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# An analytic and mathematical synchronization of micropolar nanofluid by Caputo-Fabrizio approach

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Suspension of nanoparticles in base fluid.

**Abstract.** Nanofluids and the enhancement of heat transfer in real systems have proved to be a widely researched area of nanotechnology; other areas of particular interest to researchers include the improvement of thermal conductivity, thermophoresis phenomenon, dispersion of nanoparticles volume fraction, and few others. Based on the touch of nanotechnology, this study investigates the analytic and mathematical performance of micropolar nanofluid in the enhancement of heat transfer. The base fluid is taken for the purpose of thermal conductivity subject to two types of nanoparticles: copper and silver. The mathematical analysis of the micropolar nanofluid was carried out by invoking the non-integer order derivative and transform methods. By applying a mathematical tool to the equations of micropolar nanofluid, the solutions were explored for temperature, microrotation, and velocity. In order to meet the physical aspects of the problem based on micropolar nanofluid, the comparison of velocity field of micropolar nanofluid for the suspension of ethylene glycol into silver and that of ethylene glycol into copper is made to enhance the rate of heat transfer.



## 1. Introduction

Nanoparticles are small quantities of nanometer-sized particles that are composed of metals, e.g., copper oxide, alumina, carbides, titania, gold, copper, and several others. These nanoparticles have the capability to enhance the thermophysical properties of base fluids including ethylene glycol, biofluids, oil, water, lubricants, polymer solutions, and many others. Next, kerosene, engine oil, water, and ethylene glycol are insufficient for heat transfer due to their low thermal conductivity. In order to enhance the convective heat transfer performance of such fluids, various techniques have been implemented such as boundary conditions and changing flow geometries. There is no doubt that fluids have lower thermal conductivity than metals. The thermal conductivity of base fluids can be enhanced by adding metals and, consequentially, such fluids are characterized as nanofluids [1]. Moreover, the fundamental concepts of micropolar fluid originated from Eringen's study [2] to characterize the dynamics of such fluids by considering the microscopic impacts rising from micro-motion and local structure of the fluids. A broad assessment of the micropolar fluid with its engineering applications was conducted by Ariman et al. [3,4]. Hassanien and Gorla [5] explored the effects of nonisothermal stretching sheet on micropolar with heat transfer, blowing, and suction. Mohammadein

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and Gorla [6] analyzed internal heat generation and dissipation with heat transfer in micropolar. Hussanan et al. [7] obtained a closed-form solution for Newtonian heating in micropolar fluid to the problem of free convection flow. Choi [8] studied the application of non-Newtonian fluid flow with nanoparticles to enhance the thermal conductivity of fluid. Congedo et al. [9] carried out the analysis and modeling of nanofluids with natural convection heat transfer. Ghasemi and Aminossadati [10] traced the impacts of water-CuO nanofluid along with natural convection heat transfer. Hussanan et al. [11] examined the effect of some nanofluids on accelerated plate with magnetic field and a porous medium. The water functionalized carbon nanotube flow was analyzed over a moving/static wedge in the magnetic field by Khan et al. [12]. Haq et al. [13] explored the effects of magnetic field with water functionalized metallic nanoparticles for squeezed flow over a sensor surface. In short, this study includes few latest references on nanofluids [14-16], modern fractional derivatives [17-22], heat transfer [23-27], nanoparticles [28-31], porosity and magnetic field [32-37], and few different circumstances [38-43]. Motivated by the above research work on nanofluids, the purpose of this study is to investigate the analytic and mathematical performance of micropolar nanofluid to enhance heat transfer. The base fluid is taken for the purpose of thermal conductivity subject to two types of nanoparticles, namely copper and silver, as shown in Figure 1(a)-(c). The mathematical analysis of the micropolar nanofluid has been carried out by invoking the non-integer order derivative and transform methods. By applying a mathematical tool to the equations of micropolar nanofluid, the solutions have been explored in terms of temperature, microrotation, and velocity. In order to meet the physical aspects of the problem based on micropolar nanofluid, the comparison of velocity field of micropolar nanofluid for the suspension of ethylene glycol into silver and that of ethylene glycol into copper is made to enhance the rate of heat transfer.

## 2. Mathematical equations of nanofluid

An unsteady flow of ethylene glycol based on micropolar nanofluid occupies the space lying over an oscillating plate perpendicular to the $y$-axis and is situated on the $(x, z)$ plane. Initially, due to the constant temperature of $T_w$, the fluid is considered at rest. At $t = 0^+$, the plate starts to oscillate with the velocity $UH(t)\cos\omega t$ or $U\sin\omega t$ on its plane and the level of temperature increases up to $T_w$. The governing equations for the micropolar fluid are as follows:

$$\nabla.(\rho_{nf}\mathbf{V}) = \frac{\partial \rho_{nf}}{\partial t}, \tag{1}$$

$$\begin{aligned}\nabla.p + \nabla\times(\nabla\times\mathbf{V})(K_1+\mu_{nf}) \\ -\nabla(\nabla.\mathbf{V})(K_1+2\mu_{nf}) = \rho_{nf}\mathbf{b} - \rho_{nf}\frac{d\mathbf{V}}{dt} \\ + K_1(\nabla\times\mathbf{N}),\end{aligned} \tag{2}$$

$$\begin{aligned}2K_1\mathbf{N} + \gamma_{nf}\nabla\times(\nabla\times\mathbf{N}) - \nabla(\nabla.\mathbf{N})(\gamma_{nf}+\lambda+\alpha) \\ = \rho_{nf}\mathbf{I} - \rho_{nf}j\frac{d\mathbf{N}}{dt} + K_1(\nabla\times\mathbf{V}),\end{aligned} \tag{3}$$

where $\rho_{nf}$, $\mathbf{V}$, $p$, $K_1$, $\mu_{nf}$, $\mathbf{b}$, $\mathbf{N}$, $\mathbf{I}$, $\lambda$, $\alpha$, $j$, and $\gamma_{nf}$ are the nanofluid density, velocity field, pressure, vortex viscosity, nanofluid dynamic viscosity, body force vector, microrotation vectors (gyration), body couple per unit mass vector, spin gradient viscosity coefficients, micro-inertia density, and spin gradient viscosity, respectively. Under constant viscosity, Eqs. (1)-(3) can be converted in terms of Navier-Stokes equations for micropolar fluid. Brinkman [44] expressed the relationship between base fluid and dynamic viscosity of the nanofluid written below:

$$\mu_f = \mu_{nf}(1-\varphi)^{2.5}. \tag{4}$$

Aminossadati and Ghasemi [45] and Matin et al. [46] described the viscosity of nanofluid in terms of the

(a) (b) (c)

**Figure 1.** (a) Ethylene glycol. (b) Copper metal powder. (c) Silver metal powder.

following:

$$\varphi\rho_s + \rho_f(1-\varphi) = \rho_{nf}. \tag{5}$$

The expression of $\gamma_{nf}$ is established according to a published study by Bourantas and Loukopoulos [47]:

$$j\left(\frac{K_1}{2} + \mu_{nf}\right) = \gamma_{nf}. \tag{6}$$

The continuity equation for incompressible flow is as follows:

$$\nabla.\mathbf{V} = 0. \tag{7}$$

By using the vector identity and neglecting body couple force, Eqs. (2) and (3) for free convection flow are expressed as follows:

$$\nabla.p - \nabla(\nabla.\mathbf{V})\mu_{nf} - \nabla^2.\mathbf{V}(K_1+\mu_{nf}) = \rho_{nf}\mathbf{g} - \rho_{nf}\frac{d\mathbf{V}}{dt} + K_1(\nabla\times\mathbf{N}), \tag{8}$$

$$2K_1\mathbf{N} - \gamma_{nf}\nabla^2\mathbf{N} - \nabla(\nabla.\mathbf{N})(\lambda+\alpha) = K_1(\nabla\times\mathbf{V}) - \rho_{nf}j\frac{d\mathbf{N}}{dt}. \tag{9}$$

The simplification of Eqs. (8) and (9) takes place by using mass conservation as follows:

$$\nabla.p - \nabla^2.\mathbf{V}(K_1+\mu_{nf}) - \rho_{nf}\mathbf{g} = K_1(\nabla\times\mathbf{N}) - \rho_{nf}\frac{d\mathbf{V}}{dt}, \tag{10}$$

$$2K_1\mathbf{N} - \gamma_{nf}\nabla^2\mathbf{N} - \nabla(\nabla.\mathbf{N})(\lambda+\alpha) = K_1(\nabla\times\mathbf{V}) - \rho_{nf}j\frac{d\mathbf{N}}{dt}. \tag{11}$$

In order to apply the statement of material derivative, Eqs. (10) and (11) are expressed equivalently as follows:

$$\nabla.p - \nabla^2.\mathbf{V}(\mathbf{K_1}+\mu_{nf}) - \rho_{nf}\mathbf{g} = \mathbf{K_1}(\nabla\times\mathbf{N}) - \rho_{nf}\left(\mathbf{V}(\nabla.\mathbf{V}) + \frac{d\mathbf{V}}{dt}\right), \tag{12}$$

$$2K_1\mathbf{N} - \gamma_{nf}\nabla^2\mathbf{N} - \nabla(\nabla.\mathbf{N})(\lambda+\alpha) = K_1(\nabla\times\mathbf{V}) - \rho_{nf}j\left(\mathbf{N}(\nabla.\mathbf{N}) + \frac{d\mathbf{N}}{dt}\right). \tag{13}$$

For the problem considering Cartesian coordinates ($x$, $y$, $z$), the velocity, microrotation, and gravitational fields are, respectively, assumed as follows:

$$\mathbf{V}(w(y,t),0,0), \quad \mathbf{N}(0,0,N(y,t)), \quad \text{and} \quad \mathbf{g}(g,0,0). \tag{14}$$

Simplifying Eqs. (12)-(14) gives:

$$\frac{\partial p}{\partial x} = \rho_{nf}\frac{\partial w}{\partial x} + K_1\frac{\partial N}{\partial x} + \rho_{nf}\mathbf{g} + (K_1+\mu_{nf})\frac{\partial^2 w}{\partial y^2}, \tag{15}$$

$$2K_1 N = K_1\frac{\partial w}{\partial y} - \rho_{nf}j\frac{\partial N}{\partial x} + \gamma_{nf}j\frac{\partial N}{\partial x}. \tag{16}$$

Implementing Boussinesq approximation on Eq. (15) and assuming $K_1 = 0$ in Eq. (16), we find that:

$$(T-T_\infty)g(\beta_T\rho)_{nf} + (K_1+\mu_{nf})\frac{\partial^2 w}{\partial y^2}, +K_1\frac{\partial N}{\partial x} - \rho_{nf}\frac{\partial w}{\partial x} = 0, \tag{17}$$

$$\gamma_{nf}j\frac{\partial^2 N}{\partial y^2} - \rho_{nf}j\frac{\partial N}{\partial t} = 0. \tag{18}$$

Energy equation with thermal radiation as the previously published papers [48,49] is defined as follows:

$$K_{nf}\frac{\partial^2 T}{\partial y^2} - (\rho C)_{nf}\frac{\partial T}{\partial t} - \frac{\partial q_r}{\partial y} = 0, \tag{19}$$

where $K_{nf}$ is the thermal conductivity of nanofluids, and $(\rho C)_{nf}$ is the heat capacity under constant pressure described by Khan et al. [50]:

$$\varphi(\rho C_p)_s + (\rho C_p)_f(1-\varphi) = (\rho C_p)_{nf},$$
$$\frac{K_{nf}}{K_f} = \frac{2K_f + 2\varphi(K_s-K_f) + K_s}{2K_f + \varphi(K_s-K_f) + K_s}. \tag{20}$$

The imposed conditions are set as follows:

$$T(y,0) = T_\infty, \quad T(0,t) = T_w, \quad T(\infty,t) = T_\infty, \tag{21}$$

$$N(y,0) = C_\infty, \quad N(0,t) = t, \quad N(\infty,t) = 0, \tag{22}$$

$$w(y,0) = 0, \quad w(0,t) = UH(t)\cos(\omega t) \quad \text{or} \quad U\sin(\omega t), \quad w(\infty,t) = 0. \tag{23}$$

The energy equation (19) can be implemented using Rosseland approximation [51,52]:

$$K_{nf}\left(\frac{16T^3\sigma^*}{3K_{nf}k^*}\right)\frac{\partial^2 T}{\partial y^2} - \frac{\partial T}{\partial t}(\rho C)_{nf} = 0. \tag{24}$$

Inserting the following dimensionless quantities into Eqs. (17)-(19) gives:

$$T = \frac{T-T_\infty}{T_w-T_\infty}, \qquad N^* = \frac{v_f N}{U^2}, \qquad w^* = \frac{w}{U},$$
$$t^* = \frac{U^2 t}{v_f}, \qquad y^* = \frac{Uy}{v_f},$$

and:

$$K = \frac{k_1}{\mu_f}, \qquad R = \frac{16T_\infty^3\sigma^*}{3K_f k^*},$$

$$G_r = \frac{(T_w - T)_\infty g(\beta_T)_f}{U^3}, \qquad P_r = \frac{(C_p)_f\mu_f}{K_f}.$$

(*: symbol is dropped for simplicity).

The governing partial differential equations for temperature distribution, microroration field, and velocity field are obtained in the Appendix (Eqs. (A.1) to (A.6)), respectively.

$$\frac{\partial^2 T}{\partial y^2} = \aleph_0 p_r \aleph_1^{-1}\frac{\partial T}{\partial t}, \tag{25}$$

$$\frac{\partial^2 N}{\partial y^2} = \aleph_2\aleph_3^{-1}\frac{\partial N}{\partial t}, \tag{26}$$

$$\frac{\partial^2 w}{\partial y^2} = \aleph_2\aleph_5^{-1}\frac{\partial w}{\partial t} - \aleph_5^{-1}G_r\aleph_4 T - \aleph_5^{-1}k\frac{\partial N}{\partial y}. \tag{27}$$

Here, the assumptions for Eqs. (25)-(27) are as follows:

$$T(y,0) = 0, \qquad T(0,t) = t, \qquad T(\infty,t) = 0, \tag{28}$$

$$N(y,0) = 0, \qquad N(0,t) = t, \qquad N(\infty,t) = 0, \tag{29}$$

$$w(y,0) = 0, \qquad w(0,t) = UH(t)\cos(\omega t) \quad \text{or}$$

$$U\sin(\omega t), \qquad w(\infty,t) = 0. \tag{30}$$

Finally, expressing the governing equations (25)-(27) in terms of Caputo-Fabirizio fractional derivative, we have:

$$\frac{\partial^2 T}{\partial y^2} = \aleph_1^{-1}p_r\aleph_0\frac{\partial^\alpha}{\partial t^\alpha}T, \tag{31}$$

$$\frac{\partial^2 N}{\partial y^2} = \aleph_2\aleph_3^{-1}\frac{\partial^\alpha}{\partial t^\alpha}N, \tag{32}$$

$$\frac{\partial^2 w}{\partial y^2} = \aleph_2\aleph_5^{-1}\frac{\partial^\alpha}{\partial t^\alpha}w - \aleph_5^{-1}G_r\aleph_4 T - \aleph_5^{-1}k\frac{\partial N}{\partial y}. \tag{33}$$

A fractional differential operator is defined for Eqs. (31)-(33). We have:

$$^{\mathrm{CF}}\left(\frac{\partial^\zeta}{\partial t^\zeta}\right) = {}^{\mathrm{CF}}\left(D_t^\zeta\right)$$

$$= \int_0^t \frac{G'(\eta)}{1-\zeta}\exp\left(-\frac{\zeta(t-\eta)}{1-\zeta}\right)d\eta, \qquad 0 \le \zeta \le 1, \tag{34}$$

where $D_t^\zeta$ or $\frac{\partial^\zeta}{\partial t^\zeta}$ represents the fractional operator of Caputo-Fabirizio having order $0 \le \zeta \le 1$ [53-55] defined at the normalization functions, which are $M(1) = M(0) = 1$.

## 3. Investigation of temperature distribution and microroration field

Using Laplace transform for Caputo-Fabirizio fractionalized differential equations (31)-(32) and utilizing the fact that $\lambda = \frac{1}{(1-\alpha)}$, we obtain the following:

$$\frac{\partial^2 \bar{T}}{\partial y^2} = \aleph_1^{-1}p_r\aleph_0\frac{\alpha s}{(s+\alpha\lambda)}\bar{T}, \tag{35}$$

$$\frac{\partial^2 \bar{N}}{\partial y^2} = \aleph_3^{-1}\aleph_2\frac{\alpha s}{(s+\alpha\lambda)}\bar{N}. \tag{36}$$

Expressing Eqs. (35)-(36) in a more suitable format equivalently, we have:

$$\bar{T} = s^{-2}\exp\left[-y\sqrt{\frac{p_r\aleph_0\alpha s}{\aleph_1 s + \aleph_1\alpha\lambda}}\right], \tag{37}$$

$$\bar{N} = s^{-2}\exp\left[-y\sqrt{\frac{\aleph_2\alpha s}{\aleph_3 s + \aleph_3\alpha\lambda}}\right]. \tag{38}$$

Reworking on Eqs. (37)-(38), we obtain the summation form as follows:

$$\bar{T} = \frac{1}{s^2} + \sum_{l=1}^{\infty}\frac{1}{l!}\left(-y\sqrt{\frac{\alpha p_r\aleph_0}{\aleph_1}}\right)^l \sum_{m=0}^{\infty}\frac{(-\alpha\lambda)^m\Gamma\left(m+\frac{l}{2}\right)}{m!\Gamma\left(\frac{l}{2}\right)s^{m+2}}, \tag{39}$$

$$\bar{N} = \frac{1}{s^2} + \sum_{l=1}^{\infty}\frac{1}{l!}\left(-y\sqrt{\frac{\alpha\aleph_2}{\aleph_3}}\right)^l \sum_{m=0}^{\infty}\frac{(-\alpha\lambda)^m\Gamma\left(m+\frac{l}{2}\right)}{m!\Gamma\left(\frac{l}{2}\right)s^{m+2}}. \tag{40}$$

Inverting Eqs. (39)-(40), we expressed the general solutions of temperature and microrotation field in terms of Fox-**H** function as follows:

$$T = t + (l!)^{-1}\int_0^t (t-\tau)\sum_{l=1}^{\infty}\left(-y\sqrt{\frac{\alpha p_r\aleph_0}{\aleph_1}}\right)^l \mathbf{H}_{1,3}^{1,1}\left[(\alpha\lambda t)\left|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right.\right]d\tau, \tag{41}$$

$$N = t + (l!)^{-1}\int_0^t (t-\tau)\sum_{l=1}^{\infty}\left(-y\sqrt{\frac{\alpha\aleph_2}{\aleph_3}}\right)^l \mathbf{H}_{1,3}^{1,1}\left[(\alpha\lambda t)\left|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right.\right]d\tau, \tag{42}$$

$$\sum_{f}^{\infty}\frac{(-\chi)^f\prod_{j=1}^{p}\Gamma(a_j+A_jf)}{f!\prod_{j=1}^{q}\Gamma(b_j+B_jf)}=\mathbf{H}_{p,q+1}^{1,p}\left[\chi\left|\begin{matrix}(1-a_1,A_1),(1-a_2,A_2),(1-a_3,A_3),\cdots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),(1-b_2,B_2),(1-b_3,B_3),\cdots,(1-b_q,B_q)\end{matrix}\right.\right]. \quad (43)$$

Box I

where the special function is defined by Eq. (43) as shown in Box I [56-59].

## 4. Investigation of velocity field

Using Laplace transform for Caputo-Fabirizio fractionalized differential equation (33) and utilizing the fact that $\lambda=\frac{1}{(1-\alpha)}$, we obtain the following:

$$\frac{\partial^2\bar{w}}{\partial y^2}=\frac{\aleph_2\alpha s}{(\aleph_5 s+\aleph_5\alpha\lambda)}\bar{w}-\aleph_5^{-1}G_r\aleph_4\bar{T}-\aleph_5^{-1}k\frac{\partial\bar{N}}{\partial y}. \quad (44)$$

Solving the partial differential equation (44) and using initial and boundary conditions (28-30), we get:

$$\bar{w}=\frac{Us}{s^2+\omega^2}\exp\left\{-y\sqrt{\frac{\aleph_2\alpha s}{(\aleph_5 s+\aleph_5\alpha\lambda)}}\right\}+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_5\aleph_2\alpha}{\aleph_3}-\aleph_2\right]}\frac{1}{s^2}\exp\left\{-y\sqrt{\frac{\aleph_1\alpha s}{(\aleph_3 s+\aleph_3\alpha\lambda)}}\right\}-\frac{G_r\aleph_4}{\left[\frac{p_r\aleph_0\alpha\aleph_5}{\aleph_1}-\aleph_2\right]}\frac{1}{s(s+\alpha\lambda)}\exp\left\{-y\sqrt{\frac{\aleph_0\alpha p_r s}{\aleph_1}}\right\}. \quad (45)$$

Reworking on Eq. (45), we obtain the summation form as follows:

$$\bar{w}=\frac{Us}{s^2+\omega^2}+U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\sum_{m=0}^{\infty}\frac{(-\alpha\lambda)\Gamma\left(m+\frac{l}{2}\right)}{m!\Gamma\left(\frac{l}{2}\right)}\frac{s}{s^m(s^2+\omega^2)}+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_2\aleph_5\alpha}{\aleph_3}-\aleph_2\right]}\times\frac{1}{s^2}\exp\left\{-y\sqrt{\frac{\aleph_2\alpha s}{\aleph_3(s+\alpha\lambda)}}\right\}-\frac{G_r\aleph_4}{\left[\frac{p_r\aleph_5\aleph_0\alpha}{\aleph_1}-\aleph_2\right]}\frac{1}{s(s+\alpha\lambda)}\exp\left\{-y\sqrt{\frac{p_r\aleph_0\alpha s}{\aleph_1}}\right\}. \quad (46)$$

Inverting Eq. (46), we expressed the general solution of velocity as follows:

$$w_c=UH(t)\cos\omega t+UH(t)\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t\cos\omega(t-\tau)\,\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\left|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right.\right]d\tau+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_5\aleph_2\alpha}{\aleph_3}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_2\alpha}{\aleph_3},\alpha\lambda\right)d\tau-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0\alpha p_r}{\aleph_1}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_0\alpha p_r}{\aleph_1},\alpha\lambda\right)\exp\left(-\alpha\lambda(t-\tau)\right)d\tau, \quad (47)$$

where:

$$L^{-1}\left\{\frac{1}{s^2}\exp\left(-y\sqrt{\frac{As}{s+B}}\right)\right\}=\int_0^t\phi(y,\tau,A,B)d\tau,$$

and:

$$L^{-1}\left\{\frac{1}{s}\exp\left(-y\sqrt{\frac{As}{s+B}}\right)\right\}=\phi(y,\tau,A,B).$$

The case of sine oscillations has been established by applying a similar algorithm:

$$w_s=U\sin\omega t+U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t\sin\omega(t-\tau)\times\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\left|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right.\right]d\tau+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_2\aleph_5\alpha}{\aleph_3}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_2\alpha}{\aleph_3},\alpha\lambda\right)d\tau$$

$$-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t \phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)$$

$$\exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \tag{48}$$

## 5. Limiting cases

### 5.1. Investigation of regular or conventional nanofluid, $K_1 = 0$

The solutions for velocity field to the regular or conventional nanofluid are established by assuming $K_1 = 0$ (in the absence of microrotation parameter) in Eqs. (47) and (48), as shown in the following:

$$w_c = UH(t)\cos\omega t$$

$$+UH(t)\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t \cos\omega(t-\tau)$$

$$\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\Bigg|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right]d\tau$$

$$-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t \phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)$$

$$\exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \tag{49}$$

$$w_s = U\sin\omega t$$

$$+U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t \sin\omega(t-\tau)$$

$$\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\Bigg|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right]d\tau$$

$$-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t \phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)$$

$$\exp(-\alpha\lambda(t-\tau))d\tau. \tag{50}$$

### 5.2. Investigation of regular or conventional Newtonian fluid, $K_1 = \phi = 0$

It is also pointed out that the analytic solutions to the regular or conventional Newtonian fluid can be recovered from Eqs. (47) and (48) by assuming $K_1 = \phi = 0$ (in the absence of microrotation parameter). In what follows, one can also transform the analytic solutions into an ordinary differential operator by substituting $\alpha = 1$.

## 6. Results and conclusion

In this research, micropolar nanofluid verifiably showed better thermal performance than conventional fluids based on the mathematical tools of non-integer order fractional derivative and transform. The analysis results showed vivid effects on the enhancement of high thermal conductivity subject to suspended nanoparticles in the base fluid. The graphical illustrations of the investigated solutions, which rectified physical conditions, were discussed. Various graphs are depicted in Table 1 for highlighting the effects of nanoparticles and embedded parameters of micropolar nanofluid. However, the key results are enumerated below:

(i) The analytic solutions were explored in terms of temperature, microrotation, and velocity; in addition, similar solutions for velocity field and temperature distribution to the regular or conventional nanofluid, $K_1 = 0$, and Newtonian fluid, $K_1 = \phi = 0.$, were also recovered as the limiting cases;

(ii) Figure 2 depicts the effect of nanoparticles based on the two types of solutions, i.e., fractionalized nanofluids, $\alpha = 0.4$, and ordinary nanofluids, $\alpha = 1.0$, in which the velocity field of copper-ethylene glycol is higher than that of pure ethylene glycol and silver-ethylene glycol. It is noted that the velocity field of fractionalized nanofluids, $\alpha = 0.4$, has reciprocal behavior with ordinary nanofluids, $\alpha = 1.0$. This may result from the exponential kernel in the Caputo-Fabrizio fractional derivative;

(iii) Figure 3 depicts temperature distribution with and without Caputo-Fabrizio fractional derivative for the influence of pure ethylene glycol, copper-

**Table 1.** Fundamental thermo-physical properties.

| Base fluid/nanoparticles | $\rho$ (kg/m$^3$) | $C_p$ (J/kgK) | $k$ (W/m) |
|---|---|---|---|
| Cu | 8933 | 385 | 401 |
| Ag | 10500 | 235 | 429 |
| Ethylene glycol | 1.115 | 0.58 | 0.1490 |

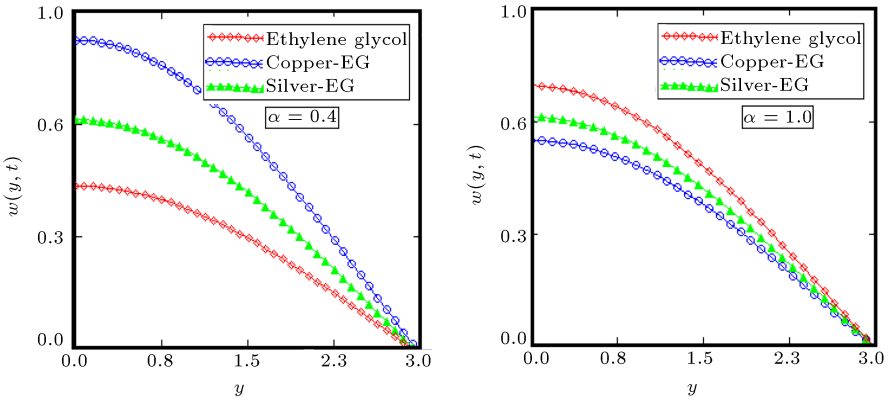


**Figure 2.** Profile of velocity field with and without Caputo-Fabrizio derivative.

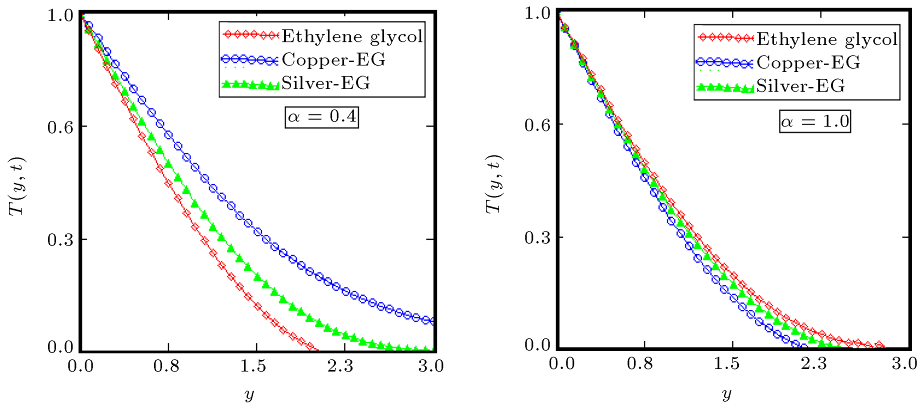


**Figure 3.** Profile of temperature distribution with and without Caputo-Fabrizio derivative.

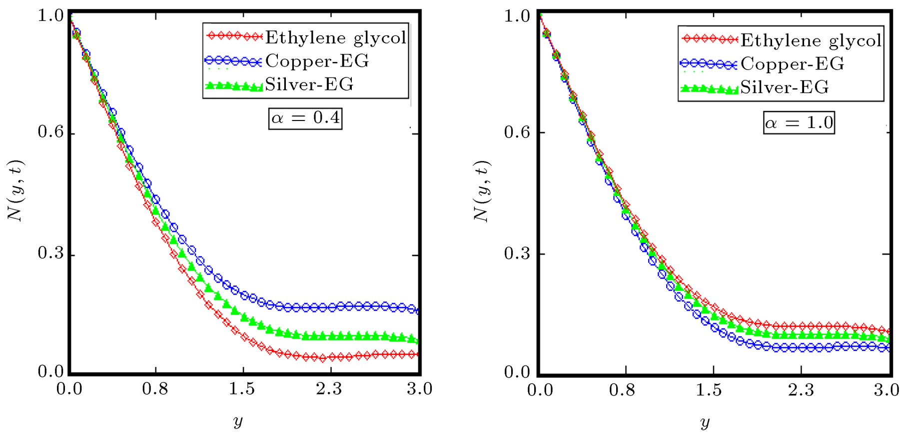


**Figure 4.** Profile of microrotation with and without Caputo-Fabrizio derivative.

ethylene glycol, and silver-ethylene glycol. In this figure, copper-ethylene glycol has scattering effects on the fractionalized temperature distribution and has opposite impacts on the ordinary temperature distribution;

(iv) The influence of nanoparticles on microrotation is underlined in Figure 4, in which the effect of microrotation is observed to be opposite near the plate. It is also pointed out that copper-ethylene glycol has accelerating behavior in comparison to all others; on the contrary, copper-ethylene glycol is of decelerating nature near the plate. This may result from the effective fractional parameter, $\alpha$;

(v) Figures 5(a), 5(b), and 5(c) are plotted to investigate the effects of different values of volume $\phi = 0.0, 0.01, 0.02$ on velocity field. Here, nanoparticles are suspended as copper-ethylene glycol and silver-ethylene glycol for the velocity field with

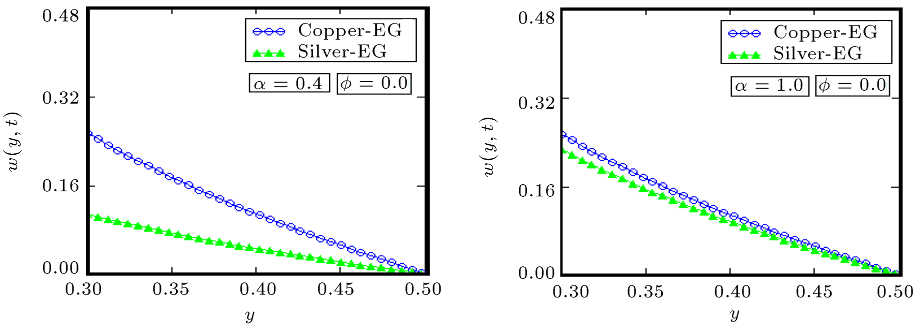


**Figure 5(a).** Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\phi = 0.0$.

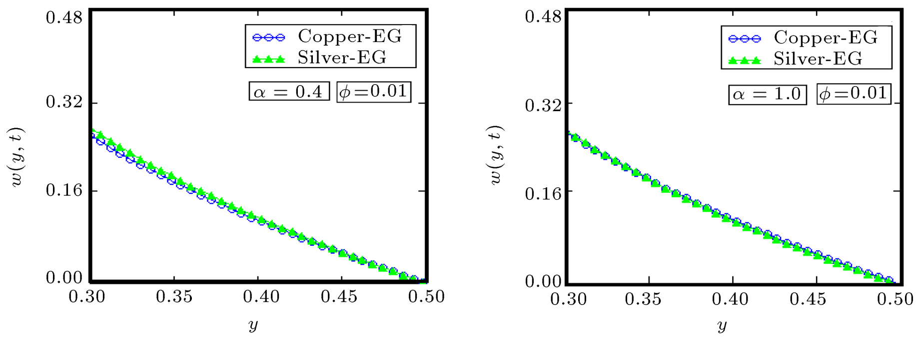


**Figure 5(b).** Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\phi = 0.01$.

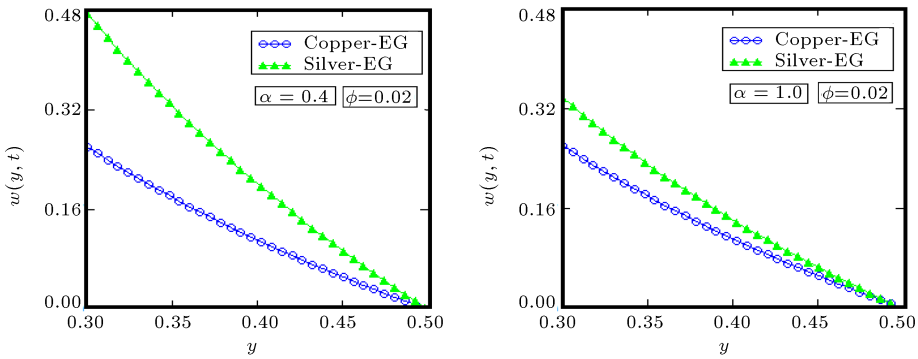


**Figure 5(c).** Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\phi = 0.02$.

and without Caputo-Fabrizio fractional derivative. An increase in volume fraction results in the scattering behavior of copper-ethylene glycol with Caputo-Fabrizio fractional operator and the sequestrating behavior of copper-ethylene glycol without Caputo-Fabrizio fractional operator. It is significantly noted that the converse phenomenon is observed as the values of volume fraction increase; in simple words silver-ethylene glycol for the velocity field with Caputo-Fabrizio fractional derivative has scattering behavior and copper-ethylene glycol has sequestrating behavior. This is due to the fact that when temperature is lower than 180°C, then an increase in volume fraction generates an increase in thermal conductivity. The same phenomenon can also be observed for temperature distribution and microrotation, too.

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## Appendix

$$\aleph_0 = \frac{(C_p\rho)_s\varphi}{(C_p\rho)_f} + (1-\varphi), \tag{A.1}$$

$$\aleph_1 = R + \frac{k_{nf}}{k_f}, \tag{A.2}$$

$$\aleph_2 = \frac{\varphi\rho_s}{\rho_f} + (1-\varphi), \tag{A.3}$$

$$\aleph_3 = \frac{1}{(1-\varphi)^{2.5}} + \frac{K}{2}, \tag{A.4}$$

$$\aleph_4 = \frac{(\beta_T\rho)_s\varphi}{(\beta_T\rho)_f} + (1-\varphi), \tag{A.5}$$

$$\aleph_5 = \frac{1}{(1-\varphi)^{2.5}} + k. \tag{A.6}$$

## Biographies

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