Optimal production policy for a closed-loop supply chain with stochastic lead time and learning in production

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Closed-loop supply chain; Stochastic lead time; Learning; Remanufacturing.

Abstract. This paper considers a closed-loop supply chain with one manufacturer and one retailer for trading a single product. On behalf of the manufacturer, the retailer collects the used items from the end customers for possible remanufacturing. The production of finished products (manufactured and remanufactured) is subject to learning. The lead time for the retailer is assumed to be stochastic. The manufacturer delivers the retailer’s order quantity in a number of equal-sized shipments. The objective is to determine the optimal number of shipments and shipment size by minimizing the average expected total cost of the closed-loop supply chain. A solution method for the model is presented, and important results are obtained for numerical examples. According to the numerical study, an impressive cost reduction due to consideration of learning in production and remanufacturing is observed. To investigate the impact of key model parameters on the optimal results, a sensitivity analysis is also carried out. The proposed model is applicable to those business firms whose production process is executed by the human beings.

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1. Introduction

Manufacturing industries all over the world are nowadays giving greater importance to remanufacturing of used items to protect the environment and the society from being over-polluted. Several companies take up the recycling process to add to their organization some value from recovery, too. In electronic industries, many products collected at the end of their useful life may have components with intrinsic economic value. In the steel manufacturing industry, production cost could be reduced by mingling the metals obtained from collected used products with the virgin raw materials. However, the collection of used products from the market and their reinsertion into the downstream flow of materials form closed-loop supply chains. Sometimes, the used products with good quality are returned for new ones to upgrade purposes with some new features or advanced technology; sometimes, these are left at the end of life in poor conditions. Therefore, there is a need for inspection, selection, or a sorting process in order to determine the quality of used items acceptable for recycling. There are plenty of ways in which used items are returned from end customers. The manufacturer can collect the used products through his own channel, and a third party can be appointed for this collection, or even the retailer can be in charge of collecting them on behalf of the manufacturer.

In competitive business environment, most of the production runs are conducted by machines to save time, energy, or achieve errorless production. However, there are also some products that cannot be designed only by a machine; a human mind is required. For
example, consider manufacturing of some sophisticated garment products or leather goods where human involvement is very much needed. When a human being is included in a manufacturing system, the erroneous (learning-forgetting) nature is also included there. Learning nature can be seen in cases of repetitive jobs. When the number of repetition increases, workers become more confident and spend less time doing their jobs. The human characteristic ‘learning’ was well defined by Wright [1] with a curve known as learning curve. According to the theory, it can be stated that when the produced quantity doubles, a certain percentage of cost reduction takes place. In the literature, learning is mostly considered in production in the forward channel. What is the impact of worker learning in the production of finished products (manufactured and remanufactured) in a closed-loop supply chain? How is the situation handled if the delivery lead time from the manufacturer to the retailer is stochastic in a closed-loop scenario? In practice, the delivery lead time cannot be always known as constant; it may vary due to many reasons such as variable transportation time, production time, or loading/unloading time. To find the answers to the above questions, in this paper, stochastic delivery lead time is considered, and its effect jointly with worker learning in production on the performance of the associated closed-loop supply chain is investigated.

2. Literature review

In this section, the literature on two key topics viz. variable/stochastic lead time and learning in production is briefly reviewed. Liao and Shyu [2] and Ben-Daya and Raouf [3] proposed inventory models where lead time was assumed to be a decision variable. Ben-Daya and Haring [4] discussed variable lead time due to production time, transferring time, setup time, etc. Ouyang et al. [5], Hoque and Goyal [6], and Mandal and Giri [7] controlled the lead time with some additional costs. Lin [8] considered variable lead time that can be reduced by investment. Taleizadeh et al. [9] assumed changeable lead time, which is linearly dependent on the lot size. Several other studies were also conducted by Taleizadeh and his coauthors [10-12] considering multi-product single/multi-constraint inventory situations, stochastic replenishment, and dynamic/fuzzy demands.

Yano [13] developed an inventory model with stochastic lead time in a two-level assembly system with the objective of minimizing the sum of the holding cost and tardiness cost. Ouyang et al. [14] extended Yano’s [13] model allowing shortages in inventory. Sajadieh et al. [15] determined an optimal policy for an integrated vendor-buyer model with stochastic lead time. They allowed shortages and assumed batches of equal sizes. Hoque [16] extended the model of Sajadieh et al. [15], assuming unequal batch deliveries from the vendor to the buyer. He considered the lead time distribution as normal instead of exponential distribution adopted by Sajadieh et al. [15]. Maity et al. [17] developed a model with probabilistic lead time and shortages. They assumed a practical pricing decision that can be much effective in real life. Isotupa and Samanta [18] obtained a model with stochastic lead time, which follows an Erlang distribution. Disney et al. [19] ignored the normal distribution of lead time and presented an optimal condition with greater generalization. Roldan et al. [20] presented a survey on inventory-routing problems with stochastic lead time and demand.

Learning is an important human factor that has attracted the attention of many researchers throughout the last few decades. Wright [1] was the first to apply the learning curve to industrial problems. The workers, the employees, even the managers who do the same work repeatedly can automatically do their subsequent jobs with a comparatively better speed. Learning curve is a curve where time per repetition decreases as the number of repetitions increases. Yelle [21] provided a comprehensive survey of works on learning curve. Like learning, forgetting is also an important human factor. Someone who learns in the production period can forget some portion in the non-production period. Elmaghraby [22] and Jaber and Bonney [23] were among the first authors who observed the forgetting phenomenon along with learning. Jaber and Bonney [23] made a comparison of their model with the forgetting model developed by Elmaghraby [22] and Carlson and Rowe [24], and found that their proposed model was more efficient than others. Chiu et al. [25] studied learning and forgetting in the case of production and, also, setup. Khan et al. [26] extended Huang’s [27] model considering learning in production and error in inspections. Burr and Pearne [28], Teng et al. [29], Van Hoof [30], and Grese et al. [31] also studied the learning process. Teyarachakul et al. [32] assumed forgetting curve and compared larger and shorter non-production periods. For a shorter non-production period, the rate of forgetting is low and, also, the batch size reduces. They found optimal policy that implies whether the smaller batch size is more effective or the bigger one. Gloc and Jaber [33] proposed a model for an imperfect production process in which items that were not good in quality were reproduced. In both production and reproduction processes, learning and forgetting both were involved. Lolli et al. [34] highlighted the learning and forgetting curve with imperfect production and rework for imperfect items like Gloc and Jaber [33]. Later, Jaber and Givi [35] assumed that the production interruption and the forgetting occurred at the break time due
to a change in the production and rework processes. Recently, Giri and Glock [36] studied learning and forgetting effects in production as well as inspection. They considered stochastic production return, and observed that more learning effect caused recovery of more used items.

In a Closed-Loop Supply Chain (CLSC), Clark and Scarf [37] first proposed multi-echelon inventory system. Guide and Wassenaar [38] developed a model to investigate profits that can be made by remanufacturing through reusable materials, which can help change many business policies made by different decision-makers in business management. Majumdar and Groenevelt [39] presented a two-period model in the context of recycling with a competing face. In their model, two vendors (one original and the other local) compete in the market for reusable items. It was shown that the local vendor gave motivations for reducing the remanufacturing cost. Koh et al. [40] assumed a model where a fixed portion of used items as remanufactured and the remanufactured items are exactly the same as the freshly manufactured products. Savaskan et al. [41] considered a model where the manufacturer determines how to take the used items from the consumer for remanufacturing. There were three possible choices: manufacturer himself, retailer, and third party. It was shown that the most fruitful way is to assemble the wastage by the retailer. Chung et al. [42] provided some effective policy for remanufacturing with the used items. Jaber and El Saadany [43] assumed that the demands for manufactured and remanufactured items are different. Hong et al. [44] proposed a model where the used items were collected by the manufacturer, the retailer, and the collector (third party) but two of them collected at the same time in all possible ways. De Giovanni and Zaccour [45] made a comparison of different ways of returning used products due to environmental consciousness. Naem et al. [46] discussed the recycling process where demand and return rate were taken as constant and variable both. Giri and Sharma [47] assumed return rate based on product quality. They [48] developed a closed-loop supply chain model with stochastic demand and random return. Cobb [49] proposed a model with an uncertain return rate. Jena et al. [50] proposed a CLSC model with uncertain return rates where advertising plays a vital role in different cases such as advertising by manufacturer, advertising by retailer, centralized advertising, decentralized advertising, etc. Masoudipour et al. [51] also discussed a bi-objective model with a quality-based return rate in a closed-loop supply chain system.

Table 1 shows a comparison of the present model and the relevant existing models. It should be noticed that although significant works have been done considering variable/stochastic lead time and learning/forgetting in production, most of them are bound in EOQ/EPQ model with an integrated or non-integrated system where the manufacturer is not allowed to produce items using the refused ones. Recently, Angius et al. [52] analyzed stochastic lead time in a closed-loop supply chain with a Markovian approach. However, learning in production along with the stochastic lead time has not yet been considered in CLSC. To fill the research gap, this paper studies learning in the production process in CLSC with stochastic lead time. The rest of the paper is organized as follows. Section 3 presents problem description and modeling assumptions. In Section 4, notations are given, which are used throughout the paper. Section 5 deals with the formulation and development of the model. Section 6 exhibits numerical analysis which ends in sensitivity analysis of some important model parameters. The paper is concluded with some managerial insights and future research directions in Section 7.

### Table 1. Comparison between the existing models and our proposed model.

<table>
<thead>
<tr>
<th>Authors</th>
<th>CLSC</th>
<th>Lead time</th>
<th>Learning in production</th>
<th>Inspection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savaskan et al. [41]</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Hoque and Goyal [6]</td>
<td>No</td>
<td>Controlled</td>
<td>No</td>
<td>No</td>
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<tr>
<td>Hoque [16]</td>
<td>No</td>
<td>Stochastic</td>
<td>No</td>
<td>No</td>
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<tr>
<td>Glock and Jaber [33]</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
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<tr>
<td>Khan et al. [26]</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
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<tr>
<td>Disney et al. [19]</td>
<td>No</td>
<td>Stochastic</td>
<td>No</td>
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<td>Angius et al. [52]</td>
<td>Yes</td>
<td>Stochastic</td>
<td>No</td>
<td>No</td>
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<tr>
<td>Giri and Glock [36]</td>
<td>Yes</td>
<td>No</td>
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<tr>
<td>This paper</td>
<td>Yes</td>
<td>Stochastic</td>
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</table>
manufacturer for possible remanufacturing. However, the manufacturer bears the purchase cost and holding cost of returned items. Besides, the following assumptions are made:

(i) The return rate of used items is influenced by the investment in collection and product demands. Here, the investment implies the economic amount of effort, which is applied to the end users to add incentive in order to receive high return;

(ii) The collected used items are inspected. A fraction $\beta$ ($0 < \beta \leq 1$) of inspected used items that qualifies for remanufacturing is accepted and returned to the manufacturer for remanufacturing;

(iii) The demand rate and the production rate of the finished product are both constant. The production rate is greater than the demand rate;

(iv) Delivery lead time is stochastic and independent of deliveries to the retailer;

(v) The retailer’s inventory is continuously reviewed, and a batch is requested to deliver when the stock level drops to a certain level;

(vi) Shortages in the retailer’s inventory are allowed and are completely backlogged;

(vii) A joint setup is made for manufacturing and remanufacturing. Remanufactured products are as good as manufactured products, i.e., it is not possible to distinguish between manufactured and remanufactured products [37];

(viii) One unit of raw material is required to produce one unit of the finished good;

(ix) Learning occurs during the production of the finished goods (manufactured and remanufactured). Learning leads to an increase in the rate of production over time and depends on the length of the production interval.

A configuration of the proposed model is depicted in Figure 1, which illustrates the flow of products throughout the closed-loop supply chain. The return rate of used items is a fraction of demand, which is influenced by the investment. A fraction of the returned items is found acceptable for remanufacturing. The rejected items are disposed with a disposal cost.

4. Notations

In this section, the notations used to develop the proposed model are described.

**Manufacturer:**

- $c_m$: Production cost per unit time
- $A_m$: Setup cost per set up
- $P$: Production rate
- $h_m$: Holding cost of the finished product per unit per unit time
- $h_{mu}$: Holding cost of returned items per unit per unit time
- $c_{mr}$: Unit purchase cost of raw material
- $c_{mu}$: Unit cost for purchase and recovery of used items selected for remanufacturing
- $c_{md}$: Unit disposal cost
- $\beta$: Fraction of used items acceptable for remanufacturing, $0 < \beta \leq 1$
- $n$: Number of shipments

![Figure 1. Product flow diagram.](image-url)
Learning exponent, \((0 < b < 1)\)

Manufacturer’s production time to produce the first unit in case of learning

Manufacturer’s production (non-production) period in cycle \(i\)

Total cost per unit time for the \(i\)th cycle

Return rate of used items

\(D\)  
Demand rate

\(A_r\)  
Ordering cost per order

\(h_r\)  
Holding cost per unit per unit time

\(s\)  
Shortage cost per unit per unit time

\(T_r\)  
Length of an ordering cycle

\(TC_r\)  
Total cost per unit time

\(Q\)  
Batch size

\(L\)  
Lead time as a random variable

\(\sigma\)  
Standard deviation of lead time

\(f_L(l)\)  
Probability density function of \(L\)

\(k\)  
Reorder level

\[ b \]

5. Model formulation and analysis

It is assumed that the manufacturer produces and delivers a lot of size \(nQ\) to the retailer in \(n\) equal-sized batches of size \(Q\). Since the lead time is random, the retailer requests the manufacturer for a batch when the stock level drops to the reorder level \(k\). Therefore, the next batch is expected to receive when the stock at the reorder point comes to zero level, i.e., after the mean lead time \(k/D\) [16]. The deviation of lead time may occur due to the variation of loading/unloading time. The used items collected by the retailer are inspected and, after inspection, only a fraction is accepted for remanufacturing. The manufacturer has to order that amount of fresh raw materials that cannot be met by the accepted used items. The manufacturer bears all the costs for holding, promotion for collection, and inspection of used items.

5.1. Retailer’s total cost

When random variable \(L\) takes value \(l\), the following three cases that may arise are considered:

- **Case (i):** A batch \(Q\) arrives early for the retailer, i.e., \(0 < l < k/D\). In this case, using Figure 2(a), the inventory holding area of the retailer for one batch can be calculated as given below:

\[
\text{Inventory holding area} = \text{Area}(\square OPNM + \triangle QRS + \square P'TSQ)
\]

\[
\frac{1}{2}[(k-Dl+k)l] + \frac{1}{2}[(k-Dl+Q-k)\left(\frac{Q-Dl}{D}\right)] + \frac{(Q-Dl)k}{D} - \frac{1}{2}\left[\frac{Q^2}{D} + 2Q\left(\frac{k}{D} - l\right)\right],
\]

where:

\[
k = \frac{DQ}{P}
\]

Thus, for the entire lot, \(Q_p\), of the manufacturer, the expected inventory holding cost of the retailer is computed as follows:

\[
h_r \int_0^{k/D} \left[\frac{Q^2}{D} + 2Q\left(\frac{k}{D} - l\right)\right] f_L(l)dl.
\]

- **Case (ii):** A batch \(Q\) arrives late for the retailer limiting the lead time \(l\) in the range \(\frac{k}{D} \leq l < \frac{k+Q}{D}\). In this case, shortages may occur. Based on Figure 2(b), the inventory shortage area for a single batch \(Q\) is obtained as \(\frac{1}{2}\pi (Dl - k)^2\). Hence

![Figure 2. Retailer’s inventory under stochastic lead time.](image)

\[\text{Figure 2. Retailer’s inventory under stochastic lead time.}\]
the retailer’s expected shortage cost for all batches is as follows:

\[ n_s \int_{k/D}^{k+Q} \left( \frac{Dl - k}{D} \right) (Dl - k) f_L(l) \, dl. \]

Similar to Case (i), the inventory holding area for one batch \( Q \) is obtained as follows:

Area (\( \triangle QMR + \triangle STU + \square POUS \))

\[ = \left[ \frac{k^2}{2D} + \frac{(Q - Dl)^2}{2D} + \frac{k(Q - Dl)}{D} \right] \]

\[ = \frac{1}{2D} (Q - Dl + k)^2. \]

Therefore, the expected inventory holding cost of the retailer for all batches is as follows:

\[ nh_r \int_{k/D}^{k+Q} \frac{1}{2D} (Q - Dl + k)^2 f_L(l) \, dl. \]

- **Case (iii):** A batch \( Q \) arrives late for the retailer limiting the lead time \( l \) in the range \( \frac{k+Q}{D} \leq l < \infty \). In this case, there is no inventory, except only shortages (see Figure 2(c)). The shortage area for one batch \( Q \) is given by:

Area (\( \square PQRS \)) = \( \frac{Q^2}{2D} \) + Area (\( \square DOQRS \))

\[ = \frac{Q^2}{2D} + Q \left( l - \frac{Q + k}{D} \right). \]

Therefore, the expected shortage cost for all batches is:

\[ n_s \int_{\frac{k+Q}{D}}^{\infty} \left[ Ql - \frac{Q^2}{2D} - \frac{Qk}{D} \right] f_L(l) \, dl. \]

Considering all the three cases, the retailer’s expected inventory holding cost for all batches is:

\[ nh_r \left[ \frac{1}{2} \int_0^{k/D} \left( \frac{Q^2}{D} + 2Q \left( \frac{k}{D} - l \right) \right) f_L(l) \, dl \right. \]

\[ + \left. \int_{\frac{k+Q}{D}}^{\infty} \frac{1}{2D} (Q - Dl + k)^2 f_L(l) \, dl \right] , \]

and the expected shortage cost for all batches is:

\[ n_s \int_{\frac{k+Q}{D}}^{\infty} \left( \frac{1}{2} \left( \frac{Dl - k}{D} \right) (Dl - k) f_L(l) \, dl \right. \]

\[ + \left. \int_{\frac{k+Q}{D}}^{\infty} \left( Ql - \frac{Q^2}{2D} - \frac{Qk}{D} \right) f_L(l) \, dl \right] . \]

Hence, the expected total cost of the retailer for one cycle is:

\[ TC_r = A_r + \frac{nhr}{2} \left[ \int_0^{k/D} \left( \frac{Q^2}{D} + 2Q \left( \frac{k}{D} - l \right) \right) f_L(l) \, dl \right. \]

\[ + \left. \int_{\frac{k+Q}{D}}^{\infty} \frac{1}{D} (Q - Dl + k)^2 f_L(l) \, dl \right] \]

\[ + \left. \frac{n}{2} \int_{\frac{k+Q}{D}}^{\infty} \left( Ql - \frac{Q^2}{2D} - \frac{Qk}{D} \right) f_L(l) \, dl \right] . \] (1)

5.2. Manufacturer’s cost

5.2.1. Change in holding cost

The manufacturer’s change in holding cost due to random delivery lead time is first calculated. In Case (ii), i.e., when batch \( Q \) arrives late by a time \( l - k/D \) limiting \( l \) to the range \( \frac{k}{D} \leq l < \frac{k+Q}{D} \), the batch is kept with the manufacturer over the delay period and it creates an extra inventory of \( nQ(l - k/D) \). Thus, in this case, the expected additional inventory holding cost of the manufacturer is:

\[ h_m \int_{k/D}^{k+Q} nQ(l - k/D) f_L(l) \, dl. \]

In Case (iii), i.e., when batch \( Q \) arrives late by a time \( l - k/D \) limiting \( l \) to the range \( \frac{k+Q}{D} \leq l < \infty \), the batch is kept with the manufacturer over the delay period and it creates an extra inventory of \( nQ(l - k/D) \). Thus, the expected additional inventory holding cost of the manufacturer, in this case, is as follows:

\[ h_m \int_{\frac{k+Q}{D}}^{\infty} nQ(l - k/D) f_L(l) \, dl. \]

Therefore, the expected net change in the manufacturer’s holding cost due to stochastic delivery lead time
at the retailer is:

\[ h_m \left( \int \frac{nQ(l-k/D)f_L(l)dl}{l} \right) \]

5.2.2. Learning in production

Since learning in production has impact on production time, in general, the ith production cycle is considered. The production time in the ith cycle [26] is given by:

\[ T_{pi} = \int_{(i-1)Q_p}^{iQ_p} T_1 x^{-b}dx = \frac{T_1}{1-b} Q_p^{1-b} \left[ (i-1)^{1-b} - (i-1)^{1-b} \right] \]

Based on the above, the production quantity in the ith cycle can be written as a function of time, \( t \), as follows:

\[ Q(t) = \left[ \frac{(1-b)t}{T_1} - \frac{1}{T_1} \right] \left[ (i-1)^{1-b} - (i-1)^{1-b} \right] \]

Then, the average inventory in the ith cycle during production is as follows:

\[ IT_{pi} = \frac{T_{pi}}{0} \int Q(t)dt = \frac{T_1}{2-b} \left[ (i-1)^{1-b} - (i-1)^{1-b} \right] (nQ)^{2-b}. \]

Therefore, the time for the first dispatch after the start of production in the ith cycle is as follows:

\[ T_{ii} = \int_{(i-1)nQ}^{Q+(i-1)nQ} T_1 x^{-b}dx \]

\[ = \frac{T_1}{1-b} Q^{1-b} \left[ (1+(i-1)n)^{1-b} - ((i-1)n)^{1-b} \right]. \]

Now, the manufacturer’s inventory in the non-production period of the ith cycle is calculated based on Figure 3 [26] as follows:

\[ IT_{di} = \text{Area (\square ABFE + \square IHGF - \square ACGE)} \]

\[ = \frac{nT_1}{1-b} Q^{2-b} \left[ (1+(i-1)n)^{1-b} - ((i-1)n)^{1-b} \right] \]

\[ + \frac{n(n-1)Q^2}{D} - \frac{T_1(i^{1-b} - (i-1)^{1-b})}{2-b}(nQ)^{2-b}. \]

The total inventory moved from the manufacturer to the retailer in a cycle is \( \frac{n(n-1)Q^2}{2D} \). Thus, the manufacturer’s average inventory in the ith cycle is:

\[ h_m \left[ \int_{k/D}^{\infty} nQ(l-k/D)f_L(l)dl \right]. \]

Figure 3a. Manufacturer’s inventory level under learning.

Figure 3b. Manufacturer’s total inventory under learning.

\[ IT_{pi} + IT_{di} - \frac{n(n-1)Q^2}{2D} \]

\[ = \frac{nT_1}{1-b} Q^{2-b} \left[ (1+(i-1)n)^{1-b} - ((i-1)n)^{1-b} \right] \]

\[ + \frac{n(n-1)Q^2}{2D} - \frac{T_1(i^{1-b} - (i-1)^{1-b})}{2-b}(nQ)^{2-b}. \]

Therefore, considering the result of Subsection 5.2.1, the manufacturer’s expected holding cost for the ith cycle is:

\[ h_m \left[ \int_{k/D}^{\infty} nQ(l-k/D)f_L(l)dl \right]. \]

Hence, for the ith cycle, the manufacturer’s expected total cost, which is the sum of the setup cost, produc-
tion cost, and expected holding cost, is as follows:

\[ TC_{mi} = A_m + c_m T p_i + b_m \int_{\frac{k}{D}}^{\infty} nQ(l - r/D) f_l(l) dl \]

\[ + \frac{nT_i Q^{1-b}}{1-b} ((i+1)(i-1)^{1-b} - ((i-1)n)^{1-b}) \]

\[ + \frac{n(n-1)Q^2}{2D} - \frac{T_i}{2-b(1-b)} \int_{k/D}^{\infty} nQ l^{n-1-b} \int_{0}^{T_i} I(t) dt \]  

(8)

5.2.3. Cost for used items

It is assumed that the used items are collected by the retailer on behalf of the manufacturer at a rate, \( R \), which is influenced by the collection investment and demand. The Collection Investment (CI) indicates the economic amount of effort (e.g., promotion, marketing, etc.), which is applied to the end users to create the necessary incentive to receive targeted return. It is assumed that \( R = \sqrt{\frac{CI}{\gamma}} D \) where \( \gamma \) is a scaling parameter and \( \sqrt{\frac{CI}{\gamma}} < 1 \). When the collected items are used for remanufacturing, the inventory level of the collected items after time \( t \) is given by:

\[ I(t) = n\beta RT r_i - Q(t) = \sqrt{\frac{CI}{\gamma}} (n\beta Q) - Q(t). \]  

(9)

If the stock of the collected used items is exhausted after a time period \( T_2 \), then we have \( I(T_2) = 0 \). This gives:

\[ Q(T_2) = \sqrt{\frac{CI}{\gamma}} (n\beta Q), \]  

(10)

which further gives:

\[ T_2 = \left[ \frac{\sqrt{\frac{CI}{\gamma}} (n\beta Q)^{1-b}}{1-b} \right] \frac{T_i}{1-b} \int_{0}^{T_i} I(t) dt \]  

(11)

Using Figure 4, the inventory holding area for collected used items is calculated as follows:

Area (\( \triangle ABC + EFC \))

\[ = \frac{1}{2} Re T r_i (nT r_i) + \int_{0}^{T_2} I(t) dt \]

\[ = \frac{1}{2} D \sqrt{\frac{CI}{\gamma}} n^2 Q^2 + \int_{0}^{T_2} I(t) dt. \]  

Therefore, the total cost for remanufacturing is:

\[ TC_{mu} = b_m \left[ \int_{0}^{T_3} I(t) dt \right] + c_{mu} nQ. \]

\[ \alpha = \sqrt{\frac{CI}{\gamma}}. \]  

(12)

5.3. Total cost of the CLSC

By combining the retailer’s cost and the manufacturer’s cost as derived from Subsections 5.1 and 5.2, respectively, the average expected cost of the closed-loop supply chain for the \( i \)th cycle is given by:

\[ TC_i(n, Q) = \frac{1}{nT r_i} [TC_r + TC_{mi} + c_{mr} (1 - \alpha) n Q \]

\[ + c_{md} (1 - \beta) \alpha n Q + TC_{mu}] = \frac{1}{nT r_i} \left[ A_r \]

\[ + \frac{nh_r}{2} \left( \int_{k/D}^{k/D} \left( \frac{Q^2}{D} + 2Q \left( \frac{k}{D} - \frac{D}{l} \right) f_l(l) dl \right) \right) \]

\[ + \frac{1}{D} \left( Q - Dl + k \right) f_l(l) dl \]  

\[ + \frac{n s}{D} \left( \int_{k/D}^{k/D} \left( \frac{Dl - k}{D} \right) (Dl - k) f_l(l) dl \right) \]

\[ + \int_{\frac{k}{D}}^{\infty} \left( Ql - \frac{Q^2}{2D} - \frac{Qk}{D} \right) f_l(l) dl \]  

\[ + A_m \]
\[ \frac{\partial^2 T C_i}{\partial n^2} = \frac{2(A_r + A_m + c_m T_p)}{n^3 T_{ri}} + \frac{2 h_{mu} T_i}{n^3 T_{ri}} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)} \frac{i}{\pi} dt + \frac{b h_{mu} T_i Q^{2-b}}{T_{ri}} \frac{[(i - 1)^{1-b} - (i - 1)^{1-b}]}{P} \]

Our objective is to find the optimal values of \( n \) and \( Q \) that minimize the average expected total cost \( T C_i(n, Q) \), \( i = 1, 2, \ldots \). Since \( n \) is discrete and \( Q \) is continuous, it is not possible to apply the calculus method to optimize \( T C_i(n, Q) \) with respect to \( n \) and \( Q \) jointly. First, it should be assumed that \( n \) is real, not just an integer. Then, the following results are presented:

**Theorem 1.** The average expected total cost function \( T C_i(n, Q) \) is convex in \( n \) for any given \( Q \), \( i = 1, 2, 3, \ldots \).

**Proof:** Differentiating Eq. (13) with respect to \( n \), we have:

\[ \frac{\partial T C_i}{\partial n} = \frac{A_r + A_m + c_m T_p}{n^2 T_{ri}} + \frac{h_{mu} Q^2}{2 D T_{ri}} + \frac{h_{mu} T_i Q^{2-b}}{2 D T_{ri}} \]

\[ \frac{T_i}{T_{ri}} \left( 1 + (i - 1) n - (i - 1) \right) \]

\[ - (i - 1)^{1-b} n^{-b} \]

\[ - \frac{h_{mu} T_i (i^{1-b} - (i - 1)^{1-b})}{(2 - b) T_{ri}} n^{-b} Q^{2-b} \]

\[ - \frac{h_{mu} T_i Q^{2-b}}{2 - b} \]

\[ \frac{2 h_{mu} [\alpha \beta(nQ)]^{2-b}}{n^3 T_{ri}(2 - b)} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

\[ \frac{2 h_{mu} [\alpha \beta(nQ)]^{2-b}}{n^3 T_{ri}(2 - b)} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

Differentiating again with respect to \( n \), we get:

\[ \frac{\partial^2 T C_i}{\partial n^2} = \frac{2(A_r + A_m + c_m T_p)}{n^3 T_{ri}} + \frac{2 h_{mu} T_i}{n^3 T_{ri}} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt + \frac{b h_{mu} T_i Q^{2-b}}{T_{ri}} \frac{[(i - 1)^{1-b} n^{-b} - (i - 1)^{1-b}]}{P} \]

\[ \frac{2 h_{mu} T_i Q^{2-b}}{T_{ri}} \frac{[(i - 1)^{1-b} n^{-b} - (i - 1)^{1-b}]}{P} \]

\[ \frac{2 h_{mu} T_i Q^{2-b}}{T_{ri}} \frac{[(i - 1)^{1-b} n^{-b} - (i - 1)^{1-b}]}{P} \]

since:

\[ \frac{2 h_{mu} T_i}{n^3 T_{ri}} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

\[ \frac{2 h_{mu} T_i}{n^3 T_{ri}} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

\[ \frac{2 h_{mu} [\alpha \beta(nQ)]^{2-b}}{n^3 T_{ri}(2 - b)} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

\[ \frac{2 h_{mu} [\alpha \beta(nQ)]^{2-b}}{n^3 T_{ri}(2 - b)} \int_0^{T_i} \frac{(1 - b) P t}{(1 - b)^i - (i - 1)^{1-b}} \frac{i}{\pi} dt \]

Since \( 0 < b < 1 \) and based on Eq. (15), the third term is positive as \((i - 1)n)^{-b} > (1 + (i - 1)n)^{-b} \) and each of the second and fourth terms is positive as \( i^{1-b} > (i - 1)^{1-b} \) for all \( i \). Therefore, \( \frac{\partial^2 T C_i}{\partial n^2} > 0 \), implying that \( T C_i(n, Q) \) is convex in \( n \).

**Theorem 2.** If \( h_{mu} > h_r/2 \), then the average expected cost function \( T C_i(n, Q) \) is convex with respect to \( Q \) for all \( Q < Q_i^1 \) where \( Q_i^1 \) is obtained by the equation shown in Box 1.
\[
Q_i^* = \frac{(b + 1)\alpha n^{-\beta}i^{\beta-1} - (i - 1)^{\beta-1}}{h_m((1 + i - 1)n)^{1 - b} - ((i - 1)n)^{1 - b}} + \frac{[i^{\beta-1} - (i - 1)^{\beta-1}] [h_{mu}(\alpha \beta)^{\beta - 1}n^{1 - b} - h_mn^{-b}]}{2 - b},
\]

\[
i = 1, 2, 3, \ldots
\]

**Proof:** Differentiating Eq. (13) twice with respect to \(Q\), we get:

\[
\frac{\partial^2 TC_i}{\partial Q^2} = \frac{2D}{nQ_i^2}(A_r + A_m) + \frac{\exp \frac{-(\alpha^{\beta})}{h_m}}{2\pi \sigma D} \left( \frac{h_m}{D - h_r} \right) + \frac{b(b + 1)\alpha n^{-\beta}i^{\beta-1} - (i - 1)^{\beta-1}}{(1 - b)}
\]

\[
- h_m DT_1 Q^{-b-1}((1 + i - 1)n)^{1 - b} - ((i - 1)n)^{1 - b})
\]

\[
+ h_m DT_1 Q^{-b-1}n^{-\beta}i^{\beta-1} - (i - 1)^{\beta-1})
\]

\[
+ \int_{-\infty}^{\infty} \frac{(Dl - k)^2(h_r + s)}{Q^3} \cdot f_L(l)dl
\]

\[
- \frac{b}{2 - b} h_{mu} DT_1 (\alpha \beta)^{\beta - 1}n^{1 - b} Q^{-b-1}
\]

\[
(i^{\beta-1} - (i - 1)^{\beta-1})),
\]

(17)

The second term in the right-hand side of Eq. (17) is positive if \(h_m > h_r/2\). The sixth term is positive as it is the sum of holding cost and shortage cost of retailer. The sum of the third, fourth, fifth, and seventh terms will be positive if:

\[
(b + 1)\alpha n^{-\beta}i^{\beta-1} - (i - 1)^{\beta-1}
\]

\[
> h_m((1 + i - 1)n)^{1 - b} - ((i - 1)n)^{1 - b})
\]

\[
+ \frac{[i^{\beta-1} - (i - 1)^{\beta-1}] [h_{mu}(\alpha \beta)^{\beta - 1}n^{1 - b} - h_mn^{-b}]}{2 - b},
\]

i.e.: \(Q < Q_i^*\), \(i = 1, 2, 3, \ldots\)

Therefore, if \(h_m > h_r/2\), then we see that \(\frac{\partial^2 TC_i}{\partial Q^2} > 0\) for all \(Q < Q_i^*, i = 1, 2, 3, \ldots\)

Hence, the theorem is proved.

**Solution algorithm:** In the following, the step by step procedure is used to find the optimal values of \(n\) and \(Q\) for successive production cycles [26].

**Step 1.** Set \(i = 1\) and \(n = 1;\)

**Step 2.** Find the optimal value of \(Q\) from (13) and, then, determine the corresponding value of \(TC_i\). Set \(n = n + 1;\)

**Step 3.** Repeat Step 2 until the value of \(TC_i\) for the \(n\)th batch is greater than that for the \((n - 1)\)th batch;

**Step 4.** Write \(Q^* = Q\) and \(TC_i^* = TC_i\) for \(n^* = n - 1;\)

**Step 5.** Set \(i = i + 1\) and \(n = 1;\)

**Step 6.** Repeat Steps 2 to 4 up to a certain value of \(i\), say, \(i = 10.\)

6. **Numerical example**

Now, it is time to demonstrate the proposed model numerically with the following parameter values: \(c_m = $100000\) per year, \(A_m = $400\) per set up, \(h_m = $4\) per unit per year, \(h_{mu} = $3\) per unit per year, \(c_{mr} = $200\) per unit, \(c_{mu} = $50\) per unit, \(c_{md} = $1\) per unit, \(b = 0.32, R = \sqrt{\frac{C_t}{D}}\) where \(\sqrt{\frac{C_t}{D}} = 0.65, \beta = 0.8,\)

\(P = 3200\) units per year, \(D = 10000\) units per year, \(A_r = $100\) per order, \(h_r = $5\) per unit per year, and \(s = $8\) per unit per year.

The probability density function of \(L\) is also assumed here:

\[
f_L(l) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(l - \mu)^2},
\]

where \(\sigma = 0.12.\)

For this data set, we check that the average expected cost function \(TC_i(n, Q)\) is convex with respect to \(Q\), see Figure 5.

For \(i = 1, the optimal solution is obtained as \(n^* = 4\) and \(Q^* = 204.245\), and the corresponding average expected cost of the closed-loop supply chain is \(TC = $137195.\) Table 2 shows the optimal results of consecutive ten cycles.

One can see from Table 2 that there is an impressive cost reduction in the first few cycles. After that, the cost reduction slows down and the cost curve almost plateaus after 8 or 9 cycles. A similar observation is made in production time \(T_{pi}\). This happens because of the human factor 'learning' in production. Initially, workers do their jobs faster than the previous
cycle; however, after a few production cycles, their efficiency reaches a threshold level, and no significant improvement in their actions is observed. Table 3 presents a comparison of the optimal results of the proposed model and the base model, ignoring learning in production. As observed from Table 3, the proposed model with learning in production provides lower cost than the model without learning in production (base model).

Now, we take $\alpha = \sqrt{\frac{2}{\pi}}$ and discuss the effect of $\alpha$ on the average expected total cost. Table 4 shows the optimal results for different values of $\alpha$. Here again, the average values of the results are calculated considering the first 10 cycles for each $\alpha$.

When $\alpha = 0$, i.e., there is no returned item, the average expected cost is higher than that of the case of returned items. As $\alpha$ increases, the average expected cost decreases. This implies that it is beneficial for the manufacturer to collect the returned items as much as possible. Based on Figure 6, it is obvious that as $\beta$ increases, the average expected total cost decreases. The average expected cost is maximum when $\beta = 0$, i.e., no used items are accepted for remanufacturing.

We now look into the ratio of $P/D$ keeping $P$ and $D$ fixed alternatively. As the ratio $P/D$ approaches 1, i.e., the demand rate is getting closer to the production rate, the average expected cost increases. Similarly, Figure 7 depicts that the average expected cost decreases steadily with the increasing value of $P/D$. Now, keeping $D$ fixed, if $P = 4000$, $5000$, $6000$, and $7000$ are considered, it is seen from Table 5 that the rate of change in the average expected cost is very small and gradually decreases. It ultimately becomes stable after a certain higher value of $P$.

### Table 2. Optimal results of consecutive ten cycles.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$n^*$</th>
<th>$Q^*$</th>
<th>$T_{p^*}$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>204.245</td>
<td>0.0139</td>
<td>137195</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>218.083</td>
<td>0.0228</td>
<td>134060</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>217.257</td>
<td>0.0186</td>
<td>134416</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>201.121</td>
<td>0.0163</td>
<td>134107</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>196.961</td>
<td>0.0148</td>
<td>133896</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>193.871</td>
<td>0.0137</td>
<td>133740</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>191.446</td>
<td>0.0129</td>
<td>133616</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>189.467</td>
<td>0.0123</td>
<td>133515</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>265.536</td>
<td>0.0112</td>
<td>133419</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>263.653</td>
<td>0.0108</td>
<td>133344</td>
</tr>
</tbody>
</table>

### Table 3. Comparison of the results of the base model and the proposed model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$n^*$</th>
<th>$Q^*$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base model</td>
<td>4</td>
<td>115.1</td>
<td>162700</td>
</tr>
<tr>
<td>Proposed model</td>
<td>3</td>
<td>222.84</td>
<td>134220.8#</td>
</tr>
</tbody>
</table>

# average of the optimal results of the first 10 cycles.

### Table 4. Optimal results for different values of $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n^*$</th>
<th>Average $Q^*$</th>
<th>Average $TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3</td>
<td>232.92</td>
<td>204917.4</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
<td>224.13</td>
<td>17733.2</td>
</tr>
<tr>
<td>0.45</td>
<td>3</td>
<td>223.88</td>
<td>155066</td>
</tr>
<tr>
<td>0.65</td>
<td>3</td>
<td>222.81</td>
<td>134220.8</td>
</tr>
<tr>
<td>0.85</td>
<td>2</td>
<td>242.59</td>
<td>112394</td>
</tr>
</tbody>
</table>

### Table 5. Sensitivity of the average expected cost with respect to $P$.

<table>
<thead>
<tr>
<th>$P$</th>
<th>Average $TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3200</td>
<td>134220.8</td>
</tr>
<tr>
<td>4000</td>
<td>133571</td>
</tr>
<tr>
<td>5000</td>
<td>133067.4</td>
</tr>
<tr>
<td>6000</td>
<td>132730</td>
</tr>
<tr>
<td>7000</td>
<td>132488.8</td>
</tr>
</tbody>
</table>

Figure 5. Convexity of $TC(n, Q)$ for the first cycle, given $n = 4$.

Figure 6. $\beta$ versus average expected total cost.

Figure 7. Average expected cost versus $P/D$ ratio.
Figure 7. $P/D$ versus average expected total cost.

Figure 8. Raw material cost versus average expected total cost.

Figure 9. Learning exponent $b$ versus average production rate.

Figure 10. Learning exponent $b$ versus average expected total cost.

Figure 11. Standard deviation of lead time versus average expected total cost.

The proposed model can be applied to several industrial problems in practice. Some real applications of the key topics of this study observed with positive results throughout the years are given below:

- Primarily, the remanufacturing process was experienced (industrial level) by tank remanufacturing in World War I. After World War II in the UK, all the car manufacturers found the recycling process
profitable. Therefore, Caterpillar and Xerox have
developed ongoing revenue opportunities from the
2nd, 3rd, ... nth life products. At the expert
remanufacturing workshop (CSISD, 2006), Xerox re-
ported that the products could be remanufactured
and useful up to the 7th life [53];
• In European Union, new regulations on end of life for
automotive products (2000/EC), electrical and elec-
tronic equipment (2003/EU) are made to protect
the environment from pollution due to waste [54];
• In 1990, Kodak initiated a single-use camera recov-
ery program to recycle and reuse the items. In 2003,
the return rate was more than 70% in the United
States and 60% throughout the whole world [55];
• Xerox Europe responsible for 25% of Xerox world-
wide business launched regulations for a waste-free
company, which results in cost saving of Xerox of
over US$76 million in 1990. Equipment recycling
was a fundamental approach to achieving Waste-
Free Product goals. Today, 90% of Xerox-designed
equipment is recyclable, which detracts about 145
million pounds of waste from landfills in 1998 [55];
• Using data from the Census of Industrial Production
(CIP) from the Department of Statistics, Ministry
of Trade and Industry in Singapore for the years
1980 to 2007, it was observed that, in the transport
equipment industry and the electronic industry,
when the gained skill doubles, the unit labor input
is reduced to 30% and 56% of the initial labor,
respectively [56].

7. Conclusion

The paper analyzed a closed-loop supply chain with a
manufacturer and a retailer where the retailer collects
the used items on behalf of the manufacturer. The
production of the finished goods (manufactured and
remanufactured) is subject to learning, i.e., the pro-
duction process is executed by workers who perform the
same job again and again with increasing speed.
The proposed model was formulated considering stochastic
lead time.

Several managerial insights can be drawn from the
findings of the proposed model:

1. As more returned items with a greater fraction of
accepted used items reduce the expected total cost of
the CLSC, managers should strive for higher
return rates. They should determine a suitable
offer price to attract more customers to sell their
rejected items. In doing so, management not only
improves the intrinsic economic value of used items,
but also protects the environment from pollution
due to possible improper disposal of used products
by the end users;

2. As observed in the numerical study, a higher value
of learning exponent leads to a significant cost
reduction of the whole system. However, after a few
production cycles, the cost curve plateaus. At
that time, the management has to make a decision
whether investment in providing supports, such as
sophisticated equipment, technology, etc., for the
workers would be a viable tool to improve further
the performance of the workers or not.

In practice, the proposed model can be applied to the
manufacturing systems where the production largely
depends on workers/laborers, e.g., garment factories,
leather goods manufacturing industries, etc. [26].
There are many ingredients that can be used for future
research. For example, the model can be extended
by incorporating stochastic demand, which may be
influenced by factors such as price, quality, availability,
etc. Further, there can be errors in inspection. As
human being is involved in inspection, there can be
two error types: wrong acceptance of bad items and
wrong rejection of good items. One may also consider
forgetting in production along with learning; the work-
ers can do a better job by repeated work; however, at
the same time, they can lose the experience if there is
a significant gap after learning.

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