An analytic and mathematical synchronization of micropolar nanofluid by Caputo-Fabrizio approach

K. A. Abro[a], A. Yıldırım[b*]

a. Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, 76062, Jamshoro, Pakistan kashif.abro@faculty.muet.edu.pk, Tel/Fax: +92333-7071191

b. Department of Mathematics, Faculty of Science, Ege University,35100, Bornova-İzmir, Turkey Tel/Fax: +902323111747, yildirim.ahmet@ege.edu.tr

Abstract. Nanofluids and enhancement of the heat transfer in real systems have proved to be a widely a research area of nanotechnology, specially, improvement in thermal conductivity, thermophoresis phenomenon, dispersion of nanoparticles volume fraction and few others. Based on the touch of nanotechnology, this research article investigates the analytic and mathematical performance of micropolar nanofluid for the enhancement of heat transfer. The base fluid is taken for the purpose of thermal conductivity subject to two types of nanoparticles so called copper and silver. The mathematical analysis of the micropolar nanofluid has been carried out by invoking non-integer order derivative and transforms methods. By applying mathematical tool on the equations of micropolar nanofluid, the solutions have been explored for temperature, microrotation and velocity. In order to meet the physical aspects of the problem based on micropolar nanofluid, the comparison of velocity field of micropolar nanofluid for the suspensions of ethylene glycol into silver and ethylene glycol into copper is characterize for to enhance the rate of heat transfer.

Key word: Heat transfer of micropolar nanofluids, Fractional derivative of non-singular kernel, Special Fox-**H** function, Suspension of nanoparticles in base fluid.

1. Introduction

Nanoparticles are the small quantities of nanometer-size particles which are composed of metals for instance, copper oxide, alumina, carbides, titania, gold, copper and several others. These nanoparticles have capability to enhance thermophysical properties of base fluids includes ethylene glycol, biofluids, oil, water, lubricants, polymer solutions and many others. In continuation, kerosene, engine oil, water and ethylene glycol are insufficient for heat transfer because they have low thermal conductivity. In order to enhance convective heat transfer performance of such fluids, various techniques have been implemented such as boundary conditions and changing flow geometries. There is no denying fact that fluids have lesser thermal conductivity than metals. The thermal conductivity of base fluids can be enhanced by adding the metals and consequentially such fluids are characterized as nanofluids [1]. Moreover, the fundamental concepts of micropolar fluid were originated by Eringen [2] that characterize the dynamics of such fluids by considering microscopic impacts rising from micro-motion and local structure of the fluids. A broad assess on micropolar fluid with its engineering applications has investigated by Ariman et al. [3, 4]. Hassanien and et al. [5] explored effects of nonisothermal stretching sheet on micropolar with heat transfer, blowing and suction. Mohammadein and et al. [6] analyzed internal heat generation and dissipation with heat transfer in micropolar. Hussanan and et al. [7] obtained closed from solution for Newtonian heating in micropolar fluid for the problem of free convection flow. Choi [8] studied applications to non-Newtonian fluid flow with nanoparticles to enhance thermal conductivity of fluid. Congedo and et al. [9] perused analysis and modeling of nanofluids with natural convection heat transfer. Ghasemi and et al. [10] traced out impacts of water-CuO nanofluid along with natural convection heat transfer. Hussanan and et al. [11] examined influence of some nanofluids for accelerated plate with magnetic field and porous medium. Water functionalized carbon nanotube flow was analyzed over a moving/static wedge with magnetic field by khan and et al. [12]. Haq and at al. [13] explored effects of magnetic field with water functionalized metallic nanoparticles for squeezed flow over a sensor surface. In brevity, we include here few very latest references on nanofluids [14-16], modern fractional derivatives [17-22], heat transfer [23-27], nanoparticles [28-31], porosity and magnetic field [32-37] and few different circumstances [38-43]. Motivating by above research work on nanofluids, our purpose is to investigate the analytic and mathematical performance of micropolar nanofluid for the enhancement of heat transfer. The base fluid is taken for the purpose of thermal conductivity subject to two types of nanoparticles so called copper and silver as shown in Fig. 1(a)-1(c). The mathematical analysis of the micropolar nanofluid has been carried out by invoking non-integer order derivative and transforms methods. By applying mathematical tool on the equations of micropolar nanofluid, the solutions have been explored for temperature, microrotation and velocity. In order to meet the physical aspects of the problem based on micropolar nanofluid, the comparison of velocity field of micropolar nanofluid for the suspensions of ethylene glycol into silver and ethylene glycol into copper is characterize for to enhance the rate of heat transfer.

2. Mathematical Equations of Nanofluid

An unsteady flow of ethylene glycol based on micropolar nanofluid occupying the space lying over an oscillating plate which is perpendicular to the y-axis and is situated in the (x, z) plane. Initially, due to constant temperature T_w the fluid is considered at rest. At $t = 0^+$, the plate starts to oscillate with the velocity $UH(t)\cos\omega t \; or \; U\sin\omega t$ in its plane and level of temperature is increased up to T_w. The governing equations for micropolar fluid are

$$\nabla.\left(\rho_{nf}\,\mathbf{V}\right)=\frac{\partial\rho_{nf}}{\partial t}, \tag{1}$$

$$\nabla.p+\nabla\times\left(\nabla\times\mathbf{V}\right)\left(K_1+\mu_{nf}\right)-\nabla\left(\nabla.\mathbf{V}\right)\left(K_1+2\mu_{nf}\right)=\rho_{nf}\,\mathbf{b}-\rho_{nf}\,\frac{d\mathbf{V}}{dt}+K_1\left(\nabla\times\mathbf{N}\right), \tag{2}$$

$$2K_1\,\mathbf{N}+\gamma_{nf}\nabla\times\left(\nabla\times\mathbf{N}\right)-\nabla\left(\nabla.\mathbf{N}\right)\left(\gamma_{nf}+\lambda+\alpha\right)=\rho_{nf}\,\mathbf{I}-\rho_{nf}\;j\,\frac{d\mathbf{N}}{dt}+K_1\left(\nabla\times\mathbf{V}\right), \tag{3}$$

Where, $\rho_{nf}, \mathbf{V}, p, K_1, \mu_{nf}, \mathbf{b}, \mathbf{N}, \mathbf{I}, \lambda, \alpha, j, \gamma_{nf}$ are nanofluid density, velocity field, pressure, vortex viscosity, nanofluid dynamic viscosity, body force vector, (gyration) microrotation vectors, body couple per unit mass vector, Spin gradient viscosity coefficients, micro-inertia density, spin-gradient viscosity respectively. Under the constant viscosity, equations (1-3) can be converted in terms of Navier-Stokes equations for micropolar fluid. The Brinkman [44] has stated relationship between base fluid and dynamic viscosity of nanofluid written below

$$\mu_f=\mu_{nf}\left(1-\phi\right)^{2.5}. \tag{4}$$

Aminossadati and et al. [45] and Matin and et al. [46] have described the viscosity of nanofluid in terms as

$$\phi\rho_s+\rho_f\left(1-\phi\right)=\rho_{nf}. \tag{5}$$

The expression of γ_{nf} is based according to published work by Bourantas and et al. [47]

$$j\left(\frac{K_1}{2}+\mu_{nf}\right)=\gamma_{nf}. \tag{6}$$

The continuity equation for incompressible flow is

$$\nabla.\mathbf{V}=0. \tag{7}$$

Using the vector identity and by neglecting body couple force, for free convection flow equations (2-3) are expressed as

$$\nabla.p-\nabla\left(\nabla.\mathbf{V}\right)\mu_{nf}-\nabla^2.\mathbf{V}\left(K_1+\mu_{nf}\right)=\rho_{nf}\,\mathbf{g}-\rho_{nf}\,\frac{d\mathbf{V}}{dt}+K_1\left(\nabla\times\mathbf{N}\right), \tag{8}$$

$$2K_1\,\mathbf{N}-\gamma_{nf}\nabla^2\mathbf{N}-\nabla\left(\nabla.\mathbf{N}\right)\left(\lambda+\alpha\right)=K_1\left(\nabla\times\mathbf{V}\right)-\rho_{nf}\;j\,\frac{d\mathbf{N}}{dt}, \tag{9}$$

Simplifying equations (8-9) take place by using mass conservation as

$$\nabla.p-\nabla^2.\mathbf{V}\left(K_1+\mu_{nf}\right)-\rho_{nf}\,\mathbf{g}=K_1\left(\nabla\times\mathbf{N}\right)-\rho_{nf}\,\frac{d\mathbf{V}}{dt}, \tag{10}$$

$$2K_1\,\mathbf{N}-\gamma_{nf}\nabla^2\mathbf{N}-\nabla\left(\nabla.\mathbf{N}\right)\left(\lambda+\alpha\right)=K_1\left(\nabla\times\mathbf{V}\right)-\rho_{nf}\;j\,\frac{d\mathbf{N}}{dt}, \tag{11}$$

In order to apply the statement of material derivative, equations (10-11) are expressed equivalently as

$$\nabla.p-\nabla^2.\mathbf{V}\left(K_1+\mu_{nf}\right)-\rho_{nf}\,\mathbf{g}=K_1\left(\nabla\times\mathbf{N}\right)-\rho_{nf}\left(\mathbf{V}\left(\nabla.\mathbf{V}\right)+\frac{d\mathbf{V}}{dt}\right), \tag{12}$$

$$2K_1\,\mathbf{N}-\gamma_{nf}\nabla^2\mathbf{N}-\nabla(\nabla.\mathbf{N})(\lambda+\alpha)=K_1\,(\nabla\times\mathbf{V})-\rho_{nf}\;j\left(\mathbf{N}(\nabla.\mathbf{N})+\frac{d\mathbf{N}}{dt}\right), \tag{13}$$

For the problem under consideration of Cartesian coordinates (x, y, z), we assume the velocity, microrotation and gravitational fields correspondingly

$$\mathbf{V}(w(y,t),0,0),\quad \mathbf{N}(0,0,N(y,t),)\ \text{and}\ \mathbf{g}(g,0,0). \tag{14}$$

Simplifying equations (12-14) gives

$$\frac{\partial p}{\partial x}=\rho_{nf}\,\frac{\partial w}{\partial x}+K_1\,\frac{\partial N}{\partial x}+\rho_{nf}\,g+\left(K_1+\mu_{nf}\right)\frac{\partial^2 w}{\partial y^2}, \tag{15}$$

$$2K_1\,N=K_1\,\frac{\partial w}{\partial y}-\rho_{nf}\;j\,\frac{\partial N}{\partial x}+\gamma_{nf}\,j\,\frac{\partial N}{\partial x}, \tag{16}$$

Implementing Boussinesq approximation on equation (15) and making $K_1\ =0$ in equation (16) respectively, we find that

$$\left(T-T_\infty\right)g\left(\beta_T\rho\right)_{nf}+\left(K_1+\mu_{nf}\right)\frac{\partial^2 w}{\partial y^2},+K_1\,\frac{\partial N}{\partial x}-\rho_{nf}\,\frac{\partial w}{\partial x}=0, \tag{17}$$

$$\gamma_{nf}\,j\,\frac{\partial^2 N}{\partial y^2}-\rho_{nf}\;j\,\frac{\partial N}{\partial t}=0. \tag{18}$$

Energy equation with thermal radiation as previously published papers [48, 49] is defined as

$$K_{nf}\,\frac{\partial^2 T}{\partial y^2}-\left(\rho C\right)_{nf}\,\frac{\partial T}{\partial t}-\frac{\partial q_r}{\partial y}=0, \tag{19}$$

Where, K_{nf} is thermal conductivity of nanofluids and $(\rho C)_{nf}$ is heat capacity at a constant pressure described by Khan et al. [50],

$$\phi\left(\rho C_p\right)_s+\left(\rho C_p\right)_f(1-\phi)=\left(\rho C_p\right)_{nf},\qquad \frac{K_{nf}}{K_f}=\frac{2K_f+2\phi\left(K_s-K_f\right)+K_s}{2K_f+\phi\left(K_s-K_f\right)+K_s}, \tag{20}$$

The imposed conditions are set as

$$T(y,0)=T_\infty,\quad T(0,t)=T_w,\quad T(\infty,t)=T_\infty, \tag{21}$$

$$N(y,0)=C_\infty,\quad N(0,t)=t,\quad N(\infty,t)=0, \tag{22}$$

$$w(y,0)=0,\quad w(0,t)=UH(t)\cos(\omega t)\ or\ U\sin(\omega t),\quad w(\infty,t)=0. \tag{23}$$

The energy equation (19) can be taken place using Rosseland approximation [51, 52]

$$K_{nf}\left(\frac{16T^3\sigma^*}{3K_{nf}k^*}\right)\frac{\partial^2 T}{\partial y^2}-\frac{\partial T}{\partial t}\left(\rho C\right)_{nf}=0, \tag{24}$$

Implementing below dimensionless quantities in equations (17-19),

$$T=\frac{T-T_\infty}{T_w-T_\infty},\ N^*=\frac{v_f N}{U^2}, w^*=\frac{w}{U},\ t^*=\frac{U^2 t}{v_f},\ \ y^*=\frac{Uy}{v_f}.$$

$$\text{and } K=\frac{k_1}{\mu_f},\ R=\frac{16T_\infty^3\sigma^*}{3K_f k^*},\ G_r=\frac{\left(T_w-T\right)_\infty g\left(\beta_T\right)_f}{U^3},\ P_r=\frac{\left(C_p\right)_f\mu_f}{K_f}.$$

(*symbol is dropped for simplicity), we obtain governing partial differential equations for temperature distribution, microroration field and velocity field by employing appendix (A1-A6) respectively

$$\frac{\partial^2 T}{\partial y^2}=\aleph_0 p_r \aleph_1^{-1}\frac{\partial T}{\partial t}, \tag{25}$$

$$\frac{\partial^2 N}{\partial y^2}=\aleph_2\aleph_3^{-1}\frac{\partial N}{\partial t}, \tag{26}$$

$$\frac{\partial^2 w}{\partial y^2}=\aleph_2\aleph_5^{-1}\frac{\partial w}{\partial t}-\aleph_5^{-1}G_r\aleph_4 T-\aleph_5^{-1}k\frac{\partial N}{\partial y}, \tag{27}$$

Here, the assumptions for equations (25-27) are

$$T(y,0)=0,\ \ T(0,t)=t,\ \ \ T(\infty,t)=0, \tag{21}$$

$$N(y,0)=0,\ \ N(0,t)=t,\ \ \ N(\infty,t)=0, \tag{22}$$

$$w(y,0)=0,\ \ w(0,t)=UH(t)\cos(\omega t)\ or\ U\sin(\omega t),\ \ \ w(\infty,t)=0. \tag{23}$$

Finally, expressing governing equations (25-27) in terms of Caputo-Fabirizio fractional derivative, we have

$$\frac{\partial^2 T}{\partial y^2}=\aleph_1^{-1}p_r\aleph_0\frac{\partial^\alpha}{\partial t^\alpha}T, \tag{31}$$

$$\frac{\partial^2 N}{\partial y^2}=\aleph_2\aleph_3^{-1}\frac{\partial^\alpha}{\partial t^\alpha}N, \tag{32}$$

$$\frac{\partial^2 w}{\partial y^2}=\aleph_2\aleph_5^{-1}\frac{\partial^\alpha}{\partial t^\alpha}w-\aleph_5^{-1}G_r\aleph_4 T-\aleph_5^{-1}k\frac{\partial N}{\partial y}. \tag{33}$$

Where, D_t^ζ or $\frac{\partial^\zeta}{\partial t^\zeta}$ represents fractional operator of Caputo-Fabirizio having order $0\le\zeta\le 1$ [53-55] defined at the normalization functions are $M(1)=M(0)=1$ as

$$^{CF}\left(\frac{\partial^\zeta}{\partial t^\zeta}\right)=\,^{CF}\left(D_t^\zeta\right)=\int_0^t\frac{G'(\eta)}{1-\zeta}\,Exp\left(-\frac{\zeta(t-\eta)}{1-\zeta}\right)d\eta,\qquad \text{for}\quad 0\le\zeta\le 1. \tag{34}$$

3. Investigation of Temperature Distribution and Microroration field

Using Laplace Transform on Caputo-Fabirizio fractionalized differential equation (31-32) and utilizing the fact $\lambda=\frac{1}{(1-\alpha)}$ we obtain,

$$\frac{\partial^2 \bar{T}}{\partial y^2} = \aleph_1^{-1} p_r \aleph_0 \frac{\alpha s}{(s+\alpha\lambda)} \bar{T}, \tag{35}$$

$$\frac{\partial^2 \overline{N}}{\partial y^2} = \aleph_3^{-1} \aleph_2 \frac{\alpha s}{(s+\alpha\lambda)} \overline{N}, \tag{36}$$

Expressing equations (35-36) in more suitable format equivalently, we have

$$\bar{T} = s^{-2}\ exp\left[-y\sqrt{\frac{p_r \aleph_0 \alpha s}{\aleph_1 s + \aleph_1 \alpha\lambda}}\right], \tag{37}$$

$$\overline{N} = s^{-2}\ exp\left[-y\sqrt{\frac{\aleph_2 \alpha s}{\aleph_3 s + \aleph_3 \alpha\lambda}}\right], \tag{38}$$

Reworking on equations (37-38), we obtain summation form as

$$\bar{T} = \frac{1}{s^2} + \sum_{l=1}^{\infty} \frac{1}{l!}\left(-y\sqrt{\frac{\alpha\ p_r \aleph_0}{\aleph_1}}\right)^l \sum_{m=0}^{\infty} \frac{(-\alpha\lambda)^m \Gamma\left(m+\frac{l}{2}\right)}{m!\ \Gamma\left(\frac{l}{2}\right) s^{m+2}}, \tag{39}$$

$$\overline{N} = \frac{1}{s^2} + \sum_{l=1}^{\infty} \frac{1}{l!}\left(-y\sqrt{\frac{\alpha \aleph_2}{\aleph_3}}\right)^l \sum_{m=0}^{\infty} \frac{(-\alpha\lambda)^m \Gamma\left(m+\frac{l}{2}\right)}{m!\ \Gamma\left(\frac{l}{2}\right) s^{m+2}}, \tag{40}$$

Inverting equations (39-40), we expressed the general solutions of temperature and microrotation field in terms of Fox-**H** function as

$$T = t + (l!)^{-1} \int_0^t (t-\tau) \sum_{l=1}^{\infty} \left(-y\sqrt{\frac{\alpha\ p_r \aleph_0}{\aleph_1}}\right)^l \mathbf{H}_{1,3}^{1,1}\left[(\alpha\lambda t)\middle| \begin{matrix} \left(1-\frac{l}{2},1\right) \\ (0,1),\left(1-\frac{l}{2},0\right),(0,1) \end{matrix}\right] d\tau, \tag{41}$$

$$N = t + (l!)^{-1} \int_0^t (t-\tau) \sum_{l=1}^{\infty} \left(-y\sqrt{\frac{\alpha \aleph_2}{\aleph_3}}\right)^l \mathbf{H}_{1,3}^{1,1}\left[(\alpha\lambda t)\middle| \begin{matrix} \left(1-\frac{l}{2},1\right) \\ (0,1),\left(1-\frac{l}{2},0\right),(0,1) \end{matrix}\right] d\tau, \tag{42}$$

Where, the special function is defined as [56-59]

$$\sum_{f}^{\infty}\frac{(-\chi)^{f}\prod_{j=1}^{p}\Gamma\left(a_{j}+A_{j}f\right)}{f!\ \prod_{j=1}^{q}\Gamma\left(b_{j}+B_{j}f\right)}=\mathbf{H}_{p,q+1}^{1,p}\left[\chi\left|\begin{matrix}(1-a_{1},A_{1}),(1-a_{2},A_{2}),(1-a_{3},A_{3}),...,(1-a_{p},A_{p})\\(0,1),(1-b_{1},B_{1}),(1-b_{2},B_{2}),(1-b_{3},B_{3}),...,(1-b_{q},B_{q})\end{matrix}\right.\right]. \tag{43}$$

4. Investigation of Velocity Field

Using Laplace transform on Caputo-Fabirizio fractionalized differential equation (33) and utilizing the fact $\lambda=\frac{1}{(1-\alpha)}$ we obtain,

$$\frac{\partial^{2}\overline{w}}{\partial y^{2}}=\frac{\aleph_{2}\alpha s}{\left(\aleph_{5}s+\aleph_{5}\alpha\lambda\right)}\overline{w}-\aleph_{5}^{-1}G_{r}\aleph_{4}\overline{T}-\aleph_{5}^{-1}k\frac{\partial\overline{N}}{\partial y}, \tag{44}$$

Solving the partial differential equation (44) and using initial and boundary conditions (28-30)**3**

$$\overline{w}=\frac{Us}{s^{2}+\omega^{2}}exp\left\{-y\sqrt{\frac{\aleph_{2}\alpha s}{\left(\aleph_{5}s+\aleph_{5}\alpha\lambda\right)}}\right\}+\frac{k\sqrt{\frac{\aleph_{2}\alpha}{\aleph_{3}}}}{\left[\frac{\aleph_{5}\aleph_{2}\alpha}{\aleph_{3}}-\aleph_{2}\right]}\frac{1}{s^{2}}exp\left\{-y\sqrt{\frac{\aleph_{1}\alpha s}{\left(\aleph_{3}s+\aleph_{3}\alpha\lambda\right)}}\right\}$$

$$-\frac{G_{r}\aleph_{4}}{\left[\frac{p_{r}\aleph_{0}\alpha\aleph_{5}}{\aleph_{1}}-\aleph_{2}\right]}\frac{1}{s\left(s+\alpha\lambda\right)}exp\left\{-y\sqrt{\frac{\aleph_{0}\alpha p_{r}s}{\aleph_{1}}}\right\}, \tag{45}$$

Reworking on equations (45), we obtain summation form as

$$\overline{w}=\frac{Us}{s^{2}+\omega^{2}}+U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_{2}\alpha}}{\aleph_{5}}\right)^{l}\sum_{m=0}^{\infty}\frac{(-\alpha\lambda)\Gamma\left(m+\frac{l}{2}\right)}{m!\Gamma\left(\frac{l}{2}\right)}\frac{s}{s^{m}\left(s^{2}+\omega^{2}\right)}+\frac{k\sqrt{\frac{\aleph_{2}\alpha}{\aleph_{3}}}}{\left[\frac{\aleph_{2}\aleph_{5}\alpha}{\aleph_{3}}-\aleph_{2}\right]}$$

$$\times\frac{1}{s^{2}}exp\left\{-y\sqrt{\frac{\aleph_{2}\alpha s}{\aleph_{3}\left(s+\alpha\lambda\right)}}\right\}-\frac{G_{r}\aleph_{4}}{\left[\frac{p_{r}\aleph_{5}\aleph_{0}\alpha}{\aleph_{1}}-\aleph_{2}\right]}\frac{1}{s\left(s+\alpha\lambda\right)}exp\left\{-y\sqrt{\frac{p_{r}\aleph_{0}\alpha s}{\aleph_{1}}}\right\}, \tag{46}$$

Inverting equations (46), we expressed the general solution of velocity as

$$w_{c}=UH(t)\cos\omega t+UH(t)\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_{2}\alpha}}{\aleph_{5}}\right)^{l}\int_{0}^{t}\cos\omega(t-\tau)\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\left|\begin{matrix}\left(1-\frac{l}{2},1\right)\\(0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right.\right]d\tau$$

$$+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_5\aleph_2\alpha}{\aleph_3}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_2\alpha}{\aleph_3},\alpha\lambda\right)d\tau-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0\alpha\, p_r}{\aleph_1}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_0\alpha\, p_r}{\aleph_1},\alpha\lambda\right)exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \quad (47)$$

Where, $L^{-1}\left\{\frac{1}{s^2}exp\left(-y\sqrt{\frac{As}{s+B}}\right)\right\}=\int_0^t\phi\left(y,\tau,A,B\right)d\tau$ and $L^{-1}\left\{\frac{1}{s}exp\left(-y\sqrt{\frac{As}{s+B}}\right)\right\}=\phi\left(y,\tau,A,B\right)$.

The case of sine oscillations has been established by applying similar algorithm

$$w_s=U\sin\omega t+U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t\sin\omega(t-\tau)\times\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\middle|\begin{matrix}\left(1-\frac{l}{2},1\right)\\ (0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right]d\tau$$

$$+\frac{k\sqrt{\frac{\aleph_2\alpha}{\aleph_3}}}{\left[\frac{\aleph_2\aleph_5\alpha}{\aleph_3}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_2\alpha}{\aleph_3},\alpha\lambda\right)d\tau-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \quad (48)$$

5. Limiting Cases

5.1 Investigation of Regular or Conventional Nanofluid $K_1=0$.

The solutions for velocity field to the regular or conventional Nanofluid are established by letting $K_1=0$ (in the absence of microrotation parameter) in the equations (47-48), we obtain

$$w_c=UH(t)cos\omega t+UH(t)\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l\int_0^t\cos\omega(t-\tau)\,\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\middle|\begin{matrix}\left(1-\frac{l}{2},1\right)\\ (0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right]d\tau$$

$$-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t\phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \quad (49)$$

$$w_s = U\sin\omega t + U\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{-y\sqrt{\aleph_2\alpha}}{\aleph_5}\right)^l \int_0^t \sin\omega(t-\tau)\,\mathbf{H}_{1,3}^{1,1}\left[\left(\frac{\alpha\lambda}{t}\right)\middle| \begin{matrix}\left(1-\frac{l}{2},1\right)\\ (0,1),\left(1-\frac{l}{2},0\right),(0,1)\end{matrix}\right]d\tau$$

$$-\frac{G_r\aleph_4}{\left[\frac{\aleph_5\aleph_0 p_r\alpha}{\aleph_1}-\aleph_2\right]}\int_0^t \phi\left(y,\tau,\frac{\aleph_0 p_r\alpha}{\aleph_1},\alpha\lambda\right)exp\left(-\alpha\lambda(t-\tau)\right)d\tau. \qquad (50)$$

5.2 Investigation of Regular or Conventional Newtonian Fluid $K_1 = \phi = 0.$

It is also pointed out that, the analytic solutions to the regular or conventional Newtonian fluid can be recovered from equations (47-48) by letting $K_1 = \phi = 0$ (in the absence of microrotation parameter). In this continuation, one can also transform the analytic solutions in to ordinary differential operator by substituting $\alpha = 1.$

6. Results and Conclusion

In this research article, micropolar nanofluid has proved to give better thermal performance than conventional fluids based on the mathematical tools of non-integer order fractional derivative and transforms. The analysis has shown vivid effects for the enhancement of high thermal conductivity subject to suspended nanoparticles in to the base fluid. The graphical illustrations have been discussed on the investigated solutions which rectify physical conditions. Various graphs have been depicted by using table 1 for highlighting the effects of nanoparticles and embedded parameters of micropolar nanofluid. However, the key results are enumerated below:

(i) The analytic solutions are explored for temperature, microrotation and velocity and the similar solutions for velocity field and temperature distribution to the regular or conventional nanofluid $K_1 = 0.$ and Newtonian fluid $K_1 = \phi = 0.$ have also been recovered as the limiting cases.

(ii) Fig. 2 is prepared for the influence of nanoparticles based on the two types of solutions i-e fractionalized nanofluids $\alpha = 0.4$ and ordinary nanofluids $\alpha = 1.0$ in which velocity field of copper- ethylene glycol is higher in comparison with pure ethylene glycol and silver- ethylene glycol. It is noted that Velocity field of fractionalized nanofluids $\alpha = 0.4$ has reciprocal behavior with ordinary nanofluids $\alpha = 1.0$. This may be due to the fact of exponential kernel in Caputo-Fabrizio fractional derivative.

(iii) Fig. 3 is depicted for temperature distribution with and without Caputo-Fabrizio fractional derivative for the influence of pure ethylene glycol, copper-ethylene glycol and silver-ethylene glycol. In this figure, copper-ethylene glycol has scattering effects for fractionalized temperature distribution and has reversal impacts for ordinary temperature distribution.

(iv) The influence of nanoparticles on microrotation is underlined in Fig. 4, in which effect of microrotation is observed opposite near the plate. It is also pointed out that copper-ethylene glycol has accelerating behavior in comparison to all others; on the contrary copper- ethylene glycol has decelerating nature near the plate. This may be due to fact of effective fractional parameter α.

(v) Figs: 5(a), 5(b) and 5(c) are plotted to investigate the effects of different values of volume fraction $\phi = 0.0, 0.01, 0.02$ on velocity field. Here, nanoparticles are suspended as copper-ethylene glycol and silver-ethylene glycol for the velocity field with and without Caputo-Fabrizio fractional derivative. The increase in volume fraction results the scattering behavior of copper-ethylene glycol with Caputo-Fabrizio fractional operator and sequestrating behavior of copper-ethylene glycol without Caputo-Fabrizio fractional operator. It is significantly noted that converse phenomenon is observed as the values of volume fraction increases, i-e (silver-ethylene glycol for the velocity field with Caputo-Fabrizio fractional derivative has scattering behavior and copper-ethylene glycol has sequestrating behavior). This is due to natural fact that, when temperature is lower than $180^0 C$ then increase in volume fraction generates increase in thermal conductivity. Same phenomenon can also be observed for temperature distribution and microrotation as well.

Acknowledgement

The author Kashif Ali Abro is highly thankful and grateful to Mehran university of Engineering and Technology, Jamshoro, Pakistan for generous support and facilities of this research work.

Ethical Statement

This material has not been published in whole or in parts elsewhere. The manuscript is not currently being considered for publication in another journal. All authors have been personally and actively involved in substantive work leading to the manuscript, and will hold themselves jointly and individually responsible for its content.

Appendix

$$\aleph_0 = \frac{(C_p\,\rho)_s\,\varphi}{(C_p\,\rho)_f} + (1-\varphi), \tag{A1}$$

$$\aleph_1 = R + \frac{k_{nf}}{k_f}, \tag{A2}$$

$$\aleph_2 = \frac{\varphi\,\rho_s}{\rho_f} + (1-\varphi), \tag{A3}$$

$$\aleph_3 = \frac{1}{(1-\varphi)^{2.5}} + \frac{K}{2}, \tag{A4}$$

$$\aleph_4 = \frac{(\beta_T\,\rho)_s\,\varphi}{(\beta_T\,\rho)_f} + (1-\varphi), \tag{A5}$$

$$\aleph_5 = \frac{1}{(1-\varphi)^{2.5}} + k. \tag{A6}$$

References

[1] Choi, S.U.S. “Enhancing thermal conductivity of fluids with nanoparticles” *ASME FED*, **231**, pp. 99-105 (1995).
[2] Eringen, A.C. “Theory of micropolar fluids” *J Math Mech*, **16**, pp. 118-125 (1966).
[3] Ariman, T., Turk, M.A. and Sylvester, N.D. “Microcontinuum fluid mechanics, A review” *International Journal of Engineering Science*, **11**, pp. 905-930 (1973).
[4] Ariman, T., Turk, M.A. and Sylvester, N.D. “Applications of microcontinuum fluid mechanics” *International Journal of Engineering Science*, **12**, pp. 273-293 (1974).

[5] Hassanien, I.A. and Gorla, R.S.R. "Heat transfer to a micropolar fluid from a nonisothermal stretching sheet with suction and blowing" *Aeta Meehanica*, **84**, pp. 191-199 (1990).

[6] Mohammadein, A.A. and Gorla, R.S.R. "Heat transfer in a micropolar fluid over a stretching sheet with viscous dissipation and internal heat generation" *International Journal of Numerical Methods for Heat and Fluid Flow*, **11**, pp. 50-58 (2001).

[7] Hussanan, A., Salleh, M.Z., Khan, I. and Tahar, R.M. "Unsteady free convection flow of a micropolar fluid with Newtonian heating: Closed form solution" *Thermal Science*, (2015). DOI: 10.2298/TSCI150221125H.

[8] Choi, S. "Enhancing thermal conductivity of fluids with nanoparticles" *Developments and applications of non-Newtonian flows*, ASME, **66**, pp. 99-105 (1995).

[9] Congedo, P.M., Collura, S. and Congedo, P.M. "Modeling and analysis of natural convection heat transfer in nanofluids" *2008 ASME Summer Heat Transfer Conference, Jacksonville*, Florida, USA, pp. 569-579 (2008).

[10] Ghasemi, B. and Aminossadati, S.M. "Natural convection heat transfer in an inclined enclosure filled with a water-CuO nanofluid" *Numerical Heat Transfer, Part A: Applications*, **55**, 807-823 (2009).

[11] Hussanan, A., Khan, I., Hashim, H., Mohamed, M.K.A., Ishak, N., Sarif, N.M. Salleh, M.Z. "Unsteady MHD flow of some nanofluids past an accelerated vertical plate embedded in a porous medium" *Jurnal Teknologi*, **78**, 121-126 (2016).

[12] Khan, W.A., Culham, R. Haq, R.U. "Heat transfer analysis of MHD water functionalized carbon nanotube flow over a static/moving wedge" *Journal of Nanomaterials*, 1-13 (2015).

[13] Haq, R.U., Nadeem, S., Khan, Z.H., Noor, N.F.M. "MHD squeezed flow of water functionalized metallic nanoparticles over a sensor surface" *Physica E: Low dimensional Systems and Nanostructures*, **73**, pp. 45-53 (2015).

[14] Kashif, A.A., Mohammad, M.R., Ilyas, K., Irfan, A.A., Asifa, T. "Analysis of stokes' second problem for nanofluids using modern fractional derivatives" *Journal of Nanofluids*, **7**, pp. 738–747 (2018).

[15] Abro, K.A., Mukarrum, H., Mirza, M.B. "An analytic study of molybdenum disulfide nanofluids using modern approach of Atangana-Baleanu fractional derivatives" *Eur. Phys. J. Plus*, **132**, pp. 439-450 (2017), DOI 10.1140/epjp/i2017-11689-y (2017)

[16] Qasem, A.M.l, Kashif, A.A., Ilyas, K. "Analytical solutions of fractional Walter's-B fluid with applications" *Complexity*, (2018), Article ID 8918541.

[17] Kashif, A.A., Shaikh. H.S., Norzieha, M., Ilyas, K., Asifa, T. "A mathematical study of magnetohydrodynamic Casson fluid via special functions with heat and mass transfer embedded in porous plate" *Malaysian Journal of Fundamental and Applied Sciences*, **14(1)**, pp. 20-38 (2018).

[18] Kashif, A.A., Ilyas, K. "Analysis of heat and mass transfer in MHD flow of generalized Casson fluid in a porous space via non-integer order derivative without singular kernel" *Chinese Journal of Physics*, **55(4)**, pp. 1583-1595 (2017).

[19] Zafar, A.A., Fetecau, C. "Flow over an infinite plate of a viscous fluid with non-integer order derivative without singular kernel" *Alexandria Engineering Journal*, **55(3)**, pp. 2789-2796 (2016).

[20] Muzaffar, H.L., Kashif, A.A., Asif, A. S. "Helical flows of fractional viscoelastic fluid in a circular pipe" *International Journal of Advanced and Applied Sciences*, **4(10)**, pp. 97-105 (2017).

[21] Kashif, A.A., Mukarrum, H., Mirza, M.B. "Analytical solution of MHD generalized Burger's fluid embedded with porosity" *International Journal of Advanced and Applied Sciences*, **4(7)**, pp. 80-89, (2017).

[22] Arshad, K., Kashif, A.A., Asifa, T., Ilyas, K. "Atangana-Baleanu and Caputo Fabrizio analysis of fractional derivatives for heat and mass transfer of second grade fluids over a vertical plate: A comparative study" *Entropy*, **19(8)** 1-12, (2017).

[23] İbiş B, Bayram M. "Numerical comparison of methods for solving fractional differential–algebraic equations (FDAEs)" *Computers & Mathematics with Applications*, **62(8),** pp. 3270-3278 (2011), https://doi.org/10.1016/j.camwa.2011.08.043**.**

[24] Kashif, A.A., Ilyas, K., Asifa, T. “Application of Atangana-Baleanu fractional derivative to convection flow of MHD Maxwell fluid in a porous medium over a vertical plate” *Mathematical Modelling of Natural Phenomena*, **13,** 1-18, (2018), https://doi.org/10.1051/mmnp/2018007.
[25] Ilyas, K. and Kashif, A.A. “Thermal analysis in Stokes’ second problem of nanofluid: Applications in thermal engineering” *Case Studies in Thermal Engineering*, **10,** (2018), DOI: https://doi.org/10.1016/j.csite.2018.04.005
[26] Wakif, A., Boulahia, A. A., Animasaun, I.L., Afridi, M.I., Qasim, M., Sehaqui, R. “Magneto-convection of alumina-water nanofluid within thin horizontal layers using the revised generalized Buongiorno’s model” *Front. Heat Mass Transf.* **12**, 1-15, (2019).
[27] Doungmo, E.F.G. “Evolution equations with a parameter and application to transport-convection differential equations” *Turkish Journal of Mathematics*, **41**, pp. 636-654 (2017). doi:10.3906/mat-1603-107.
[28] Abderrahim, W., Zoubair, B., Rachid, S. ”A semi-analytical analysis of electro-thermo-hydrodynamic stability in dielectric nanofluids using Buongiorno’s mathematical model together with more realistic boundary conditions” *Results in physics*, **9**, (2018), 1438-1454.
[29] Abro, A.K., Ali, D.C., Irfan, A.A., Ilyas, K. “Dual thermal analysis of magnetohydrodynamic flow of nanofluids via modern approaches of Caputo–Fabrizio and Atangana–Baleanu fractional derivatives embedded in porous medium” *Journal of Thermal Analysis and Calorimetry*, **18**, pp. 1-11 (2018). https://doi.org/10.1007/s10973-018-7302-z.
[30] Kashif, A.A., Anwer, A.M., Shahid, H.A., Ilyas, I. Tlili. “Enhancement of heat transfer rate of solar energy via rotating Jeffrey nanofluids using Caputo–Fabrizio fractional operator: An application to solar energy” *Energy Reports*, **5,** pp. 41–49, (2019). https://doi.org/10.1016/j.egyr.2018.09.00
[31] Abderrahim, W., Zoubair, B., Rachid, S. “Numerical study of the onset of convection in a Newtonian nanofluid layer with spatially uniform and non-uniform internal heating” *Journal of Nanofluids*, **6(1)**, pp. 136-148, (2017).
[32] Doungmo, E.F.G. “Strange attractor existence for non-local operators applied to four-dimensional chaotic systems with two equilibrium points” *Chaos*, **29**, 023117 (2019); https://doi.org/10.1063/1.5085440.
[33] Ullah, H., Islam S., Khan I., Sharidan S., Fiza M. “MHD boundary layer flow of an incompressible upper-convected Maxwell fluid by optimal homotopy asymptotic method” Scientia Iranica B, **24(1)**, pp. 202-210, (2017).
[34] Ambreen, S., Kashif, A.A., Muhammad, A.S. “Thermodynamics of magnetohydrodynamic Brinkman fluid in porous medium: Applications to thermal science” *Journal of Thermal Analysis and Calorimetry*, (2018), DOI: 10.1007/s10973-018-7897-0.
[35] Kashif A A, Ilyas Khan, J.F. G´omez-Aguilar. “A mathematical analysis of a circular pipe in rate type fluid via Hankel transform” Eur. Phys. J. Plus (2018) 133: 397, DOI 10.1140/epjp/i2018-12186-7
[36] Abderrahim, W., Zoubair, B., Mishra, S.R., Mohammad, M. R., Rachid, S. “Influence of a uniform transverse magnetic field on the thermo-hydrodynamic stability in water-based nanofluids with metallic nanoparticles using the generalized Buongiorno’s mathematical model” *Eur. Phys. J. plus*, (2018).
[37] Doungmo, E.F.G. “Solvability of chaotic fractional systems with 3D four-scroll attractors” *Chaos, Solitons & Fractals*,**104**, 443-451, (2017).
[38] Kashif, A.A. and Ahmet Y. “Fractional treatment of vibration equation through modern analogy of fractional differentiations using integral transforms” *Iranian Journal of Science and Technology, Transaction A: Science*, **43**, (2019). https://doi.org/10.1007/s40995-019-00687-4,
[39] Abro, K.A., J.F. Gomez-Aguilar. Dual fractional analysis of blood alcohol model via non-integer order derivatives, *Fractional Derivatives with Mittag-Leffler Kernel, Studies in Systems, Decision and Control*, **194**, https://doi.org/10.1007/978-3-030-11662-0_5.

[40] Abro, K.A., Ali, A.M., Anwer, A.M. "Functionality of circuit via modern fractional differentiations" *Analog Integrated Circuits and Signal Processing*, **18**, pp. 1-11, (2018). https://doi.org/10.1007/s10470-018-1371-6.

[41] Kashif, A.A., Mukarrum, H., Mirza, M. B. "A Mathematical analysis of magnetohydrodynamic generalized Burger fluid for permeable oscillating plate" *Punjab University Journal of Mathematics*, **50(2)**, pp. 97-111 (2018).

[42] Kashif, A.A., Muhammad, A.S. "Heat transfer in magnetohydrodynamics second grade fluid with porous impacts using Caputo-Fabrizoi fractional derivatives" *Punjab University Journal of Mathematics*, **49(2)**, pp. 113-125 (2017).

[43] Doungmo E.F.G., Khan, Y, Mugisha, S. "Control parameter & solutions to generalized evolution equations of stationarity, relaxation and diffusion" *Results in Physics*, **9**, pp. 1502-1507 (2018), https://doi.org/10.1016/j.rinp.2018.04.051.

[44] Brinkman, H.C. "The viscosity of concentrated suspensions and solution" *Journal of Chemical Physics*, **20**, pp. 571-581 (1952).

[45] Aminossadati, S.M. and Ghasemi, B. "Natural convection cooling of a localised heat source at the bottom of a nanofluid-filled enclosure" *European Journal of Mechanics B/Fluids*, **28**, pp. 630-640 (2009).

[46] Matin, M.H. and Pop, I. "Forced convection heat and mass transfer flow of a nanofluid through a porous channel with a first order chemical reaction on the wall" *International Communications in Heat and Mass Transfer*, **46**, pp. 134-141 (2013).

[47] Bourantas, G.C. and Loukopoulos, V.C. "MHD natural-convection flow in an inclined square enclosure filled with a micropolar-nanofluid" *International Journal of Heat and Mass Transfer*, **79**, pp. 930-944 (2014).

[48] Turkyilmazoglu, M. and Pop, I. "Heat and mass transfer of unsteady natural convection flow of some nanofluids past a vertical infinite flat plate with radiation effect" *International Journal of Heat and Mass Transfer*, **59**, pp. 167-171 (2013).

[49] Hussanan, A., Khan, I., Hashim, H., Mohamed, M.K.A., Ishak, N., Sarif, N.M. Salleh, M.Z. "Unsteady MHD flow of some nanofluids past an accelerated vertical plate embedded in a porous medium" *Jurnal Teknologi*, **78**, pp. 121-126 (2016).

[50] Khan, W.A., Khan, Z.H. and Haq, R.U. "Flow and heat transfer of ferrofluids over a flat plate with uniform heat flux" *The European Physical Journal Plus*, **130**, pp. 1-10 (2015).

[51] Hussanan, A., Khan, I. and Shafie, S. "An exact analysis of heat and mass transfer past a vertical plate with Newtonian heating" *Journal of Applied Mathematics*, **12**, pp. 1-9 (2013).

[52] Hussanan, A., Ismail, Z., Khan, I., Hussein, A.G. and Shafie, S. "Unsteady boundary layer MHD free convection flow in a porous medium with constant mass diffusion and Newtonian heating" *The European Physical Journal Plus*, **129**, pp. 1-16, (2014).

[53]. Caputo, M. and Fabrizio M. "A new definition of fractional derivative without singular kernel" *Progr. Fract. Differ. Appl.* **1(2)**, pp. 73–85, (2015).

[54] Shah, N.A and Khan, I. "Heat transfer analysis in a second grade fluid over and oscillating vertical plate using fractional Caputo–Fabrizio derivatives" *Eur. Phys. J. C*, **(4)**, pp. 362-376 (2016).

[55] Kashif, A.A., Mukarrum, H., Mirza, M.B. "Slippage of Fractionalized Oldroyd-B Fluid with Magnetic Field in Porous Medium" *Progr. Fract. Differ. Appl.*, **3(1),** pp. 69-80 (2017).

[56] Mathai, A.M, Saxena, R.K., Haubold, H.J. "The H-Functions: Theory and Applications" (2010) Springer, New York.

[58] Kashif, A.A., Gomez-Aguilar, J.F. "A comparison of heat and mass transfer on a Walter's-B fluid via Caputo-Fabrizio versus Atangana-Baleanu fractional derivatives using the Fox-H function" *Eur. Phys. J. Plus*, **134**, pp. 101-114 (2019), DOI 10.1140/epjp/i2019-12507-4.

[57] Debnath, L and Bhatta, D. "Integral Transforms and Their Applications" (Second Edition) (2007), Chapman & Hall/CRC.

[59] Abro, K.A, Ilyas, K, José F G A. “Thermal effects of magnetohydrodynamic micropolar fluid embedded in porous medium with Fourier sine transform technique” *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, **41**, pp. 174-181 (2019)

Biographies

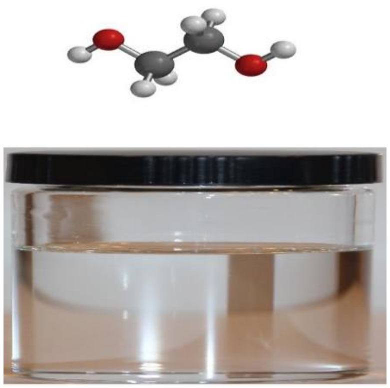
Kashif Alli Abro was born in 1988 in Dadu, Sindh, Pakistan. He is currently Assistant Professor in Mehran University of Engineering and Technology, Jamshoro, Pakistan. He received his Bachelor’s degree in Mathematics from University of Sindh Jamshoro, Pakistan and M.Phil. and PhD in Applied Mathematics from the NED University of Engineering and Technology, Karachi, Pakistan. His research interests are: fractional calculus (modern fractional differential operators), heat and mass transfer, nanofluids, fluid mechanics and modern control engineering.

Ahmet Yildirim was born in 1980. He received his MSc degree in Mathematics from Ege University, Izmir, Turkey in 2002. His field of research includes applied mathematics and mathematical physics.

Table: 1. Fundamental Thermo-physical Properties

Base Fluid/Nanoparticles	ρ **(Kg/m^3)**	C_p **(J/Kg K)**	k **(W/m.**
Cu	8933	385	401
Ag	10500	235	429
Ethylene Glycol	1.115	0.58	0.1490

Fig. 1(a). Ethylene Glycol



1(b). Copper Metal Powder

1(c). Silver Metal Powder

Fig.2. Profile of velocity field with and without Caputo-Fabrizio derivative.

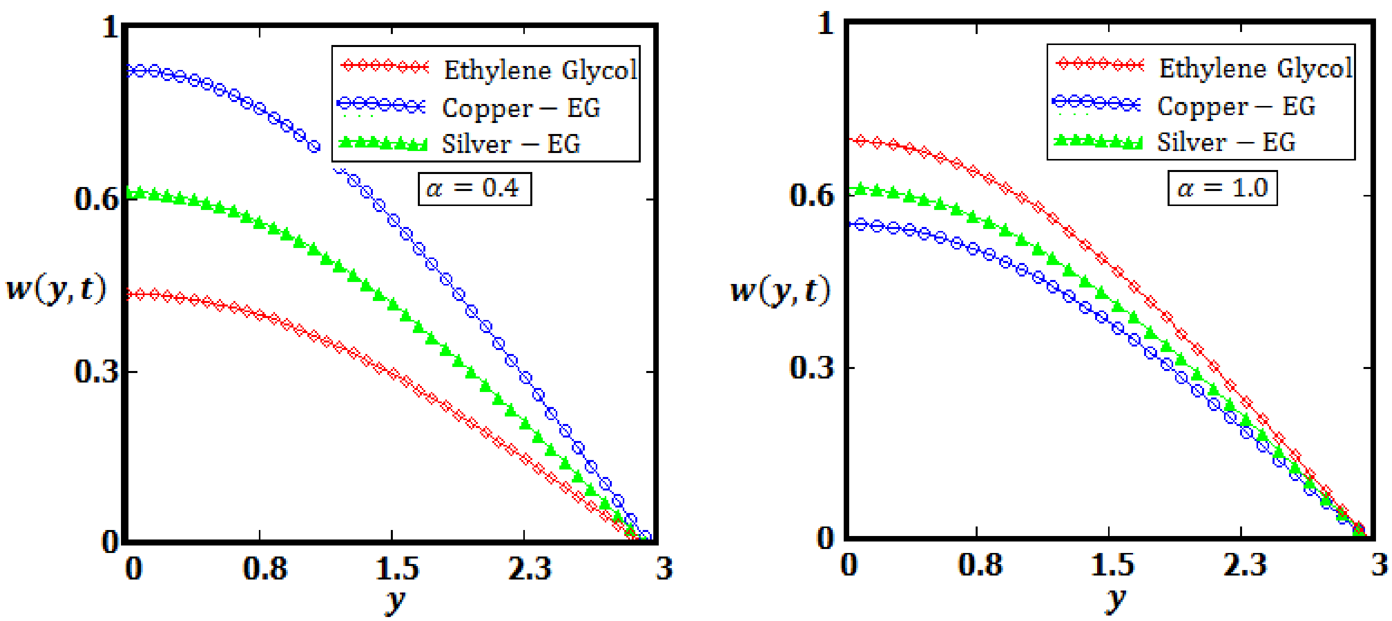


Fig.3. Profile of temperature distribution with and without Caputo-Fabrizio derivative.

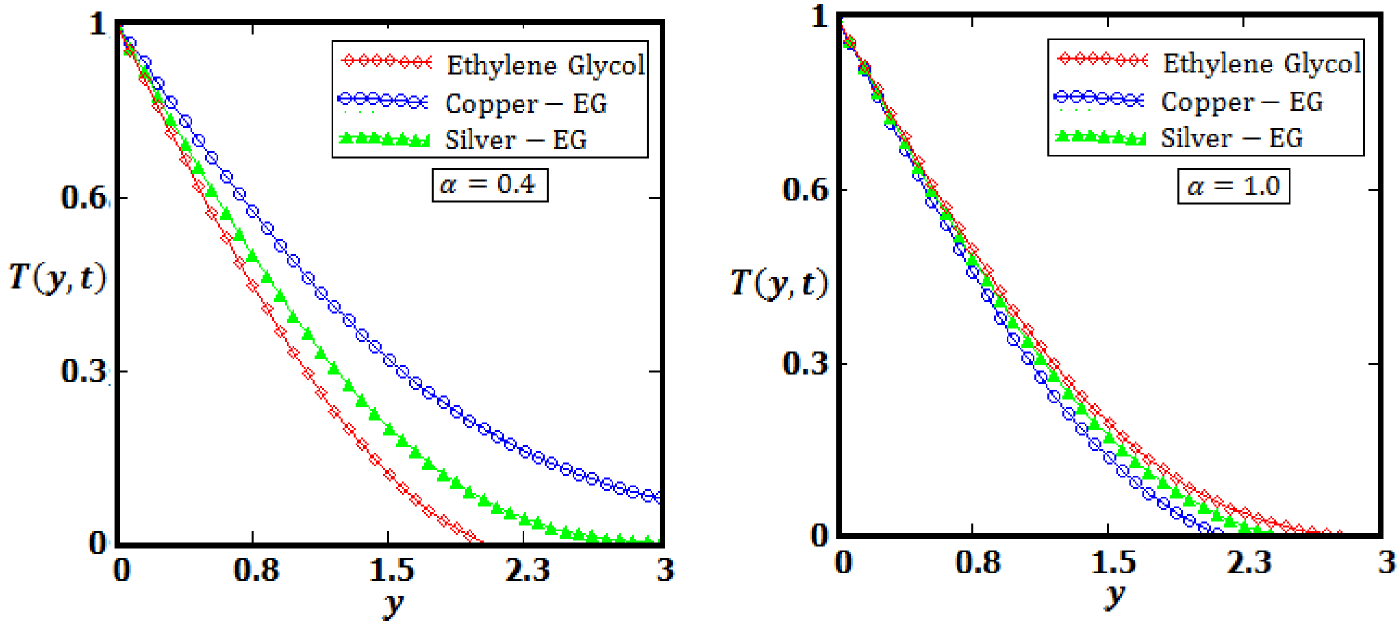


Fig.4. Profile of microrotation with and without Caputo-Fabrizio derivative.

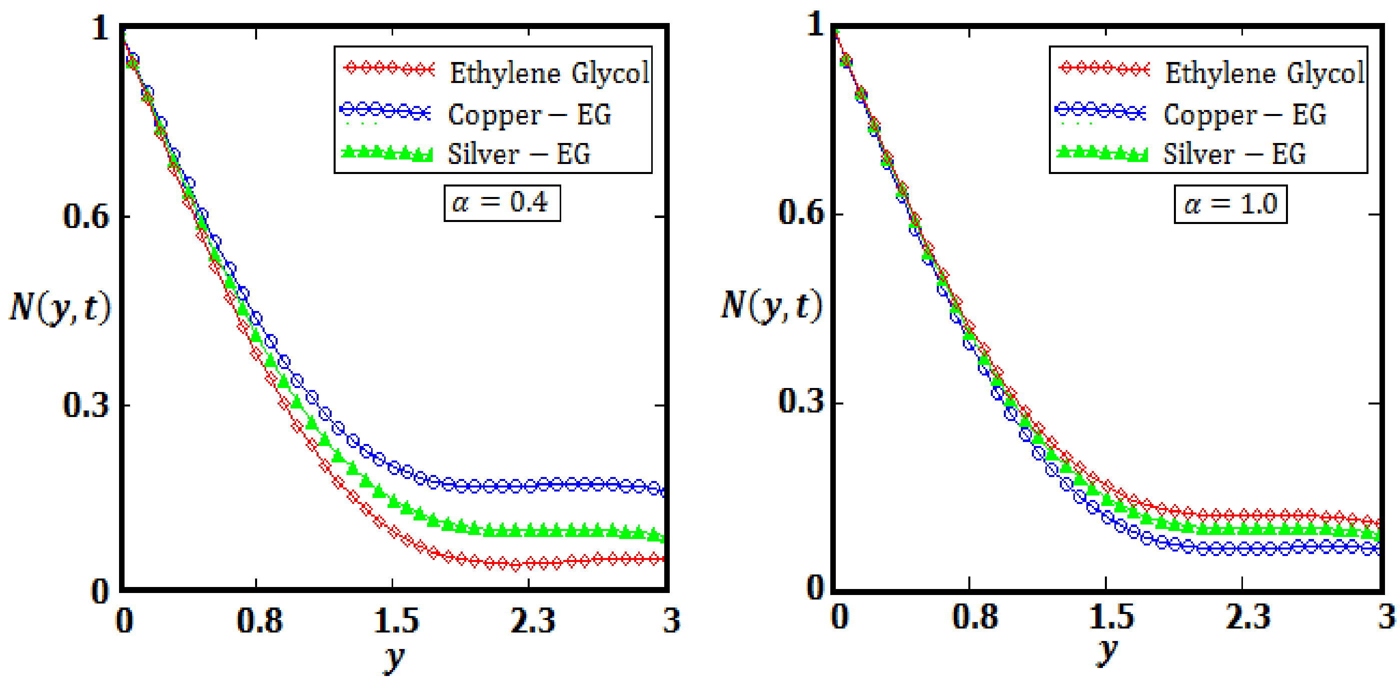


Fig.5(a). Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\emptyset = 0.0$.

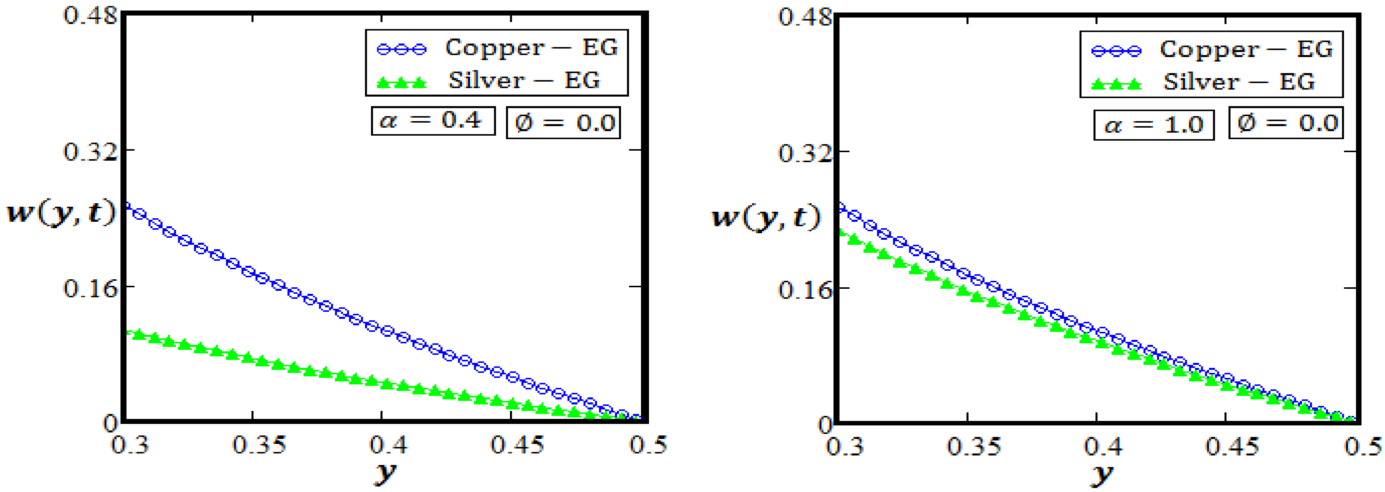


Fig.5(b). Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\emptyset = 0.01$.

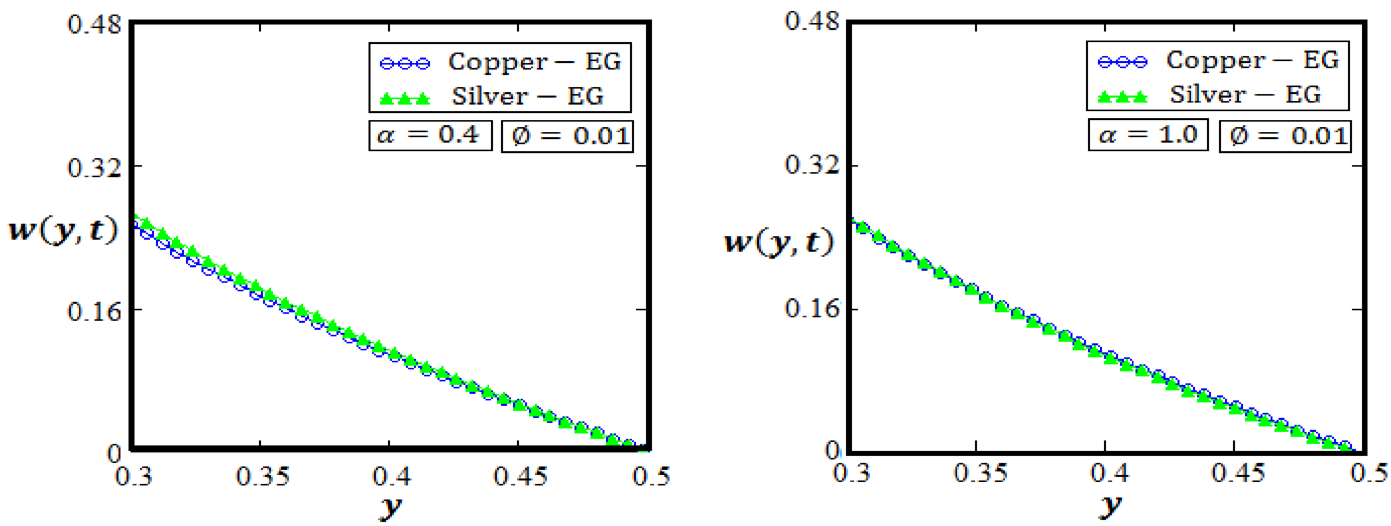
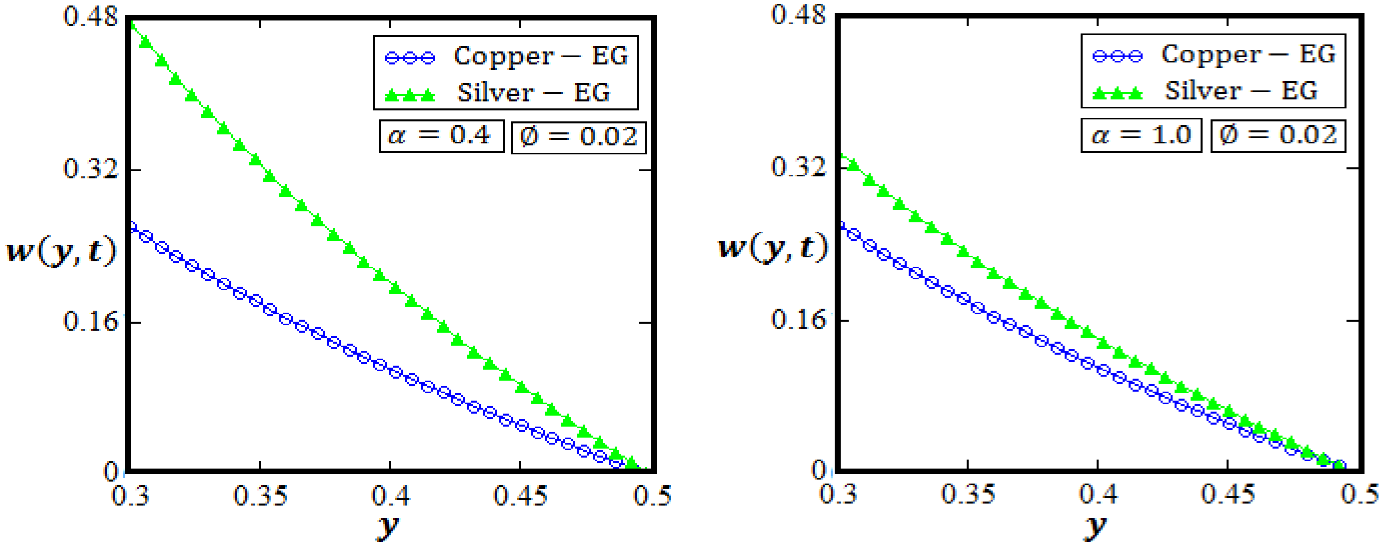


Fig.5(c). Profile of velocity field with and without Caputo-Fabrizio derivative when nanoparticle volume fraction is $\emptyset = 0.02$.



0.48
0.32
0.16
0
w(y, t)
0.3
0.35
0.4
0.45
0.5
y
Copper − EG
Silver − EG
α = 0.4
∅ = 0.02
0.48
0.32
0.16
0
w(y, t)
0.3
0.35
0.4
0.45
0.5
y
Copper − EG
Silver − EG
α = 1.0
∅ = 0.02