N,\beta_N,\lambda_N)$ and discard first *M* observations as the burn-in period.
7. The approximate values of $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1$, $\hat{\beta}_2,\hat{\lambda}_1$ and $\hat{\lambda}_2$ are:

$$\hat{\alpha}_1=\frac{\sum_{i=M+1}^{N}\alpha_{1i}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})},\quad \hat{\beta}_1=\frac{\sum_{i=M+1}^{N}\beta_{1i}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}$$

$$\hat{\lambda}_1=\frac{\sum_{i=M+1}^{N}\lambda_1^{(i)}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})},\hat{\alpha}_2=\frac{\sum_{i=M+1}^{N}\alpha_{2i}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}$$

$$\hat{\beta}_2=\frac{\sum_{i=M+1}^{N}\beta_{2i}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})},\hat{\lambda}_2=\frac{\sum_{i=M+1}^{N}\lambda_2^{(i)}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}{\sum_{i=M+1}^{N}k(\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i},\lambda_{1i},\lambda_{2i},p_{1i})}$$

In the next section, we evaluate the performance of Bayes estimators on the basis of posterior risk for the informative and the noninformative priors under different loss functions.

4. A Simulation Study and Some Comparisons

In this section, a comprehensive simulation study has been carried out to assess the performance of the Bayes estimators assuming different loss functions, sample sizes, mixing weights, parameter values, and censoring rates using non-informative and informative priors in terms of posterior risks. Samples of sizes *n= 20, 40, 60* and *100* have been generated by the inverse transformation methods from the two-component mixture of the transmuted Weibull distribution with parameters $\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1$ and λ_2 such that $(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2)\in$ (1.5, 2, 2, 1.4, 0.4, 0.5), (2.5, 3, 1.6, 1.3, 0.3, 0.4), (2, 1.5, 1.4, 1.3, 0.4, 0.6) and probabilistic mixing weights $p_1\in\{(0.3, 0.5, 0.7)\}$. The $p_1 n$ observations were generated randomly from the first component density $f_1(x_1;\alpha_1,\beta_1,\lambda_1)$ and the remaining $(1-p_1)n$ were generated from the second component density $f_2(x_2;\alpha_2,\beta_2,\lambda_2)$, respectively. For a fixed sample size, test termination time, we used the aforementioned algorithm, and the Bayes estimates (BEs) and posterior risks (PRs) were computed using the UP and the IP under the SELF, PLF and QLF. It is worth mentioning that right censoring was considered, all the observations which were greater than a fixed censoring time *T* were considered as the censored ones. In each case, only failure items can be identified to be a member of either subpopulation-I or subpopulation-II of the mixture. In this study, we considered different censoring rates, for example, 20% and 40%, to evaluate their impact on the Bayes estimators. For each of the combinations of parameters, different sample sizes and censoring rates, the steps of the algorithm are repeated N=10,000 times using R software, R[40]. The simulated results are then averaged by considering M=2000 as the burn-in period. The simulated Bayes estimates and the posterior risks using the UP and IP under SELF, PLF and QLF are tabulated in Tables 1 & A1 –A2 (Tables A1 – A2 given in Appendix A), where the posterior risks have been presented in the parentheses below the Bayes estimates.

The simulation study indicates that the estimated values of each parameter converge to the true value by increasing the sample size and magnitude of corresponding PRs decreases at a fixed test termination time. This pattern is not restricted to any specific loss function or prior but observed for all the considered loss functions. The simulation study gives us some interesting features of the Bayes estimates. By comparing the results of 20% and 40% censored rates, one can easily observe that the PRs for 20% censoring are smaller than 40% censoring as shown in Tables 1 & A1-A2. The reason for this is the availability of more information for 20% censored data than 40% censored data; and due to this PRs of 20% are smaller than the 40%. Furthermore, we also observed a direct effect of censoring rate on the posterior risk, i.e., the posterior risk increased as the censoring rate increased. From Tables 1 & A1-A2, it is noticed that the IP is more accurate than the UP. Also, we observed that the convergence of Bayes estimates to the nominal value is faster in the case of IP than the UP for the all assumed loss functions.

Next, we compare the performance of different loss functions. From Table 1, it can be observed that the Bayes estimates are found to be more efficient for the first set of parameter values under the IP as compared to the UP under SELF than PLF and QLF. In the case of 20% censoring rate and assuming IP, the transmuted parameters (λ_1, λ_2) of both components are over estimated under the assumed loss functions. It is clear from Table A1 that if $\alpha_1 < \alpha_2$, the estimated values of the shape and of the transmuted parameters are over-estimated under the considered loss functions for 20% and 40% censoring rates. It is also noticed that IP gives us more precise estimates using the SELF for the second set of parametric values by considering 20% censoring rate than the PLF and the QLF. From Table A2, it is clear that IP produced more accurate estimates for all parameters under QLF when 40% censoring was considered. Also, for the third set of parameters, if $\alpha_1 > \alpha_2$ the estimated values of the shape and the scale parameters are overestimated for the first component while under-estimated for the second component under SELF. Among all set of parametric values, the QLF is the most suitable for both 20% and 40% censoring rates (cf. Tables 1 & A1-A2). In the case of 20% censoring rate, to estimate the mixing component parameter p_1, the SELF is observed more superior than the other two loss functions.

5. Bayesian Credible Intervals

The goal of this section is to obtain the Bayesian credible intervals for the unknown parameters. The Bayesian credible intervals are obtained by utilizing the marginal posterior densities of the parameter of interest. A 95% Bayesian credible interval (sometimes also known as the posterior interval) is analogous to the familiar 95% frequentist confidence interval which provides a range of likely values for a parameter [41].

Given the data, the 95% credibility interval (μ_L, μ_U) includes the true μ with probability 95%, whereas the frequentist intervals may include it or not, i.e., its inclusion probability is either zero or one. According to Eberly and Casella [42] the 100 (1-k) % credible intervals can be obtained by solving the following two equations is:

$$\int_0^L g(\alpha \mid \mathbf{x}) d\alpha = \frac{k}{2} = \int_U^\infty g(\alpha \mid \mathbf{x}) d\alpha$$

where $g(\alpha \mid \mathbf{x})$ is the posterior distribution of α given data, *L* and *U* denote the lower and upper limits of the credible interval, respectively, and *k* is the significance level.

In the case of interval estimation (Tables 2 and A3-A4) it is observed that the width of 95% credible intervals reduces as the sample size increases. Also, all the credible intervals contain the nominal value of the respective unknown parameters. The least widths for Bayesian credible intervals has been observed for 20% censoring rate than the 40% and the reason is that we have more information in the case of a small censoring rate than the large one.

6. Real Life Application

In this section, we analyzed two real-life data sets in order to illustrate the practicality of the mixture distribution in practice. For the sake of space, the results of the first data set are presented here, whereas the second data set is discussed in Appendix A.

6.1 Real Data Set-I: Hepatitis Analysis

We analyzed a real data set to illustrate the methodology discussed in the previous section. The data set has been collected from the Combined Military Hospital (CMH) Rawalpindi, Pakistan. In order to show the usefulness of the proposed mixture distribution, we performed Bayesian analysis on the life time data of hepatitis B & C patients to estimate the unknown parameters assuming UP and IP under SELF, PLF and QLF. This data set consists of 80 values regarding recovery time period (in days) of hepatitis (B & C) patients treated with antiviral medications such as lamivudine for hepatitis B or ribavirin (trade name Copegus, Rebetol, Ribasphere) for hepatitis C. For the sake of privacy, we report admitted date, discharge date and number of recovery days from 2016 to 2017. Since the patients have two different types of hepatitis, the data can easily be divided into two subpopulations. In Table 3, the hepatitis B patients have been classified into subpopulation-1 while the hepatitis C patients into subpopulation 2. The appropriateness of the two-component mixture of the transmuted Weibull distribution is tested against the Weibull and transmuted Weibull by using χ^2- statistic and the P-value is 0.4559. Thus, it is evident that the mixture distribution is a good fit against the other assumed models.

For the analysis, we categorized the data (Table 3) into two groups with probability mixing weight $p_1 = 0.5$ and T= 30. The reason of considering T= 30 is if a patient does not recover within one month then he is referred to the CMH Lahore. The summary of the data is as follows:

$$n_1=40,\ r_1=40,\ r_2=8,\ p_1=0.5\ ,\sum_{j=1}^{r_1} x_{1j}=680\ ,\sum_{j=1}^{r_1}\log(1/x_{1j})=-112.88,$$

$$n_2=40,\ r_2=19,,\sum_{j=1}^{r_2} x_{2j}=523,\sum_{j=1}^{r_2}\log(1/x_{2j})=-62.96..$$

The Bayes estimates (BEs), the PRs (in parenthesis) and interval estimates for the parameters of the subject distribution are obtained assuming UP and IP under SELF, PLF and QLF.

Tables 4-5 contain the Bayesian estimation of parameters of the mixture of the transmuted Weibull distribution. The examination of the results confirms our previous conclusion that the BEs and credible intervals based on IP under SELF provides comparatively better results than the UP for estimating component parameters. Further, it is noticed that the estimates have the minimum amount of PRs for 20% censoring rate than the 40% for the both priors and under the assumed loss functions.

7. Conclusion

In this article, we introduced two-component mixture of the transmuted Weibull distribution and estimated its parameters using the Bayesian method. An extensive simulation study is conducted to compare and highlight some important and interesting properties of the Bayes estimators of the two-component mixture of the transmuted Weibull distribution using the noninformative (uniform) and informative (gamma) priors under SELF, PLF and QLF. First, we derived the expression for posterior densities and noticed that the densities are not in closed forms. Thus, we proposed an MCMC technique to obtain the posterior summaries. The second objective of this study was the choice of an appropriate loss function and prior for the estimation of mixture's parameters assuming type-I right censored data at different sample sizes and test termination times. To this end, we obtained different posterior summaries, like Bayes estimates and their respective posterior risk, and Bayesian credible intervals assuming different sample sizes and test termination times. Two different censoring rates, i.e., 20% and 40%, were considered. From Tables 1 & A1-A2, it is clear that the estimated values become very close to the nominal value of the parameters and the PRs decreased with the increase of sample size. Thus, our simulated results follow the consistency property. It is also observed that the 40% censoring rate have large amount of PRs than the 20% censored data. Furthermore, the results obtained from the real-life data sets showed the same pattern which confirmed that the proposed MCMC algorithm performed well to estimate the unknown parameters in the Bayesian framework. The PLF was observed the most preferable choice for the estimation of mixing component (proportion) than the SELF and the QLF. Also, the IP is more efficient prior for estimating the shape and the scale parameters. To show the application of the proposed model, two real life data sets have been analyzed and its appropriateness was tested through χ^2-statistic. In future, truncated mixture can be studied.

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Appendix A

For $f(x;\Omega) = p_1 f_1(x;\Omega_1) + p_2 f_2(x;\Omega_2)$ & $p_1 + p_2 \leq 1$, some algebraic manipulations yields the following likelihood form.

$$L(\mathbf{x};\Omega) \propto p_1^{r_1+1-1}(1-p_1)^{r_2+1-1}\alpha_1^{r_1+1-1} exp\left\{-\alpha_1\sum_{j=1}^{r_1} log\left(\frac{1}{x_{1j}}\right)\right\}\alpha_2^{r_2+1-1} exp\left\{-\alpha_2\sum_{j=1}^{r_2} log\left(\frac{1}{x_{2j}}\right)\right\}\frac{1}{(\beta_1)^{r_1-1+1}} exp\left(-\frac{\sum_{j=1}^{r_1} x_{1j}^{\alpha_2}}{\beta_1}\right)\frac{1}{(\beta_2)^{r_2-1+1}}$$

$$exp\left(-\frac{\sum_{j=1}^{r_1} x_{2j}^{\alpha_2}}{\beta_2}\right) exp\left[\sum_{j=i}^{r_1} log\left\{1-\lambda_1+2\lambda_1 exp\left(-\frac{x_{1j}^{\alpha_1}}{\beta_1}\right)\right\}\right] exp\left[\sum_{j=i}^{r_2} log\left\{1-\lambda_2+2\lambda_2 exp\left(-\frac{x_{2j}^{\alpha_2}}{\beta_2}\right)\right\}\right]\left[1-p_1 exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\right.$$

$$\left.\left\{1-\lambda_1+2\lambda_1 exp\left(-\frac{T^{\alpha_1}}{\beta_1}\right)\right\}-(1-p_1) exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\left\{1-\lambda_{2-}+\lambda_2 exp\left(-\frac{T^{\alpha_2}}{\beta_2}\right)\right\}\right]^{n-r}$$

Figure A1 shows graphs of the marginal posterior densities of parameters ($\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, p_1$) assuming the UP and the 1P for the hepatitis data. From Figures A1a to A1b & A1e to A1f, we noticed that the plots of marginal posterior densities for the scale and the shape component for the scale and the shape parameters (α_2, β_2) tend to be more peaked for 20% censoring than the 40%. This is because we lost more information in the case of a high censoring rate (Figures A1c to A1d & A1g to A1h). Also, the graphs of marginal posterior densities for transmuted parameters (λ_1, λ_2) are also symmetrical as shown in Figures A1i to A1l and the same is true for the posterior density of the mixing weight (Figures A1m & A1n).

Real Data Set-II

The second data set consist of 50 observations (in the unit of millimeter), the hole diameter is 12mm and the sheet thickness is 3.15mm reported by Dasgupta [43]. The data are: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16. Dasgupta analyzed the Burr distribution by using the same data set and compared the process stability by bootstrapping the distribution of coefficient of variation under extreme value model. He concludes that an extreme value distribution may explain well the data. Recently, Bakouch et al. [44] used this data set to evaluate the performance of the transmuted general (GT) family of distributions by fitting four sub-models namely transmuted Burr (TB) distribution, transmuted Weibull (TW) distribution, transmuted Gamma (TGa) distribution, and transmuted Gompterz (TGz) distribution. In order to perform the Bayesian analysis assuming the two-component mixture of TWD, we randomly assembled the data into two subpopulations using probabilistic mixing weight p_1=0.5 and T=0.24. The summary of our results are as follows:

$$n_1=25,\ r_1=18,\ p_1 = 0.5,\ \sum_{j=1}^{r_1} x_{1j} = 2.22,\ \sum_{j=1}^{r_1} \log x_{1j} = -40.51,$$

$$n_2=25,\ r_2=19,\ \sum_{j=1}^{r_2} x_{2j} = 2.48,\ \sum_{j=1}^{r_2} \log x_{2j} = -41.53.$$

The goodness of fit for the distribution is tested by using χ^2- statistic against the Weibull and transmuted Weibull distributions and the obtained P-value is 0.2290 at the 5% level of significance. Thus, it is safe to conclude that the proposed distribution provides the best fit for the data set. We applied the two-component mixture of TWD to the data set given in Table A5 by adopting the methodology discussed previously. More specifically, we performed the Bayesian analysis to estimate the unknown parameters assuming UP and IP under SELF, PLF and QLF. The results of the BEs, the PRs (in parenthesis) and interval estimates for the parameters have been summarized in Tables intervals (Tables A6-A7). The BEs and the credible intervals (Tables A6-A7) based on IP are more efficient than the UP prior. For both priors, the SELF provided more efficient estimates for transmuted parameters (λ_1, λ_2) and proportion parameter (p_1).

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Table 1: BEs of Mixture for Two -Component of TWD along with PRs (in parentheses) under UP and IP

Table 2: 95% Bayesian Credible Intervals of Mixture for Two-Component of TWD using UP and IP with hyperparameters are a_1=0.5, a_2=1.5, b_1=1, b_2=1, c_1=0.5, c_2=1, d_1=1, d_2=2, e_1=0.5, e_2=1, l_1=0.1 and l_2=0.2

Table 3: Real life data of the Survival Time (in days) of Hepatitis (B & C) Patients

Table 4: B Es of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

Table 5: 95%Bayesian Credible Intervals of Mixture of the Two-Component of TWD using UP and IP

Table A1: BEs of Mixture for Two -Component of TWD along with PRs (in parentheses) under UP and IP

Table A2: BEs of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

Table A3: 95%Bayesian Credible Intervals of Mixture for Two-Component of TWD of using UP and IP with hyperparameters are a_1=1, a_2=0.5, b_1=2, b_2=1, c_1=0.5, c_2=1.5, d_1=1, d_2=2, e_1=1, e_2=1, l_1=0.1 and l_2=0.2

Table A4: 95% Bayesian Credible Intervals of Mixture for Two -Component of TWD using UP and IP with hyperparameters are a_1=0.5, a_2=1, b_1=1, b_2=2, c_1=1.5, c_2=0.5, d_1=2, d_2=1, e_1=0.5, e_2=1, l_1=0.1 and l_2=0.2

Table A5: Measurements of Hole and Sheet Thickness (12 mm and 3.15 mm)

Table A6: BEs of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

Table A7: 95%Bayesian Credible Intervals of Mixture of the Two-Component of TWD using UP and IP

Figure A1: Graphical depiction of posterior desnities assuming different priors and censoring rates.

Table 1: BEs of Mixture for Two -Component of TWD along with PRs (in parentheses) under UP and IP

LF		20 % Censoring													
		UP							IP						
	n	$\alpha_1=1.5$	$\beta_1=2.0$	$\lambda_1=0.4$	$p_1=0.3$	$\alpha_2=2.0$	$\beta_2=1.4$	$\lambda_2=0.5$	$\alpha_1=1.5$	$\beta_1=2.0$	$\lambda_1=0.4$	$p_1=0.3$	$\alpha_2=2.0$	$\beta_2=1.4$	$\lambda_2=0.5$
SELF	20	1.6473 (0.3747)	1.8156 (0.4174)	0.2861 (0.0410)	0.3884 (0.0122)	1.8353 (0.2400)	1.6098 (0.0145)	0.4798 (0.0530)	1.6192 (0.2045)	1.6201 (0.1376)	0.3413 (0.0518)	0.3339 (0.0107)	2.3074 (0.2883)	1.6918 (0.0214)	0.4792 (0.0078)
	40	1.6435 (0.2239)	1.9977 (0.2278)	0.3601 (0.0323)	0.3527 (0.0064)	1.9369 (0.1750)	1.5917 (0.0097)	0.4893 (0.0444)	1.5749 (0.1942)	1.8646 (0.1156)	0.3520 (0.0204)	0.3282 (0.0065)	2.2123 (0.2010)	1.5321 (0.0078)	0.4860 (0.0061)
	60	1.5239 (0.1232)	2.0064 (0.0726)	0.3897 (0.0308)	0.3220 (0.0043)	2.0087 (0.1174)	1.4557 (0.0049)	0.4979 (0.0416)	1.5238 (0.1201)	1.9773 (0.0834)	0.3878 (0.0107)	0.3205 (0.0041)	2.1283 (0.1232)	1.4452 (0.0045)	0.4890 (0.0047)
	100	1.5016 (0.0781)	2.0549 (0.0201)	0.4183 (0.0236)	0.3011 (0.0028)	2.0605 (0.0787)	1.4260 (0.0030)	0.5144 (0.0342)	1.5069 (0.0903)	1.9913 (0.0756)	0.4000 (0.0095)	0.3016 (0.0027)	2.0569 (0.1130)	1.4000 (0.0027)	0.5230 (0.0015)
PLF	20	1.7573 (0.2201)	1.9736 (0.2160)	0.2879 (0.0936)	0.4038 (0.0308)	1.7071 (0.1436)	1.7604 (0.1011)	0.5131 (0.0666)	1.7396 (0.1409)	1.6607 (0.1812)	0.3439 (0.0053)	0.3496 (0.0313)	2.3901 (0.1653)	1.6037 (0.0239)	0.4772 (0.0161)
	40	1.7103 (0.1335)	1.9881 (0.1140)	0.3625 (0.0849)	0.3616 (0.0179)	1.9815 (0.1393)	1.5979 (0.0123)	0.5054 (0.0621)	1.6354 (0.1210)	1.8798 (0.1030)	0.3542 (0.0042)	0.3612 (0.0181)	2.2572 (0.0893)	1.5368 (0.0093)	0.4823 (0.0127)
	60	1.5702 (0.0926)	2.0132 (0.0669)	0.3912 (0.0830)	0.3286 (0.0134)	2.0377 (0.0580)	1.4883 (0.0062)	0.5052 (0.0545)	1.5654 (0.0831)	1.9553 (0.0761)	0.3891 (0.0037)	0.3270 (0.0129)	2.1570 (0.0575)	1.4581 (0.0057)	0.4900 (0.0118)
	100	1.5274 (0.0516)	2.0256 (0.0209)	0.4050 (0.0535)	0.3049 (0.0076)	2.0795 (0.0380)	1.4279 (0.0038)	0.5060 (0.0276)	1.5443 (0.0660)	1.9850 (0.0513)	0.4000 (0.0003)	0.3056 (0.0080)	2.0944 (0.0350)	1.4017 (0.0034)	0.5245 (0.0029)
QLF	20	1.5756 (0.1625)	1.7411 (0.0650)	0.4757 (0.0686)	0.3128 (0.1107)	2.1202 (0.0975)	1.8968 (0.0210)	0.5093 (0.0948)	1.6255 (0.1626)	1.8763 (0.0789)	0.3329 (0.0152)	0.3607 (0.0968)	2.4133 (0.0858)	1.4906 (0.0250)	0.5911 (0.0516)
	40	1.5551 (0.0879)	1.9277 (0.0408)	0.4727 (0.0647)	0.3112 (0.0605)	2.1694 (0.0486)	1.6863 (0.0099)	0.5077 (0.0916)	1.5510 (0.0806)	1.9490 (0.0345)	0.3571 (0.0059)	0.3437 (0.0516)	2.2156 (0.0446)	1.4672 (0.0099)	0.5646 0.0491)
	60	1.5333 (0.0630)	1.9893 (0.0307)	0.4648 (0.0622)	0.3110 (0.0452)	2.0915 (0.0304)	1.4795 (0.0069)	0.5069 (0.0846)	1.5062 (0.0560)	1.9664 (0.0255)	0.3881 (0.0034)	0.3310 (0.0395)	2.1353 (0.0310)	1.4244 (0.0066)	0.5460 (0.0466)
	100	1.5542 (0.0360)	1.9629 (0.0155)	0.4321 (0.0242)	0.3391 (0.0236)	2.0757 (0.0184)	1.3936 (0.0040)	0.5604 (0.0363)	1.4862 (0.0390)	2.0375 (0.0213)	0.4076 (0.0021)	0.3119 (0.0256)	2.0933 (0.0274)	1.4049 (0.0038)	0.5110 (0.0382)
LF		40 % Censoring													
		UP							IP						

	n	$\alpha_1 = 1.5$	$\beta_1 = 2.0$	$\lambda_1 = 0.4$	$p_1 = 0.3$	$\alpha_2 = 2.0$	$\beta_2 = 1.4$	$\lambda_2 = 0.5$	$\alpha_1 = 1.5$	$\beta_1 = 2.0$	$\lambda_1 = 0.4$	$p_1 = 0.3$	$\alpha_2 = 2.0$	$\beta_2 = 1.4$	$\lambda_2 = 0.5$
SELF	20	1.4002 (0.3757)	1.6544 (0.4189)	0.4554 (0.0527)	0.3514 (0.0165)	2.1488 (0.2722)	1.3244 (0.0923)	0.5371 (0.0576)	1.2608 (0.2052)	2.2982 (0.1584)	0.3121 (0.0652)	0.4839 (0.0151)	1.5685 (0.2947)	1.2502 (0.0495)	0.4263 (0.0137)
	40	1.4778 (0.2384)	1.7801 (0.2854)	0.4378 (0.0458)	0.3434 (0.0084)	2.1245 (0.2019)	1.3602 (0.0634)	0.5225 (0.0569)	1.3159 (0.1949)	1.6476 (0.1248)	0.3495 (0.0212)	0.4194 (0.0087)	1.9599 (0.2252)	1.3401 (0.0356)	0.4417 (0.0107)
	60	1.4877 (0.1429)	1.8890 (0.1056)	0.4350 (0.0360)	0.3386 (0.0057)	2.0539 (0.1229)	1.3847 (0.0263)	0.5179 (0.0544)	1.5022 (0.1392)	1.7568 (0.1071)	0.3845 (0.0123)	0.3186 (0.0050)	2.0118 (0.1426)	1.3831 (0.0102)	0.4801 (0.0076)
	100	1.5120 (0.0873)	1.9624 (0.0979)	0.4215 (0.0245)	0.3115 (0.0038)	2.0466 (0.0933)	1.4042 (0.0104)	0.5092 (0.0428)	1.5098 (0.0974)	2.0073 (0.0960)	0.3996 (0.0108)	0.3012 (0.0032)	2.0602 (0.1292)	1.4227 (0.0098)	0.4927 (0.0053)
PLF	20	1.5955 (0.2404)	1.8577 (0.3787)	0.5002 (0.0595)	0.3776 (0.0325)	2.1160 (0.2776)	1.4592 (0.1249)	0.5584 (0.1025)	1.3397 (0.1578)	1.8077 (0.2189)	0.3123 (0.0098)	0.4993 (0.0338)	1.6568 (0.1766)	1.5207 (0.0410)	0.4267 (0.0286)
	40	1.5664 (0.1571)	1.9850 (0.3558)	0.4664 (0.0978)	0.3554 (0.0341)	2.1014 (0.1827)	1.4321 (0.0637)	0.5549 (0.0648)	1.3698 (0.1379)	1.9848 (0.1744)	0.3497 (0.0075)	0.4296 (0.0205)	2.0166 (0.1133)	1.4771 (0.0349)	0.4418 (0.0225)
	60	1.5448 (0.1141)	2.0347 (0.0912)	0.4439 (0.0860)	0.3470 (0.0268)	2.0957 (0.0834)	1.4290 (0.0286)	0.5301 (0.0644)	1.5067 (0.0917)	1.9957 (0.0568)	0.3805 (0.0061)	0.3263 (0.0193)	2.0469 (0.0702)	1.4486 (0.0110)	0.4811 (0.0214)
	100	1.5003 (0.0766)	2.0854 (0.0460)	0.4285 (0.0840)	0.3156 (0.0190)	2.0693 (0.0454)	1.4010 (0.0115)	0.5104 (0.0424)	1.5320 (0.0843)	2.0017 (0.0453)	0.4065 (0.0045)	0.3040 (0.0155)	2.0770 (0.0335)	1.4244 (0.0073)	0.5027 (0.0185)
QLF	20	1.3838 (0.1998)	2.1261 (0.1244)	0.4754 (0.0806)	0.3295 (0.1416)	1.7859 (0.1477)	1.6830 (0.0736)	0.5106 (0.0985)	1.2541 (0.1778)	1.6349 (0.1201)	0.3648 (0.0155)	0.3404 (0.1098)	1.6409 (0.1258)	1.5981 (0.0702)	0.5510 (0.0576)
	40	1.4899 (0.1128)	2.1082 (0.0851)	0.4654 (0.0691)	0.3157 (0.0718)	1.9479 (0.0679)	1.5769 (0.0301)	0.5017 (0.0960)	1.3518 (0.0942)	1.8474 (0.0735)	0.3829 (0.0080)	0.3336 (0.0670)	1.9199 (0.0597)	1.4741 (0.0353)	0.5376 (0.0538)
	60	1.5064 (0.0788)	2.0813 (0.0731)	0.4616 (0.0645)	0.3043 (0.0599)	1.9745 (0.0414)	1.4599 (0.0218)	0.5005 (0.0927)	1.4299 (0.0770)	1.9676 (0.0611)	0.3954 (0.0041)	0.3326 (0.0457)	2.0044 (0.0444)	1.4471 (0.0244)	0.5288 (0.0514)
	100	1.5091 (0.0528)	2.0045 (0.0578)	0.4564 (0.0516)	0.2905 (0.0349)	2.0760 (0.0280)	1.4063 (0.0103)	0.4991 (0.0792)	1.4896 (0.0452)	1.9934 (0.0516)	0.4085 (0.0035)	0.3017 (0.0331)	2.0804 (0.0246)	1.4226 (0.0176)	0.5084 (0.0404)

Table 2: 95% Bayesian Credible Intervals of Mixture for Two-Component of TWD using UP and IP with hyperparameters are a_1=0.5, a_2=1.5, b_1=1, b_2=1, c_1=0.5, c_2=1, d_1=1, d_2=2, e_1=0.5, e_2=1, l_1=0.1 and l_2=0.2

Censoring Rate	Size	Parameters	UP		IP	
			Lower Limit	Upper Limit	Lower Limit	Upper Limit
20 %	20	$\alpha_1 = 1.5$	1.0193	10.8590	1.5076	6.8866
		$\alpha_2 = 2.0$	1.0457	9.9829	1.4628	6.0349
		$\beta_1 = 2.0$	0.6648	6.4781	0.8483	3.0799
		$\beta_2 = 1.4$	0.6321	4.8065	0.6979	2.6335
		$\lambda_1 = 0.4$	0.3482	0.9361	0.3451	0.4481
		$\lambda_2 = 0.5$	0.4184	0.9681	0.4165	0.6683
		$p_1 = 0.3$	0.2273	3.0082	0.2031	1.8047
40 %		$\alpha_1 = 1.5$	1.0237	10.9073	1.5085	7.7015
		$\alpha_2 = 2.0$	1.0469	10.5421	1.4895	7.9162
		$\beta_1 = 2.0$	0.7477	6.8479	0.8618	8.2817
		$\beta_2 = 1.4$	0.6688	6.0649	0.7412	2.8065
		$\lambda_1 = 0.4$	0.3513	0.9430	0.3541	0.4507
		$\lambda_2 = 0.5$	0.4283	0.9804	0.4254	0.6784
		$p_1 = 0.3$	0.2489	3.0695	0.2667	2.2364
20%	40	$\alpha_1 = 1.5$	0.9370	8.1197	0.9224	6.7606
		$\alpha_2 = 2.0$	1.0357	9.6256	0.9959	5.4217
		$\beta_1 = 2.0$	0.5157	6.4623	0.5905	2.8047
		$\beta_2 = 1.4$	0.4350	4.5378	0.4692	2.2673
		$\lambda_1 = 0.4$	0.3396	0.8948	0.3046	0.4225
		$\lambda_2 = 0.5$	0.4034	0.9466	0.4207	0.6655
		$p_1 = 0.3$	0.1524	2.3285	0.1541	1.0640
40%		$\alpha_1 = 1.5$	0.9497	9.3651	0.9588	6.6882
		$\alpha_2 = 2.0$	1.0404	10.2397	0.9995	6.6487
		$\beta_1 = 2.0$	0.5951	6.4667	0.6371	3.7396
		$\beta_2 = 1.4$	0.4452	4.9274	0.4770	2.3125

		$\lambda_1 = 0.4$	0.3443	0.9032	0.3059	0.4412
		$\lambda_2 = 0.5$	0.3976	0.9474	0.4217	0.6717
		$p_1 = 0.3$	0.1660	2.6785	0.1799	2.0907
20%	60	$\alpha_1 = 1.5$	0.6360	7.7392	0.5754	6.5649
		$\alpha_2 = 2.0$	0.6404	8.8064	0.7967	5.1786
		$\beta_1 = 2.0$	0.3972	5.8619	0.4299	2.5934
		$\beta_2 = 1.4$	0.3200	3.8016	0.3524	1.9025
		$\lambda_1 = 0.4$	0.3315	0.8822	0.3023	0.4146
		$\lambda_2 = 0.5$	0.3923	0.9396	0.4139	0.5786
		$p_1 = 0.3$	0.1135	2.3008	0.1326	1.0598
40%		$\alpha_1 = 1.5$	0.7982	8.8417	0.5820	6.5996
		$\alpha_2 = 2.0$	0.6982	9.5806	0.8052	5.7471
		$\beta_1 = 2.0$	0.4086	6.3806	0.4332	3.5954
		$\beta_2 = 1.4$	0.3983	4.5960	0.3614	2.2963
		$\lambda_1 = 0.4$	0.3360	0.8923	0.3053	0.4411
		$\lambda_2 = 0.5$	0.3966	0.9426	0.4192	0.5992
		$p_1 = 0.3$	0.1291	2.6386	0.1457	1.8416
20%	100	$\alpha_1 = 1.5$	0.2180	5.9429	0.3957	5.9407
		$\alpha_2 = 2.0$	0.2834	8.0151	0.5317	5.1241
		$\beta_1 = 2.0$	0.2475	5.5634	0.3043	2.2083
		$\beta_2 = 1.4$	0.1861	3.7751	0.2424	1.9010
		$\lambda_1 = 0.4$	0.3280	0.8779	0.3013	0.4110
		$\lambda_2 = 0.5$	0.3918	0.9318	0.4120	0.5763
		$p_1 = 0.3$	0.0698	1.8032	0.0960	1.0038
40%		$\alpha_1 = 1.5$	0.2792	5.9633	0.4002	5.9556
		$\alpha_2 = 2.0$	0.2931	8.2385	0.5459	5.5638
		$\beta_1 = 2.0$	0.2506	6.3151	0.3174	3.0312
		$\beta_2 = 1.4$	0.2186	4.5808	0.2532	2.1005
		$\lambda_1 = 0.4$	0.3314	0.8806	0.3018	0.4286
		$\lambda_2 = 0.5$	0.3981	0.9417	0.4130	0.5898

		$p_1 = 0.3$	0.0702	2.0866	0.1086	1.5733

Table 3: Real life data of the Survival Time (in days) of Hepatitis (B & C) Patients

Disease(Hepatitis B)			Disease(Hepatitis C)		
Admit Date	Discharge Date	No of Recovery days	Admit Date	Discharge Date	No of Recovery days
02-01-2016	16-01-2016	15	02-01-2016	30-01-2016	29
03-02-2016	18-02-2016	16	06-01-2016	04-02-2016	30
03-03-2016	19-03-2016	17	03-02-2016	01-03-2016	28
04-03-2016	18-03-2016	15	06-04-2016	04-05-2016	29
05-04-2016	21-04-2016	17	16-04-2016	16-05-2016	31
08-05-2016	24-05-2016	17	02-05-2016	31-05-2016	30
10-05-2016	30-05-2016	21	08-06-2016	10-07-2016	33
12-06-2016	30-06-2016	19	15-06-2016	16-06-2016	32
03-07-2016	19-07-2016	17	21-07-2016	20-08-2016	30
06-07-2016	19-07-2016	14	22-07-2016	18-08-2016	27
01-08-2016	16-08-2016	15	18-08-2016	16-09-2016	30
22-08-2016	07-09-2016	17	28-08-2016	24-09-2016	28
09-09-2016	28-09-2016	20	28-08-2016	26-09-2016	30
15-09-2016	02-10-2016	18	02-09-2016	02-10-2016	31
04-10-2016	20-10-2016	17	10-09-2016	12-10-2016	33
28-10-2016	16-11-2016	20	25-09-2016	22-10-2016	28
16-11-2016	03-12-2016	19	15-10-2016	16-11-2016	33
25-11-2016	16-12-2016	23	22-10-2016	23-11-2016	33
12-12-2016	26-12-2016	14	28-10-2016	24-11-2016	27
09-01-2017	18-01-2017	10	07-11-2016	04-12-2016	28
13-02-2017	28-02-2017	16	15-12-2016	16-12-2016	32
18-03-2017	29-03-2017	12	24-12-2016	21-01-2017	29
24-03-2017	08-04-2017	16	02-01-2017	30-01-2017	29
05-04-2017	21-04-2017	17	08-01-2017	06-02-2017	30
08-05-2017	24-05-2017	17	04-02-2017	01-03-2017	24
12-05-2017	30-05-2017	19	16-04-2017	10-05-2017	25
14-06-2017	30-06-2017	17	18-04-2017	16-05-2017	29
03-07-2017	18-07-2017	16	01-05-2017	31-05-2017	31
08-07-2017	22-07-2017	15	05-06-2017	08-07-2016	34
01-08-2017	16-08-2017	16	15-06-2017	13-07-2017	29
24-08-2017	11-09-2017	19	20-07-2017	14-08-2017	26
19-09-2017	04-10-2017	16	21-07-2017	15-08-2017	25
23-09-2017	12-10-2017	20	18-08-2017	16-09-2017	30
04-10-2017	20-10-2017	16	26-08-2017	24-09-2017	30
28-10-2017	16-11-2017	20	27-08-2017	22-09-2017	27

26-11-2017	13-12-2017	19	04-09-2017	02-10-2017	29
28-11-2017	16-12-2017	20	11-09-2017	11-10-2017	31
02-12-2017	18-12-2017	17	23-09-2017	19-10-2017	27
08-12-2017	21-12-2017	14	10-10-2017	08-11-2017	30
14-02-2017	30-12-2017	17	21-11-2017	23-12-2017	33

Table 4: B Es of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

LF	20 % Censoring													
	UP							IP						
	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	2.4705 (0.0032)	1.5832 (0.0927)	0.3384 (0.0384)	0.4829 (0.0036)	3.0773 (0.0016)	1.2740 (0.0151)	0.3820 (0.0405)	2.4956 (0.0034)	1.6223 (0.0893)	0.3024 (0.0174)	0.4827 (0.0037)	3.0216 (0.0049)	1.2907 (0.2454)	0.4136 (0.0201)
PLF	2.4748 (0.0086)	1.5944 (0.0525)	0.2910 (0.1052)	0.4756 (0.0054)	3.0818 (0.0089)	1.2799 (0.0118)	0.4055 (0.1070)	2.4801 (0.0090)	1.6177 (0.0509)	0.3483 (0.0518)	0.4754 (0.0055)	3.0290 (0.0149)	1.3258 (0.1296)	0.4357 (0.0442)
QLF	2.4534 (0.0235)	1.6289 (0.0229)	0.2992 (0.1087)	0.4815 (0.0087)	3.0594 (0.0533)	1.2533 (0.0078)	0.3957 (0.1093)	2.4891 (0.0228)	1.6059 (0.0218)	0.3263 (0.0886)	0.4812 (0.0850)	3.0927 (0.0493)	1.3091 (0.0401)	0.4459 (0.0572)
LF	40 % Censoring													
	UP							IP						
	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	2.4658 (0.0038)	1.5938 (0.1138)	0.2947 (0.0486)	0.5212 (0.0043)	3.0780 (0.0009)	1.3103 (0.0164)	0.4022 (0.0496)	2.4932 (0.0042)	1.6006 (0.0992)	0.2968 (0.0180)	0.5166 (0.0056)	3.0468 (0.0119)	1.3220 (0.2576)	0.4146 (0.0204)
PLF	2.4702 (0.0092)	1.6253 (0.0629)	0.3397 (0.1201)	0.5226 (0.0098)	3.0860 (0.0112)	1.3155 (0.0123)	0.4277 (0.1190)	2.4774 (0.0104)	1.6279 (0.0547)	0.3202 (0.0648)	0.5182 (0.0062)	3.0636 (0.0334)	1.3259 (0.1357)	0.4220 (0.0447)
QLF	2.4799 (0.0217)	1.6379 (0.0249)	0.2891 (0.1184)	0.5150 (0.0039)	3.0463 (0.0667)	1.2922 (0.0082)	0.4227 (0.1234)	2.4561 (0.0234)	1.6106 (0.0245)	0.3066 (0.916)	0.5010 (0.0041)	3.0816 (0.1021)	1.3106 (0.0423)	0.4484 (0.1182)

Table 5: 95%Bayesian Credible Intervals of Mixture of the Two-Component of TWD using UP and IP

Censoring Rate	Parameters	UP		IP	
		Lower Limit	Upper Limit	Lower Limit	Upper Limit
20%	$\alpha_1 = 2.5$	0.0587	2.8914	0.0695	2.8940
	$\alpha_2 = 3.0$	0.0286	3.7778	0.0599	3.0195
	$\beta_1 = 1.6$	0.2986	1.7778	0.3563	1.6008
	$\beta_2 = 1.3$	0.2184	1.6721	0.3549	1.8350
	$\lambda_1 = 0.3$	0.1696	0.6763	0.2144	0.5770
	$\lambda_2 = 0.4$	0.1797	0.6833	0.3210	0.6832
	$p_1 = 0.5$	0.1125	3.3063	0.1304	3.5657
40%	$\alpha_1 = 2.5$	0.0655	2.7424	0.1035	3.6075
	$\alpha_2 = 3.0$	0.0338	4.5974	0.0945	3.9551
	$\beta_1 = 1.6$	0.3038	2.1947	0.3697	1.6238
	$\beta_2 = 1.3$	0.2290	3.7483	0.3920	1.9520
	$\lambda_1 = 0.3$	0.1774	0.6814	0.2158	0.5875
	$\lambda_2 = 0.4$	0.1832	0.6875	0.3227	0.6857
	$p_1 = 0.5$	0.1289	4.9770	0.1455	4.5908

Table A1: BEs of Mixture for Two -Component of TWD along with PRs (in parentheses) under UP and IP

LF	20 % Censoring														
		UP							IP						
	n	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	20	1.8925 (0.2593)	1.5085 (0.1023)	0.3769 (0.0339)	0.3894 (0.0124)	2.5369 (0.2827)	1.4061 (0.0306)	0.4595 (0.0442)	2.7200 (0.2309)	1.4753 (0.2420)	0.3621 (0.0052)	0.4479 (0.0116)	3.4721 (0.2860)	1.3678 (0.1676)	0.4669 (0.0338)
	40	2.1738 (0.1817)	1.5511 (0.0248)	0.3480 (0.0309)	0.4408 (0.0070)	2.8638 (0.1358)	1.3640 (0.0082)	0.4464 (0.0396)	2.5753 (0.2002)	1.5673 (0.2195)	0.3287 (0.0009)	0.4443 (0.0069)	3.0435 (0.1649)	1.3837 (0.1372)	0.4291 (0.0277)
	60	2.3918 (0.1094)	1.5815 (0.0102)	0.3412 (0.0306)	0.4581 (0.0047)	3.0648 (0.1026)	1.3499 (0.0047)	0.4402 (0.0392)	2.5371 (0.1136)	1.5871 (0.1280)	0.3262 (0.0007)	0.4792 (0.0047)	3.0657 (0.1084)	1.3491 (0.1048)	0.4161 (0.0205)
	100	2.5919 (0.0948)	1.6008 (0.0036)	0.3339 (0.0290)	0.4999 (0.0030)	3.0679 (0.0988)	1.3186 (0.0029)	0.4305 (0.0378)	2.4842 (0.0931)	1.6046 (0.0937)	0.30000 (0.0002)	0.4997 (0.0030)	3.0024 (0.0946)	1.3125 (0.0827)	0.4190 (0.0183)
PLF	20	2.0225 (0.2602)	1.5933 (0.0468)	0.4194 (0.1050)	0.4051 (0.0313)	2.8609 (0.2147)	1.4032 (0.0304)	0.5053 (0.0915)	2.3992 (0.2662)	1.4577 (0.2471)	0.3591 (0.0053)	0.4129 (0.0280)	3.3125 (0.2708)	1.3847 (0.0160)	0.4479 (0.0365)
	40	2.2444 (0.1411)	1.6023 (0.0244)	0.3899 (0.0937)	0.4488 (0.0160)	2.9389 (0.1502)	1.3983 (0.0087)	0.4795 (0.0863)	2.6519 (0.1531)	1.5748 (0.1683)	0.3301 (0.0027)	0.4520 (0.0153)	3.1490 (0.1748)	1.3420 (0.0072)	0.4201 (0.0318)
	60	2.4434 (0.1032)	1.6103 (0.0099)	0.3835 (0.0846)	0.4632 (0.0102)	3.1186 (0.1076)	1.3623 (0.0048)	0.4526 (0.0849)	2.5866 (0.1290)	1.5811 (0.1082)	0.3272 (0.0021)	0.4841 (0.0098)	3.1236 (0.1023)	1.3215 (0.0049)	0.4167 (0.0241)
	100	2.5216 (0.0594)	1.6195 (0.0037)	0.3747 (0.0817)	0.5030 (0.0061)	3.1034 (0.0709)	1.3201 (0.0029)	0.4223 (0.0836)	2.5129) (0.0973)	1.6065 (0.0839)	0.3000 (0.0018)	0.5027 (0.0060)	3.0379 (0.0910)	1.3040 (0.0028)	0.4191 0.0163)
QLF	20	2.4997 (0.1213)	1.5489 (0.0202)	0.3665 (0.1092)	0.4378 (0.0708)	2.6539 (0.1224)	1.3978 (0.0185)	0.3905 (0.1485)	2.2865 (0.1128)	1.5296 (0.0234)	0.3493 (0.0244)	0.4431 (0.0647)	2.6142 (0.1157)	1.4586 (0.0160)	0.4503 (0.0071)
	40	2.5161 (0.0609)	1.5609 (0.0101)	0.3586 (0.0963)	0.4687 (0.0312)	2.8030 (0.0609)	1.3423 (0.0078)	0.3912 (0.1229)	2.5871 (0.0635)	1.5487 (0.0102)	0.3221 (0.0167)	0.4628 (0.0384)	2.8986 (0.0522)	1.3841 (0.0060)	0.4458 (0.0044)
	60	2.5395 (0.0389)	1.5878 (0.0059)	0.3518 (0.0423)	0.5006 (0.0202)	3.0402 (0.0421)	1.3264 (0.0047)	0.3979 (0.1090)	2.4498 (0.0426)	1.5673 (0.0069)	0.3172 (0.0145)	0.4802 (0.0244)	2.9842 (0.0377)	1.3498 (0.0041)	0.4261 (0.0033)
	100	2.5488 (0.0235)	1.6028 (0.0036)	0.3132 (0.0049)	0.5049 (0.0127)	3.0721 (0.0247)	1.3150 (0.0018)	0.4020 (0.0907)	2.4861 (0.0243)	1.5896 (0.0030)	0.3063 (0.0070)	0.5084 (0.0119)	3.0612 (0.0246)	1.3180 (0.0024)	0.4071 (0.0021)
LF	40 % Censoring														
		UP							IP						

	n	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	20	2.8919 (0.2990)	1.7084 (0.1237)	0.3585 (0.0438)	0.4312 (0.0162)	3.2158 (0.2994)	1.3584 (0.0637)	0.4557 (0.0591)	2.6422 (0.2512)	1.5230 (0.2754)	0.2521 (0.0091)	0.4366 (0.0142)	2.8254 (0.2998)	1.3882 (0.1736)	0.3519 (0.0412)
	40	2.7424 (0.2300)	1.6936 (0.0392)	0.3470 (0.0402)	0.4617 (0.0093)	3.0766 (0.1581)	1.3605 (0.0337)	0.4424 (0.0457)	2.5942 (0.2153)	1.5418 (0.2425)	0.2976 (0.0045)	0.4680 (0.0084)	2.8824 (0.1759)	1.3467 (0.1465)	0.3831 (0.0296)
	60	2.5540 (0.1157)	1.6632 (0.0248)	0.3407 (0.0387)	0.4624 (0.0070)	3.1220 (0.1046)	1.3777 (0.0236)	0.4274 (0.0439)	2.5626 (0.1267)	1.5961 (0.1378)	0.2993 (0.0033)	0.4748 (0.0061)	2.9157 (0.1182)	1.3145 (0.1284)	0.3997 (0.0233)
	100	2.4784 (0.0987)	1.6082 (0.0046)	0.3336 (0.0310)	0.5248 (0.0035)	3.0180 (0.0998)	1.4043 (0.0177)	0.4249 (0.0388)	2.5245 (0.0969)	1.6004 (0.0985)	0.3089 (0.0024)	0.4874 (0.0049)	3.0146 (0.0972)	1.3024 (0.0870)	0.4003 (0.0181)
PLF	20	2.9072 (0.3305)	1.6869 (0.0634)	0.3987 (0.1280)	0.4496 (0.0368)	3.4031 (0.3745)	1.4914 (0.1344)	0.4865 (0.1017)	2.7303 (0.2461)	1.4959 (0.2559)	0.2539 (0.0236)	0.4526 (0.0320)	2.8574 (0.2811)	1.3819 (0.0465)	0.3535 (0.0418)
	40	2.8549 (0.2251)	1.6304 (0.0436)	0.3881 (0.1123)	0.4718 (0.0299)	3.1817 (0.2103)	1.4564 (0.1156)	0.4706 (0.0964)	2.6861 (0.1538)	1.5535 (0.1743)	0.3081 (0.0112)	0.4770 (0.0170)	2.9753 (0.1859)	1.3683 (0.0232)	0.3898 (0.0325)
	60	2.6322 (0.1564)	1.6223 (0.0183)	0.3806 (0.0998)	0.4699 (0.0215)	3.1799 (0.1157)	1.4429 (0.0130)	0.4546 (0.0884)	2.5477 (0.1142)	1.5844 (0.1186)	0.3068 (0.0110)	0.5065 (0.0119)	2.9829 (0.1062)	1.3417 (0.0142)	0.3992 (0.0287)
	100	2.4877 (0.0970)	1.6424 (0.0106)	0.3724 (0.0835)	0.4856 (0.0074)	3.0696 (0.0865)	1.4003 (0.0082)	0.4459 (0.0864)	2.5032 (0.0856)	1.6088 (0.0865)	0.3040 (0.0103)	0.5002 (0.0098)	3.0453 (0.0923)	1.3149 (0.0080)	0.4039 (0.0179)
QLF	20	2.3005 (0.1631)	1.5041 (0.0346)	0.3630 (0.1098)	0.4143 (0.1036)	2.8977 (0.1747)	1.1013 (0.0346)	0.3922 (0.1490)	2.2507 (0.1410)	1.5285 (0.0377)	0.3165 (0.0340)	0.4293 (0.0797)	3.1282 (0.1591)	1.4260 (0.0191)	0.4435 (0.0152)
	40	2.3472 (0.0816)	1.5464 (0.0264)	0.3536 (0.1048)	0.4640 (0.0420)	2.9048 (0.0820)	1.2741 (0.0187)	0.3935 (0.1449)	2.3767 (0.0742)	1.5409 (0.0171)	0.3120 (0.0235)	0.4611 (0.0418)	3.0614 (0.0907)	1.3857 (0.0121)	0.4354 (0.0143)
	60	2.4336 (0.0554)	1.5816 (0.0114)	0.3498 (0.0633)	0.4974 (0.0255)	3.0568 (0.0601)	1.2936 (0.0173)	0.3981 (0.1118)	2.4887 (0.0458)	1.5690 (0.0092)	0.3010 (0.0110)	0.5011 (0.0259)	3.0440 (0.0679)	1.3450 (0.0096)	0.4281 (0.0075)
	100	2.5018 (0.0380)	1.6086 (0.0037)	0.3134 (0.0066)	0.5043 (0.0131)	3.0078 (0.0328)	1.3017 (0.0078)	0.4001 (0.0943)	2.5355 (0.0327)	1.5910 (0.0051)	0.3000 (0.0097)	0.5197 (0.0196)	3.0152 (0.0512)	1.3257 (0.0058)	0.4065 (0.0040)

Table A2: BEs of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

LF	**20 % Censoring**														
		UP							**IP**						
	N	$\alpha_1 = 2$	$\beta_1 = 1.4$	$\lambda_1 = 0.4$	$p_1 = 0.7$	$\alpha_2 = 1.5$	$\beta_2 = 1.3$	$\lambda_2 = 0.6$	$\alpha_1 = 2$	$\beta_1 = 1.4$	$\lambda_1 = 0.4$	$p_1 = 0.7$	$\alpha_2 = 1.5$	$\beta_2 = 1.3$	$\lambda_2 = 0.6$
SELF	20	1.7365 (0.2696)	1.3246 (0.0466)	0.4419 (0.0177)	0.6094 (0.0126)	1.4573 (0.3051)	1.2451 (0.1226)	0.6276 (0.0355)	1.6826 (0.2285)	1.5461 (0.1369)	0.3212 (0.0431)	0.6412 (0.0110)	1.2515 (0.2241)	1.2084 (0.1559)	0.5034 (0.0338)

	40	1.8127 (0.1534)	1.3542 (0.0110)	0.4320 (0.0086)	0.6166 (0.0068)	1.4633 (0.1583)	1.2452 (0.1210)	0.6090 (0.0317)	1.7881 (0.1334)	1.4882 (0.0965)	0.3667 (0.0168)	0.6617 (0.0061)	1.3838 (0.1566)	1.2464 (0.1230)	0.5516 (0.0214)
	60	1.9367 (0.1103)	1.3875 (0.0049)	0.4200 (0.0072)	0.6595 (0.0044)	1.4840 (0.1184)	1.2769 (0.0536)	0.6063 (0.0303)	1.9234 (0.1011)	1.4599 (0.0472)	0.3694 (0.0161)	0.6901 (0.0040)	1.4654 (0.1144)	1.2652 (0.1036)	0.5681 (0.0208)
	100	2.0420 (0.0729)	1.4087 (0.0023)	0.4143 (0.0056)	0.6812 (0.0026)	1.5068 (0.0838)	1.2948 (0.0320)	0.5925 (0.0276)	2.0459 (0.0715)	1.4277 (0.0216)	0.4074 (0.0135)	0.7125 (0.0025)	1.4918 (0.0829)	1.2982 (0.0929)	0.5998 (0.0132)
PLF	20	1.8125 (0.1519)	1.4775 (0.0612)	0.4615 (0.0391)	0.6197 (0.0205)	1.5585 (0.2024)	1.2390 (0.1590)	0.6553 (0.0554)	1.7492 (0.1332)	1.5796 (0.1436)	0.3410 (0.0696)	0.6497 (0.0170)	1.3380 (0.1731)	1.4364 (0.1266)	0.5170 (0.0571)
	40	1.8545 (0.0836)	1.4615 (0.0146)	0.4532 (0.0223)	0.6221 (0.0109)	1.5479 (0.1091)	1.2556 (0.1206)	0.6341 (0.0502)	1.8250 (0.0738)	1.4951 (0.0825)	0.3890 (0.0446)	0.6663 (0.0092)	1.4393 (0.1110)	1.3556 (0.1085)	0.5706 (0.0381)
	60	1.9650 (0.0565)	1.4309 (0.0067)	0.4399 (0.0199)	0.6628 (0.0067)	1.5255 (0.0831)	1.2881 (0.0838)	0.6308 (0.0489)	1.9496 (0.0522)	1.4530 (0.0562)	0.3905 (0.0423)	0.6930 (0.0058)	1.4665 (0.0825)	1.3394 (0.0983)	0.5876 (0.0374)
	100	2.0598 (0.0356)	1.4130 (0.0031)	0.4142 (0.0086)	0.6831 (0.0038)	1.4951 (0.0566)	1.3014 (0.0632)	0.6153 (0.0457)	2.0633 (0.0348)	1.4291 (0.0288)	0.4129 (0.0310)	0.7042 (0.0034)	1.4905 (0.0673)	1.3116 (0.0773)	0.6108 (0.0218)
QLF	20	1.7749 (0.0917)	1.3447 (0.0287)	0.4947 (0.0882)	0.6253 (0.0335)	1.2353 (0.1956)	1.4082 (0.1210)	0.5660 (0.1108)	1.7364 (0.0795)	1.5418 (0.0229)	0.3776 (0.0871)	0.6582 (0.0272)	1.4199 (0.1969)	1.2108 (0.2121)	0.5256 (0.0557)
	40	1.9107 (0.0513)	1.3768 (0.0131)	0.4799 (0.0786)	0.6436 (0.0198)	1.3125 (0.0831)	1.3579 (0.0542)	0.6073 (0.0615)	1.9821 (0.0416)	1.4887 (0.0098)	0.3914 (0.0633)	0.6713 (0.0141)	1.4426 (0.1000)	1.2547 (0.1471)	0.5422 (0.0445)
	60	2.0327 (0.0303)	1.3803 (0.0068)	0.4476 (0.0206)	0.6661 (0.0105)	1.4826 (0.0653)	1.3312 (0.0433)	0.6014 (0.0592)	1.9914 (0.0299)	1.4490 (0.0068)	0.3944 (0.0547)	0.6857 (0.0115)	1.4751 (0.0605)	1.2822 (0.1097)	0.5576 (0.0387)
	100	2.0252 (0.0180)	1.4071 (0.0040)	0.4244 (0.0197)	0.6917 (0.0064)	1.5158 (0.0387)	1.3113 (0.0210)	0.6001 (0.0545)	2.0301 (0.0171)	1.4150 (0.0036)	0.4068 (0.0487)	0.6919 (0.0073)	1.4950 (0.0406)	1.3062 (0.0876)	0.6086 (0.0181)
LF	**40 % Censoring**														
	UP								**IP**						
	N	$\alpha_1 = 2$	$\beta_1 = 1.4$	$\lambda_1 = 0.4$	$p_1 = 0.7$	$\alpha_2 = 1.5$	$\beta_2 = 1.3$	$\lambda_2 = 0.6$	$\alpha_1 = 2$	$\beta_1 = 1.4$	$\lambda_1 = 0.4$	$p_1 = 0.7$	$\alpha_2 = 1.5$	$\beta_2 = 1.3$	$\lambda_2 = 0.6$
SELF	20	1.8528 (0.2949)	1.5270 (0.0533)	0.4610 (0.0278)	0.6612 (0.0174)	1.4360 (0.3355)	1.4961 (0.1396)	0.5788 (0.0449)	1.5916 (0.2658)	1.5479 (0.1511)	0.3508 (0.0514)	0.6138 (0.0144)	1.2940 (0.2794)	1.4670 (0.1598)	0.5497 (0.0377)
	40	1.9251 (0.1606)	1.4812 (0.0453)	0.4422 (0.0263)	0.6515 (0.0081)	1.4654 (0.1659)	1.4054 (0.1245)	0.5812 (0.0431)	1.7675 (0.1769)	1.4867 (0.1098)	03614 (0.0234)	0.6357 (0.0085)	1.3711 (0.1673)	1.4296 (0.1286)	0.5682 (0.0235)
	60	2.0264 (0.1249)	1.4491 (0.0150)	0.4305 (0.0247)	0.6840 (0.0057)	1.4987 (0.1202)	1.3519 (0.0692)	0.5605 (0.0415)	1.9690 (0.1180)	1.4462 (0.0694)	0.3809 (0.0194)	0.6684 (0.0054)	1.4404 (0.1289)	1.3572 (0.1057)	0.5803 (0.0214)
	100	2.0713 (0.0959)	1.4190 (0.0087)	0.4266 (0.0153)	0.6913 (0.0031)	1.5055 (0.0957)	1.3292 (0.0345)	0.5977 (0.0394)	1.9923 (0.0848)	1.4158 (0.0475)	0.4000 (0.0149)	0.6901 (0.0035)	1.5086 (0.0973)	1.3254 (0.0963)	0.6028 (0.0154)

PLF	20	1.8451 (0.1645)	1.6202 (0.0665)	0.4902 (0.0485)	0.6743 (0.0262)	1.6117 (0.2514)	1.3651 (0.1620)	0.6164 (0.0651)	1.6730 (0.2129)	1.4663 (0.1617)	0.3529 (0.0729)	0.6254 (0.0232)	1.3978 (0.2076)	1.4782 (0.1654)	0.5567 (0.0639)
	40	1.9738 (0.0974)	1.5674 (0.0323)	0.4711 (0.0378)	0.6577 (0.0124)	1.5405 (0.1503)	1.3457 (0.1306)	0.6078 (0.0533)	1.8168 (0.1187)	1.4493 (0.1257)	0.3643 (0.0584)	0.6476 (0.0140)	1.4308 (0.1194)	1.3831 (0.1222)	0.5794 (0.0420)
	60	2.0667 (0.0806)	1.4476 (0.0170)	0.4584 (0.0257)	0.6881 (0.0083)	1.5209 (0.1243)	1.3298 (0.1210)	0.5964 (0.0517)	2.0037 (0.0895)	1.4234 (0.0939)	0.3818 (0.0441)	0.6724 (0.0080)	1.4609 (0.1010)	1.3402 (0.1097)	0.5958 (0.0393)
	100	2.0943 (0.0460)	1.4165 (0.0102)	0.4253 (0.0174)	0.6936 (0.0045)	1.5045 (0.0979)	1.3098 (0.0989)	0.5920 (0.0486)	2.0209 (0.0572)	1.4080 (0.0685)	0.4000 (0.0326)	0.6931 (0.0061)	1.5019 (0.0867)	1.3862 (0.0916)	0.6056 (0.0256)
QLF	20	1.6962 (0.1240)	1.4883 (0.0502)	0.4739 (0.0916)	0.5816 (0.0535)	1.3819 (0.2486)	1.2254 (0.1606)	0.6018 (0.1218)	1.7812 (0.1053)	1.5001 (0.0329)	0.4286 (0.0935)	0.6263 (0.0400)	1.4380 (0.2110)	1.4387 (0.2331)	0.6254 (0.0682)
	40	1.8451 (0.0625)	1.4609 (0.0210)	0.4504 (0.0827)	0.6250 (0.0240)	1.4337 (0.1223)	1.3222 (0.1228)	0.6011 (0.0668)	1.8641 (0.0584)	1.4700 (0.0188)	0.4153 (0.0811)	0.6584 (0.0242)	1.4611 (0.1127)	1.3851 (0.1573)	0.6276 (0.0581)
	60	1.9819 (0.0407)	1.4214 (0.0114)	0.4356 (0.0211)	0.6702 (0.0140)	1.4974 (0.0791)	1.3259 (0.0787)	0.6005 (0.0638)	1.9186 (0.0362)	1.4363 (0.0109)	0.4132 (0.0780)	0.6817 (0.0145)	1.4842 (0.0968)	1.3429 (0.1206)	0.6179 (0.0468)
	100	2.0711 (0.0219)	1.4037 (0.0056)	0.4017 (0.0203)	0.7028 (0.0076)	1.5075 (0.0533)	1.3029 (0.0511)	0.5973 (0.0578)	1.9764 (0.0227)	1.4054 (0.0072)	0.4000 (0.0506)	0.7005 (0.0080)	1.5202 (0.0462)	1.3153 (0.0987)	0.6073 (0.0255)

Table A3: 95%Bayesian Credible Intervals of Mixture for Two-Component of TWD of using UP and IP with hyperparameters are a_1=1, a_2=0.5, b_1=2, b_2=1, c_1=0.5, c_2=1.5, d_1=1, d_2=2, e_1=1, e_2=1, l_1=0.1 and l_2=0.2

Censoring Rate	Size	Parameters	UP		IP	
			Lower Limit	Upper Limit	Lower Limit	Upper Limit
20 %	20	$\alpha_1 = 2.5$	1.4024	13.8814	1.9556	8.8013
		$\alpha_2 = 3.0$	1.5739	7.6740	2.1231	14.1630
		$\beta_1 = 1.6$	0.7097	2.1776	0.8392	2.2626
		$\beta_2 = 1.3$	0.7231	2.1907	0.7660	2.2391
		$\lambda_1 = 0.3$	0.2426	0.8693	0.3102	0.4916
		$\lambda_2 = 0.4$	0.3099	0.9310	0.4552	0.5467
		$p_1 = 0.5$	0.2663	2.1117	0.3179	1.4747
40%		$\alpha_1 = 2.5$	1.4336	14.7699	1.9684	9.9844
		$\alpha_2 = 3.0$	1.6647	8.3043	2.3110	15.4317
		$\beta_1 = 1.6$	0.7177	4.4872	0.8487	5.7680
		$\beta_2 = 1.3$	0.7352	4.3795	0.7769	4.7679
		$\lambda_1 = 0.3$	0.2521	0.8759	0.3438	0.9665
		$\lambda_2 = 0.4$	0.3389	0.9450	0.4561	1.0698
		$p_1 = 0.5$	0.2736	2.2102	0.3224	1.7848
20 %	40	$\alpha_1 = 2.5$	1.4012	12.3951	1.1265	8.6167
		$\alpha_2 = 3.0$	1.0164	6.9280	1.3375	9.9317
		$\beta_1 = 1.6$	0.5029	2.1642	0.5760	2.2518
		$\beta_2 = 1.3$	0.4676	2.1694	0.5158	2.2191
		$\lambda_1 = 0.3$	0.2348	0.8653	0.3011	0.3737
		$\lambda_2 = 0.4$	0.3076	0.9205	0.4086	0.5000
		$p_1 = 0.5$	0.2511	1.8660	0.2102	1.3802
40%		$\alpha_1 = 2.5$	1.4073	14.5029	1.3358	8.9419
		$\alpha_2 = 3.0$	1.0631	8.2743	1.5659	13.5041
		$\beta_1 = 1.6$	0.5146	4.1274	0.6094	5.2276
		$\beta_2 = 1.3$	0.4944	4.8022	0.5269	4.5021

		$\lambda_1 = 0.3$	0.2351	0.8660	0.3057	0.4977
		$\lambda_2 = 0.4$	0.3332	0.9366	0.4098	0.5079
		$p_1 = 0.5$	0.2728	2.1452	0.2257	1.7002
20 %	60	$\alpha_1 = 2.5$	0.9431	12.1943	0.6911	8.5076
		$\alpha_2 = 3.0$	1.0106	6.2809	1.2292	9.5548
		$\beta_1 = 1.6$	0.3780	1.7228	0.4210	2.2097
		$\beta_2 = 1.3$	0.3568	1.5870	0.3834	2.1807
		$\lambda_1 = 0.3$	0.2240	0.8355	0.3002	0.3284
		$\lambda_2 = 0.4$	0.3099	0.9124	0.4060	0.4536
		$p_1 = 0.5$	0.1599	1.8515	0.1609	1.3089
40%		$\alpha_1 = 2.5$	0.9592	14.5153	0.7058	8.9087
		$\alpha_2 = 3.0$	1.0530	8.7726	1.2757	9.9611
		$\beta_1 = 1.6$	0.3823	5.1488	0.4336	5.1898
		$\beta_2 = 1.3$	0.3708	5.5830	0.3935	4.3529
		$\lambda_1 = 0.3$	0.2264	0.8381	0.3025	0.3951
		$\lambda_2 = 0.4$	0.3270	0.8659	0.4095	0.4957
		$p_1 = 0.5$	0.1808	2.1218	0.1781	1.6148
20 %	100	$\alpha_1 = 2.5$	0.4291	11.2170	0.3840	8.5019
		$\alpha_2 = 3.0$	0.5522	5.8514	0.4947	9.0473
		$\beta_1 = 1.6$	0.2249	1.6703	0.2565	2.2021
		$\beta_2 = 1.3$	0.2105	1.5804	0.2498	2.0233
		$\lambda_1 = 0.3$	0.2219	0.8312	0.2967	0.3240
		$\lambda_2 = 0.4$	0.3064	0.9093	0.4042	0.4503
		$p_1 = 0.5$	0.0960	1.3365	0.1193	1.0372
40%		$\alpha_1 = 2.5$	0.4342	13.8626	0.3992	8.7897
		$\alpha_2 = 3.0$	0.5631	7.9133	0.6913	9.5036
		$\beta_1 = 1.6$	0.2436	5.0806	0.4279	4.8949
		$\beta_2 = 1.3$	0.2396	5.0557	0.3829	3.0396
		$\lambda_1 = 0.3$	0.2246	0.8325	0.3008	0.3945
		$\lambda_2 = 0.4$	0.3104	0.8101	0.4088	0.4862

		$p_1 = 0.5$	0.0981	2.0300	0.1600	1.2988

Table A4: 95% Bayesian Credible Intervals of Mixture for Two -Component of TWD using UP and IP with hyperparameters are a_1=0.5, a_2=1, b_1=1, b_2=2, c_1=1.5, c_2=0.5, d_1=2, d_2=1, e_1=0.5, e_2=1, l_1=0.1 and l_2=0.2

Censoring Rate	Size	Parameters	UP		IP	
			Lower Limit	Upper Limit	Lower Limit	Upper Limit
20 %	20	$\alpha_1 = 2.0$	1.5563	7.4482	2.0609	6.5614
		$\alpha_2 = 1.5$	1.4418	6.7518	1.2249	4.7279
		$\beta_1 = 1.4$	0.6353	1.5172	0.7507	2.5019
		$\beta_2 = 1.3$	0.6547	2.5691	0.9130	4.2485
		$\lambda_1 = 0.4$	0.3420	0.8566	0.3548	0.9030
		$\lambda_2 = 0.6$	0.4771	0.9727	0.4988	0.7884
		$p_1 = 0.7$	0.3908	2.4093	0.4626	2.2613
40%		$\alpha_1 = 2.0$	1.5583	8.1332	2.0720	6.8684
		$\alpha_2 = 1.5$	1.4478	7.7247	1.3260	4.8073
		$\beta_1 = 1.4$	0.6448	3.7247	0.7602	2.5273
		$\beta_2 = 1.3$	0.6737	3.6482	0.9558	4.3560
		$\lambda_1 = 0.4$	0.3445	0.8896	0.4096	0.9241
		$\lambda_2 = 0.6$	0.4819	0.9839	0.5487	0.7950
		$p_1 = 0.7$	0.4201	3.3895	0.4709	2.2066
20%	40	$\alpha_1 = 2.0$	1.0541	7.3238	0.9344	5.2861
		$\alpha_2 = 1.5$	0.9926	6.6156	0.5999	3.5661
		$\beta_1 = 1.4$	0.4450	1.9675	0.4947	1.7989
		$\beta_2 = 1.3$	0.5651	2.3371	0.7091	4.0337
		$\lambda_1 = 0.4$	0.3366	0.8189	0.3498	0.6876
		$\lambda_2 = 0.6$	0.4751	0.9609	0.4832	0.7790
		$p_1 = 0.7$	0.3651	2.0714	0.3771	1.8717
40%		$\alpha_1 = 2.0$	1.0630	8.1174	0.9506	6.2553
		$\alpha_2 = 1.5$	1.0216	6.7656	0.6086	4.6074
		$\beta_1 = 1.4$	0.4511	3.5917	0.5110	2.3623

		$\beta_2 = 1.3$	0.5910	2.7975	0.7586	4.2187
		$\lambda_1 = 0.4$	0.3407	0.8873	0.3500	0.6953
		$\lambda_2 = 0.6$	0.4791	0.9613	0.4928	0.7864
		$p_1 = 0.7$	0.3780	3.2743	0.4040	2.0120
20%	60	$\alpha_1 = 2.0$	0.6630	6.3061	0.6099	5.0410
		$\alpha_2 = 1.5$	0.5833	5.3553	0.4965	3.0524
		$\beta_1 = 1.4$	0.3403	1.8660	0.3667	1.6764
		$\beta_2 = 1.3$	0.4109	2.1858	0.4645	3.4177
		$\lambda_1 = 0.4$	0.3372	0.7947	0.3361	0.6647
		$\lambda_2 = 0.6$	0.4720	0.9520	0.4775	0.7519
		$p_1 = 0.7$	0.2763	2.0671	0.2899	1.6742
40%		$\alpha_1 = 2.0$	0.6735	8.4398	0.6345	5.8870
		$\alpha_2 = 1.5$	0.5908	6.0337	0.5054	4.3055
		$\beta_1 = 1.4$	0.3570	3.3484	0.3775	2.2124
		$\beta_2 = 1.3$	0.4223	2.5866	0.4883	4.4674
		$\lambda_1 = 0.4$	0.3384	0.8793	0.3454	0.6876
		$\lambda_2 = 0.6$	0.4718	0.9560	0.4887	0.7739
		$p_1 = 0.7$	0.2873	3.2230	0.3058	2.0050
20%	100	$\alpha_1 = 2.0$	0.4331	6.2290	0.2281	4.2312
		$\alpha_2 = 1.5$	0.3464	4.9695	0.2299	3.0495
		$\beta_1 = 1.4$	0.2289	1.7479	0.2064	1.5485
		$\beta_2 = 1.3$	0.3314	2.1521	0.2802	3.3880
		$\lambda_1 = 0.4$	0.3355	0.7823	0.3214	0.6569
		$\lambda_2 = 0.6$	0.4714	0.9494	0.4653	0.7478
		$p_1 = 0.7$	0.2271	2.0472	0.1776	1.5812
40%		$\alpha_1 = 2.0$	0.4478	7.1552	0.2498	5.1813
		$\alpha_2 = 1.5$	0.3563	6.0243	0.2350	4.2080
		$\beta_1 = 1.4$	0.2312	3.3539	0.3753	2.1444
		$\beta_2 = 1.3$	0.3416	2.5712	0.2993	4.3073
		$\lambda_1 = 0.4$	0.3373	0.8655	0.3321	0.6078

		$\lambda_2 = 0.6$	0.4646	0.9557	0.4763	0.7645
		$p_1 = 0.7$	0.2305	2.2243	0.1807	1.9776

Table A5: Measurements of Hole and Sheet Thickness (12 mm and 3.15 mm)

Subpopulation-I	Subpopulation-II
0.04,0.06,0.12,0.22,0.08,0.26,0.14,0,0.08,0.32 0.14,0.16,0.12,0.24,0.16,0.08,0.16,0.32,0.24,0.22 0.24,0.02,0.22,0.06,0.14,0.26	0.02,0.14,0.08,0.12,0.24,0.04,0.16,0.26,0.28, 0.24,0.22,0.18,0.32,0.14,0.24,0.16,0.18,0.16, 0.12, 0.06,0.18,0.14,0.04, 0.18, 0.16

Table A6: BEs of Mixture for Two-Component of TWD along with PRs (in parentheses) under UP and IP

LF	20 % Censoring													
	UP							IP						
	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	2.4811 (0.0121)	1.5778 (0.0213)	0.2998 (0.0132)	0.4993 (0.0062)	3.0539 (0.0129)	1.2854 (0.0222)	0.3998 (0.0134)	2.4947 (0.0109)	1.5813 (0.0179)	0.2977 (0.0002)	0.4970 (0.0062)	3.0810 (0.0112)	1.2925 (0.0181)	0.4177 (0.0075)
PLF	2.4935 (0.0248)	1.6267 (0.0258)	0.3207 (0.0428)	0.4956 (0.0126)	3.0180 (0.0252)	1.2995 (0.0280)	0.4214 (0.0431)	2.4861 (0.0227)	1.5819 (0.0211)	0.2981 (0.0007)	0.4933 (0.0126)	2.9925 (0.0230)	1.2730 (0.0211)	0.4160 (0.0166)
QLF	2.4315 (0.0540)	1.6129 (0.0468)	0.3016 (0.0883)	0.4823 (0.0289)	3.0538 (0.0540)	1.3129 (0.0526)	0.4013 (0.0894)	2.4890 (0.0519)	1.5917 (0.0320)	0.2951 (0.0052)	0.4895 (0.0297)	2.9848 (0.0501)	1.2802 (0.0324)	0.4139 (0.0388)
LF	40 % Censoring													
	UP							IP						
	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$	$\alpha_1 = 2.5$	$\beta_1 = 1.6$	$\lambda_1 = 0.3$	$p_1 = 0.5$	$\alpha_2 = 3.0$	$\beta_2 = 1.3$	$\lambda_2 = 0.4$
SELF	2.4636 (0.0137)	1.5911 (0.0287)	0.3013 (0.0137)	0.5129 (0.0078)	3.0274 (0.0146)	1.3116 (0.0287)	0.3989 (0.0138)	2.4911 (0.0114)	1.5914 (0.0237)	0.3196 (0.0074)	0.5148 (0.0076)	2.9915 (0.0127)	1.2990 (0.0275)	0.4086 (0.0094)
PLF	2.4781 (0.0290)	1.6272 (0.0312)	0.3222 (0.0437)	0.5205 (0.0152)	3.0408 (0.0268)	1.2855 (0.0292)	0.4205 (0.0436)	2.4636 (0.0249)	1.5974 (0.0245)	0.3201 (0.0209)	0.5221 (0.0147)	2.9654 (0.0277)	1.2821 (0.0266)	0.4567 (0.0173)
QLF	2.4546 (0.0676)	1.6337 (0.0477)	0.3043 (0.0896)	0.4884 (0.0362)	3.0721 (0.0693)	1.2721 (0.0551)	0.3996 (0.0919)	2.4002 (0.0601)	1.5905 (0.0337)	0.3057 (0.0061)	0.4823 (0.0336)	2.9577 (0.0656)	1.2955 (0.0341)	0.4149 (0.0397)

Table A7: 95%Bayesian Credible Intervals of Mixture of the Two-Component of TWD using UP and IP

Censoring Rate	Parameters	UP		IP	
		Lower Limit	Upper Limit	Lower Limit	Upper Limit
20 %	$\alpha_1 = 2.5$	0.1283	2.6787	0.1639	2.9583
	$\alpha_2 = 3.0$	0.1342	3.3523	0.1701	3.9277
	$\beta_1 = 1.6$	0.2666	3.3633	0.3523	3.0044
	$\beta_2 = 1.3$	0.2618	3.2837	0.3397	3.1605
	$\lambda_1 = 0.3$	0.1991	0.4885	0.1733	0.4310
	$\lambda_2 = 0.4$	0.1996	0.4907	0.3733	0.5910
	$p_1 = 0.5$	0.1368	2.2408	0.1763	1.9733
40%	$\alpha_1 = 2.5$	0.1386	2.8999	0.1722	3.1662
	$\alpha_2 = 3.0$	0.1362	3.6547	0.1745	4.1778
	$\beta_1 = 1.6$	0.2964	3.4257	0.3631	3.1373
	$\beta_2 = 1.3$	0.2712	3.7151	0.3592	3.5716
	$\lambda_1 = 0.3$	0.2034	0.4903	0.2764	0.4933
	$\lambda_2 = 0.4$	0.2065	0.4990	0.3746	0.5920
	$p_1 = 0.5$	0.1452	3.1401	0.1805	2.4102

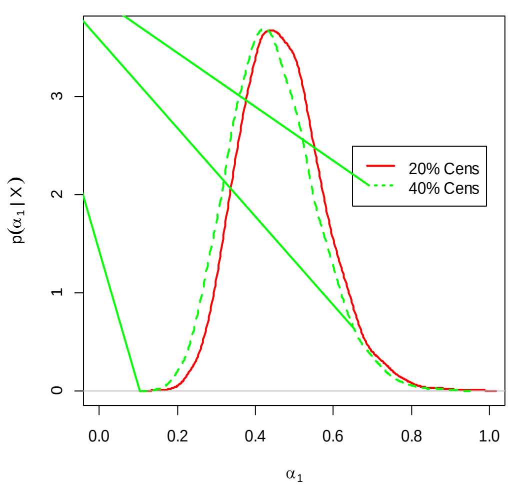

Figure A1a: Posterior Density of $\alpha_1$using UP

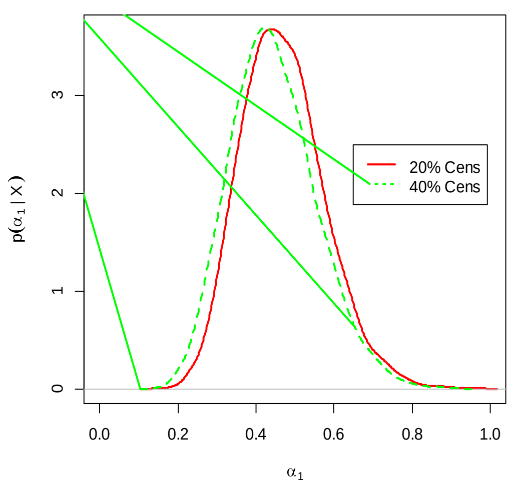

Figure A1b: Posterior Density of α_1 using IP

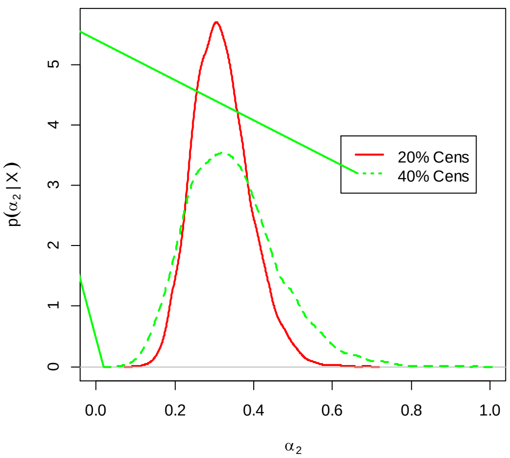

Figure A1c: Posterior Density of $\alpha_2$using UP

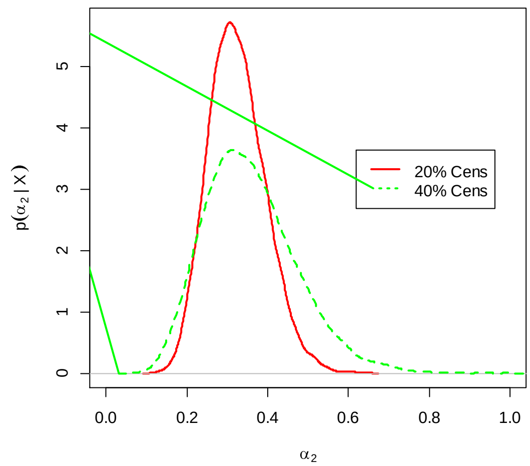

Figure A1d: Posterior Density of α_2 using IP

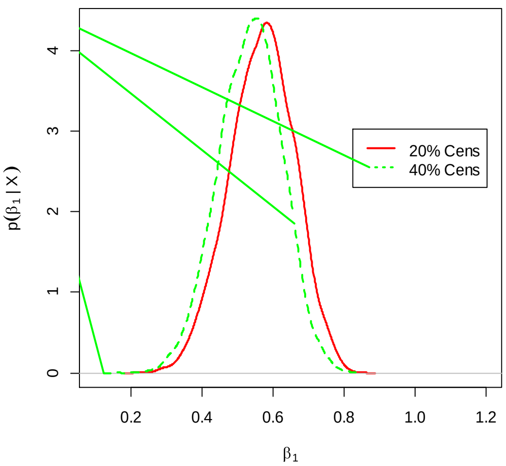

Figure A1e: Posterior Density of β_1 using UP

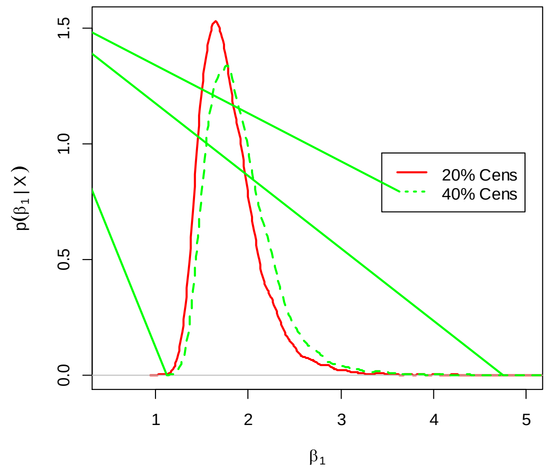

Figure A1f: Posterior Density of $\beta_1$using IP

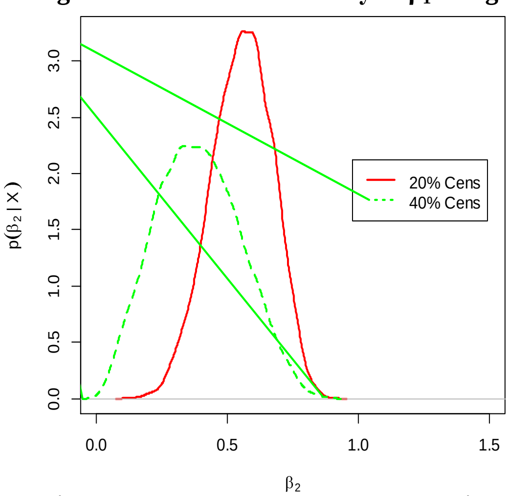

Figure A1g: Posterior Density of $\beta_2$using UP

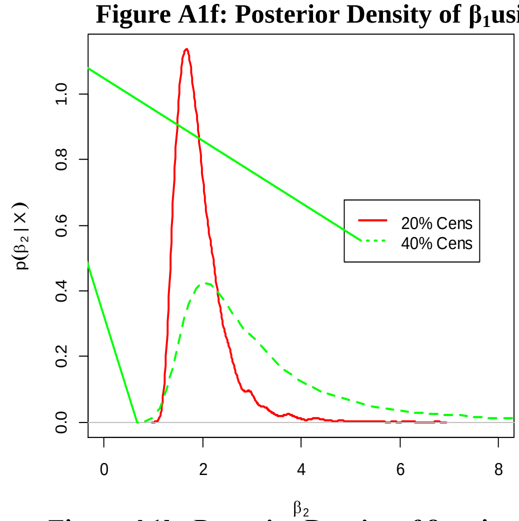

Figure A1h: Posterior Density of β_2 using IP

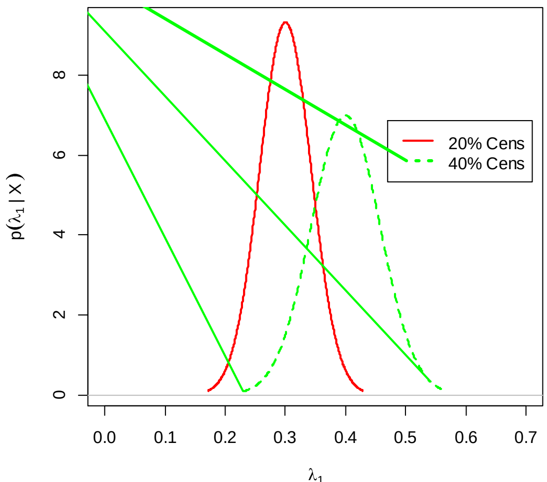

Figure A1i: Posterior Density of λ_1 using UP

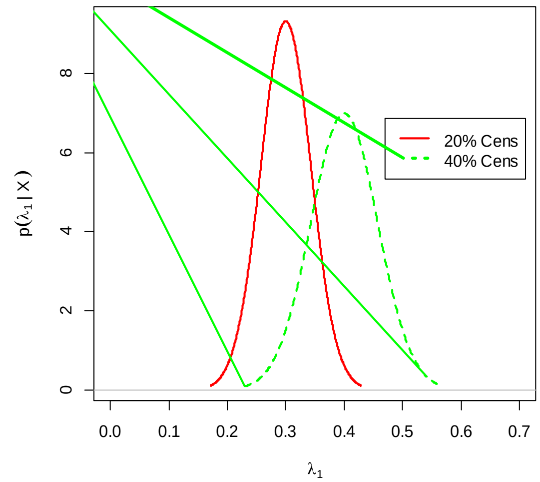

Figure A1j: Posterior Density of λ_1 using IP

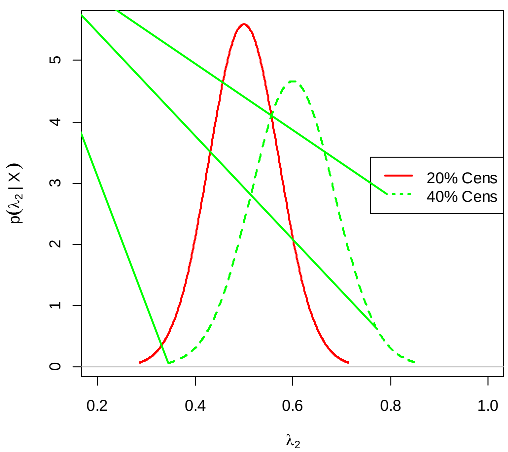

Figure A1k: Posterior Density of λ_2 using UP

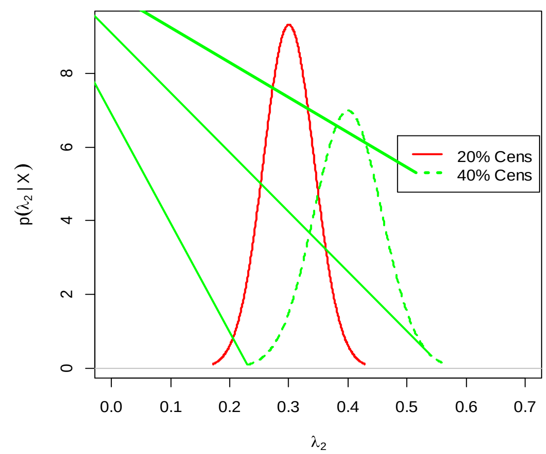

Figure A1l: Posterior Density of λ_2 using IP

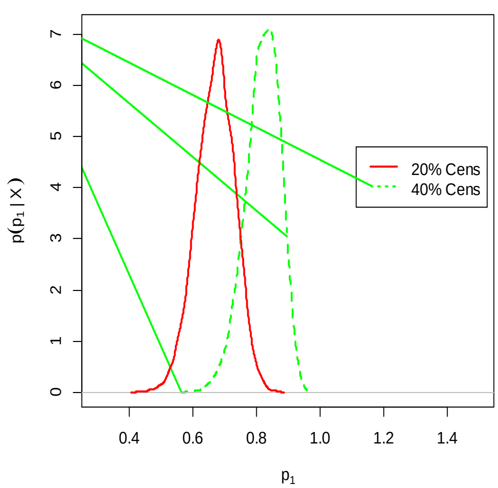

Figure A1m: Posterior Density of p_1 using UP

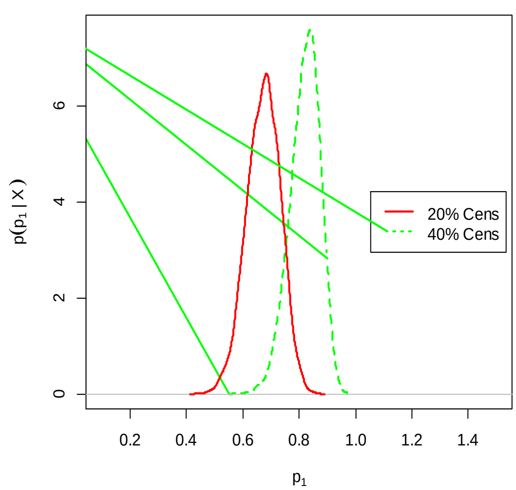

Figure A1n: Posterior Density of p_1 using IP

Figure A1: Graphical depiction of posterior desnities assuming different priors and censoring rates.