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Calculation of coupled modes of fluid-structure systems by pseudo symmetric subspace iteration method

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KEYWORDS

Fluid-structure interaction; Eigen-problem; Pseudo symmetric technique; Subspace iteration method; Coupled modes. **Abstract.** An efficient technique is proposed for calculation of coupled modes of fluidstructure interaction systems. The algorithm was developed with the symmetric matrix operation mentality such that apparently a symmetric eigen-problem was being solved. It was proven that each left eigen-vector was related to the corresponding right eigen-vector through a simple relation. Therefore, the subsequent transient analysis could readily be performed. Overall, the method seemed very efficient and was ideal to employ in generalpurpose finite element programs for solving the above-mentioned eigen-problems.

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1. Introduction

There are many different approaches to formulating fluid-structure interaction problems. In all of them, the structure is always formulated in terms of displacement degrees of freedom. However, several alternatives are available for the fluid domain. Some researches that have preferred to work with displacement degrees of freedom for the fluid domain include Wilson and Khalvati [1], Bathe and Hahn [2], and Hamdi et al. [3]. In a different approach, velocity potential has been introduced as the unknown variable for the fluid region. This can be followed in the work of Everstine [4] or Olson and Bathe [5]. Finally, formulation based on pressure variable was initially given by Zienkiewicz and Bettess [6] and Zienkiewicz et al. [7], and followed by many recent studies [8-12]. All the above-mentioned methods have their own advantages and disadvantages.

*. Corresponding author. E-mail address: vahlotfi@aut.ac.ir (V. Lotfi) However, let us assume that the problem is formulated based on displacement and pressure degrees of freedom for the solid and fluid domains, respectively. Then, it is well known that one is encountered with an eigenvalue problem having asymmetric matrices. Therefore, standard eigen-value computation methods are not directly applicable. This can be overcome by introducing additional variables, which help us to arrive at a symmetric generalized eigen-problem [13-15]. In this case, one can apply different alternatives such as the subspace iteration method to extract eigen-values and eigen-vectors [16]. However, this is not very efficient due to an increase in the number of degrees of freedom.

In the present study, an efficient technique is proposed for the calculation of coupled modes of fluidstructure systems. It is referred to as pseudo symmetric subspace iteration method. The method is explained in full details in the next section. It can be employed to obtain the eigen-values and the corresponding right eigen-vectors. Furthermore, it will be proven that each left eigen-vector is related to the corresponding right eigen-vector through a simple relation. Finally, a numerical example is considered and a brief discussion of the efficiency of the method is presented in the latter part of this paper.

2. Theoretical concepts

The eigen-value problem corresponding to a typical fluid-structure interaction system can be written as:

$$\bar{\mathbf{K}} (\mathbf{X}^R)_j = \bar{\lambda}_j \bar{\mathbf{M}} (\mathbf{X}^R)_j, \qquad (1)$$

with the following definitions:

$$\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & -\mathbf{B}^T \\ \mathbf{0} & \mathbf{H} \end{bmatrix}, \qquad (2a)$$

$$\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix},$$
(2b)

$$(\mathbf{X}^R)_j = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}_j,$$
 (2c)

where \mathbf{M} and \mathbf{K} represent the mass and stiffness matrices of the structure, respectively, and \mathbf{G} and \mathbf{H} are assembled matrices of fluid domain. Furthermore, \mathbf{B} in the above relations is often referred to as interaction matrix. The right eigen-vector is composed of \mathbf{r} , which is the vector of nodal relative displacements, and the vector \mathbf{p} , which includes nodal pressures. Physically, it is clear that the eigen-values of these systems are real and free vibration modes exist. However, it is noted from the forms of matrices $\mathbf{\bar{K}}$, $\mathbf{\bar{M}}$ (Eqs. (2a) and (2b)) that the eigen-problem does not have symmetric matrices and the subspace iteration method is not directly applicable.

Let us now define matrix $\hat{\mathbf{K}}$ as follows:

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}.$$
(3)

Then, one can obtain an equivalent symmetric eigenvalue problem through multiplying Eq. (1) by $\hat{\mathbf{K}} \bar{\mathbf{K}}^{-1}$:

$$\hat{\mathbf{K}} (\mathbf{X}^R)_j = \bar{\lambda}_j \, \hat{\mathbf{M}} (\mathbf{X}^R)_j, \qquad (4)$$

with the following definition being employed:

$$\hat{\mathbf{M}} = \hat{\mathbf{K}} \, \bar{\mathbf{K}}^{-1} \, \bar{\mathbf{M}} \, . \tag{5}$$

It is clear that matrix $\hat{\mathbf{K}}$ is symmetric by its definition (i.e., Eq. (3)). This is not so obvious for matrix $\hat{\mathbf{M}}$. However, let us carry out the matrix multiplication (5) to investigate this fact, which yields:

$$\hat{\mathbf{M}} = \begin{bmatrix} (\mathbf{M} + \mathbf{B}^T \mathbf{H}^{-1} \mathbf{B}) & \mathbf{B}^T \mathbf{H}^{-1} \mathbf{G} \\ \mathbf{G}^T \mathbf{H}^{-1} \mathbf{B} & \mathbf{G}^T \mathbf{H}^{-1} \mathbf{G} \end{bmatrix}.$$
(6)

It is now apparent that matrix $\hat{\mathbf{M}}$ is also symmetric. It is worthwhile to add that the following identity is employed in the above expansion, which can easily be verified:

$$\bar{\mathbf{K}}^{-1} = \begin{bmatrix} \mathbf{K}^{-1} & \mathbf{K}^{-1} \mathbf{B}^T \mathbf{H}^{-1} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix}.$$
 (7)

It should be mentioned that the above symmetric eigen-value problem (i.e., Eq. (4)) could have been obtained by introducing additional variables and then, using static condensation to eliminate those variables. It is also very interesting to note that matrix $\hat{\mathbf{M}}$ can be written in different forms such as:

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^T \\ \mathbf{G}^T \end{bmatrix} \begin{bmatrix} \mathbf{H}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{G} \end{bmatrix}, \quad (8)$$

or alternatively as:

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{I}^T & \mathbf{B}^T \\ \mathbf{0} & \mathbf{G}^T \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix}, \quad (9)$$

where ${\bf I}$ is an identity matrix. The latter form will be quite useful.

2.1. The initial explanation

The method is described in its initial form in this section for simplification purposes. However, It will be subsequently presented in an efficient format, which is referred to as pseudo symmetric technique.

Consider the asymmetric eigen-value problem related to fluid-structure interaction systems (i.e., Eq. (1)). The method commences by defining an estimate of the right modal matrix, \mathbf{X}^{R} . Then, Eq. (10) is used for matrix $\tilde{\mathbf{X}}$:

$$\bar{\mathbf{K}} \, \tilde{\mathbf{X}} = \bar{\mathbf{M}} \, \mathbf{X}^R. \tag{10}$$

At this stage, a new smaller eigen-value problem should be defined the matrices of which are obtained by projecting matrices $\mathbf{\tilde{K}}, \mathbf{\tilde{M}}$ onto $\mathbf{\tilde{X}}$. However, the resulting eigen-problem will have asymmetric matrices. Thus, a problem with smaller dimensions is alternatively defined by projecting matrices $\mathbf{\hat{K}}, \mathbf{\hat{X}}$ onto $\mathbf{\tilde{X}}$. This is like the case that one is trying to solve the equivalent symmetric eigen-value problem (4). Therefore, the projected matrices will be as follows:

$$\mathbf{K}^* = \tilde{\mathbf{X}}^T \tilde{\mathbf{K}} \tilde{\mathbf{X}}, \tag{11a}$$

$$\mathbf{M}^* = \tilde{\mathbf{X}}^T \, \hat{\mathbf{M}} \, \tilde{\mathbf{X}}. \tag{11b}$$

The modal matrix of this new system is denoted by \mathbf{Q} and its eigen-values are stored in a diagonal matrix $\mathbf{\overline{\Lambda}}$. Obviously, this will satisfy Eq. (12):

$$\mathbf{K}^* \mathbf{Q} = \mathbf{M}^* \mathbf{Q} \, \bar{\mathbf{\Lambda}}. \tag{12}$$

Subsequently, a better estimate of modal matrix, \mathbf{X}^{R} , is obtained through the following relation:

$$\mathbf{X}^R = \tilde{\mathbf{X}} \mathbf{Q}. \tag{13}$$

This iterative process continues up to the point of convergence. Finally, matrices $\bar{\mathbf{A}}$, and \mathbf{X}^R will have the smallest eigen-values of the original asymmetric problem (1) and its corresponding right eigen-vectors.

2.2. Intermediate description

The explanation presented in the previous section is not readily suitable for the pseudo symmetric approach. Therefore, some of the relations should be expanded and prepared for the final formulation.

Assume that matrix $\hat{\mathbf{X}}$ is partitioned such that the upper and lower parts correspond to solid and fluid degrees of freedom, respectively.

$$\tilde{\mathbf{X}} = \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \tilde{\mathbf{X}}_2 \end{bmatrix}.$$
(14)

Then, one can rewrite Eq. (11a) as follows:

$$\mathbf{K}^* = \tilde{\mathbf{X}}_1^T \mathbf{K} \tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_2^T \mathbf{G} \tilde{\mathbf{X}}_2.$$
(15)

Similarly, Eq. (11b) can be written in an expanded form by employing Eq. (9):

$$\mathbf{M}^{*} = \begin{bmatrix} \tilde{\mathbf{X}}_{1}^{T} & \tilde{\mathbf{X}}_{2}^{T} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{T} & \mathbf{B}^{T} \\ \mathbf{0} & \mathbf{G}^{T} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}_{1} \\ \tilde{\mathbf{X}}_{2} \end{bmatrix}.$$
(16)

Clearly, this can also be calculated as follows:

$$\mathbf{M}^* = \mathbf{Y}^T \mathbf{Z},\tag{17}$$

with the following definitions:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \tilde{\mathbf{X}}_2 \end{bmatrix}, \qquad (18)$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \mathbf{Y}.$$
(19)

Let us now assume that matrix \mathbf{Y} is partitioned similarly to $\tilde{\mathbf{X}}$. Thus, Eq. (18) is written in the following form, which is more suitable from programming point of view:

$$\begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \tilde{\mathbf{X}}_2 \end{bmatrix}. \quad (20)$$

Moreover, to avoid a normally large matrix inversion (i.e., \mathbf{H}^{-1} calculation), Eq. (19) is also written as:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \tilde{\mathbf{X}}_2 \end{bmatrix}, \quad (21)$$

where \mathbf{Z}_2 is calculated by solving the following equation:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Y}_2 \end{bmatrix}.$$
(22)

Certainly, the upper part of this equation is trivial and it has to be disregarded in the actual calculation process.

2.3. Pseudo symmetric technique

The method is explained now in an efficient form, which is referred to as pseudo symmetric subspace iteration approach. In this process, it is assumed that one is working with $\overline{\mathbf{K}}$, and $\overline{\mathbf{M}}$ matrices rather than $\overline{\mathbf{K}}$, and $\overline{\mathbf{M}}$. The former matrices are symmetric forms of the latter asymmetric matrices. These can be explicitly defined as follows:

$$\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & -\mathbf{B}^T \\ -\mathbf{B} & \mathbf{H} \end{bmatrix}, \qquad (23a)$$

$$\mathbf{\bar{\bar{M}}} = \begin{bmatrix} \mathbf{M} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{G} \end{bmatrix}.$$
(23b)

It should be mentioned that these matrices are assumed to be stored in a symmetric upper skyline form and one is working with those forms throughout the whole actual calculation process. Nevertheless, equations are still written such that the full matrices are employed. This is done merely to write equations conveniently. In the remaining part of this section, a special matrix multiplication notation is used, which is defined in the Appendix. There are 6 different types of this matrix multiplication. In each case, one or several parts of the first matrix do not participate in the multiplication process. Obviously, a programming subroutine can easily be designed for each special type of matrix multiplication. We can also take into account that the first matrix is stored in a symmetric skyline form to perform the operation more efficiently. Having in mind the above-mentioned notation, Eq. (10) can now be readily written as:

$$\bar{\mathbf{K}} \stackrel{1}{*} \tilde{\mathbf{X}} = \bar{\mathbf{M}} \stackrel{2}{*} \mathbf{X}^{R}.$$
(24)

The right-hand-side operation of Eq. (24) is easily carried out by a special efficient subroutine written for that purpose, which takes into account the skyline form of matrix $\overline{\mathbf{M}}$ as mentioned above. Moreover, the actual equation solving of Eq. (24) should be slightly modified due to the special matrix multiplication of type 1, which is used on the left-hand side. This would affect the reduction process and its corresponding operation on the right-hand side in the Gaussian elimination method.

When matrix $\hat{\mathbf{X}}$ is calculated, one can obtain the projected stiffness matrix by an alternative form of Eq. (15) by utilizing special matrix multiplication of types 3 and 4 as follows:

$$\mathbf{K}^{*} = \tilde{\mathbf{X}}^{T} \left(\mathbf{\bar{K}}^{3} \mathbf{\tilde{X}} + \mathbf{\bar{M}}^{4} \mathbf{\tilde{X}} \right).$$
(25)

The projected mass matrix is written as before (Eq. (17)):

$$\mathbf{M}^* = \mathbf{Y}^T \mathbf{Z}$$

However, matrices \mathbf{Y} , and \mathbf{Z} are now defined by a variation of Eqs. (20) and (21) based on the special multiplication notation:

$$\mathbf{Y} = \begin{bmatrix} \tilde{\mathbf{X}}_1 \\ \mathbf{0} \end{bmatrix} + \mathbf{\bar{M}} \stackrel{6}{*} \tilde{\mathbf{X}}, \qquad (26)$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{bmatrix} + \mathbf{\bar{M}} ^{3} \mathbf{\tilde{X}}.$$
 (27)

Furthermore, matrix \mathbf{Z}_2 is obtained by the following relation, which is equivalent to Eq. (22):

$$\bar{\bar{\mathbf{K}}} \stackrel{4}{*} \begin{bmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Y}_2 \end{bmatrix}.$$
(28)

It is worthwhile to mention that in solving Eq. (28), reduction in merely the lower part of matrix $\mathbf{\bar{K}}$ is required. Note that the calculation has already been performed by carrying out the reduction process for solving Eq. (24). However, changes in the right-hand side corresponding to the reduction should still be made. Certainly, the upper part of the matrix is not really used in the back-substitution phase due to trivial nature of the upper part of Eq. (28), which is more apparent in its equivalent form (i.e., Eq. (22)). Thus, this step can also be carried out very efficiently while we have avoided any inverse matrix calculation as mentioned before.

When projected stiffness and mass matrices are calculated, the remaining steps are as explained previously. That is, the eigen-problem with smaller dimension is solved (i.e., Eq. (12)). Subsequently, Eq. (13) is utilized to obtain a better estimate of modal matrix, \mathbf{X}^{R} .

2.4. Left eigen-vectors

The pseudo symmetric technique was discussed in the previous section for the calculation of right eigenvectors. To complete the solution, one will also need to calculate the left eigen-vectors. A technique is presented in this section for this purpose.

Based on Eq. (1), it is clear that the modal matrix corresponding to the right eigen-vectors satisfies the following equation:

$$\bar{\mathbf{K}} \mathbf{X}^R = \bar{\mathbf{M}} \mathbf{X}^R \bar{\mathbf{\Lambda}},\tag{29}$$

where $\overline{\mathbf{A}}$ is the diagonal matrix which stores eigenvalues. Let us now multiply both sides of this relation by transposing the modal matrix corresponding to the left eigen-vectors, which yields:

$$(\mathbf{X}^{L})^{T} \, \bar{\mathbf{K}} \, \mathbf{X}^{R} = (\mathbf{X}^{L})^{T} \, \bar{\mathbf{M}} \, \mathbf{X}^{R} \, \bar{\mathbf{\Lambda}}.$$
(30)

Eq. (30) can also be written as:

$$\mathbf{K}^* = \mathbf{M}^* \, \bar{\mathbf{\Lambda}}. \tag{31}$$

Obviously, both matrices \mathbf{K}^* , and \mathbf{M}^* are diagonal ones due to orthogonality condition of eigen-vectors. On the other hand, modal matrix \mathbf{X}^R does also satisfy the following equation based on Eq. (4):

$$\hat{\mathbf{K}} \mathbf{X}^{R} = \hat{\mathbf{M}} \mathbf{X}^{R} \bar{\mathbf{\Lambda}}. \tag{32}$$

However, modal matrix \mathbf{X}^R can also represent the left modal matrix of $\hat{\mathbf{K}}$, and $\hat{\mathbf{M}}$. This is due to the fact that they are symmetric matrices, as shown. Thus, when Eq. (32) is multiplied by transpose of the right modal matrix, we have:

$$(\mathbf{X}^R)^T \, \hat{\mathbf{K}} \, \mathbf{X}^R = (\mathbf{X}^R)^T \, \hat{\mathbf{M}} \, \mathbf{X}^R \, \bar{\mathbf{\Lambda}}.$$
(33)

This will also result in diagonal matrices on both sides due to orthogonality relation of eigen-vectors. Eq. (33) may also be written as Eq. (31) with the same \mathbf{K}^* , and \mathbf{M}^* . Certainly, this requires proper scaling of left eigen-vectors, which form the columns of the left modal matrix, \mathbf{X}^L . Since both Eqs. (30) and (33) can lead to Eq. (31), one can conclude the following equation:

$$(\mathbf{X}^L)^T \, \bar{\mathbf{M}} = (\mathbf{X}^R)^T \, \hat{\mathbf{M}}. \tag{34}$$

Or alternatively:

$$(\mathbf{X}^L)^T = (\mathbf{X}^R)^T \mathbf{D}, \qquad (35)$$

with the following definition:

$$\mathbf{D} = \hat{\mathbf{M}} \, \bar{\mathbf{M}}^{-1}. \tag{36}$$

By substituting Eq. (5) in Eq. (36), we have:

$$\mathbf{D} = \hat{\mathbf{K}} \, \bar{\mathbf{K}}^{-1}. \tag{37}$$

Although a simple relation (i.e., Eq. (35)) is obtained between left and right modal matrices, let us try to simplify it even further. For this purpose, one can expand the right-hand side of Eq. (37) by employing Eqs. (3) and (7), which yields:

$$\mathbf{D} = \begin{bmatrix} \mathbf{I} & \mathbf{B}^T \mathbf{H}^{-1} \\ \mathbf{0} & \mathbf{G}^T \mathbf{H}^{-1} \end{bmatrix}.$$
(38)

It is easy to verify that the above relation can also be written as:

$$\mathbf{D} = \begin{bmatrix} \mathbf{I} & \mathbf{B}^T \\ \mathbf{0} & \mathbf{G}^T \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix}.$$
(39)

Substituting Eq. (39) in Eq. (35) and transposing both sides results in:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \mathbf{X}^{R}.$$
(40)

The following relation can also be easily verified through Eq. (40) by partitioning both left and right modal matrices into two parts, which correspond to the structure and fluid degrees of freedom, respectively:

$$\mathbf{X}_{1}^{L} = \mathbf{X}_{1}^{R}. \tag{41}$$

Moreover, Eq. (40) may be written as:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \mathbf{X}^{R}, \quad (42)$$

or alternatively as:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{B} & \mathbf{G} \end{bmatrix} \mathbf{X}^{R}.$$
(43)

By noting the definition of $\overline{\mathbf{M}}$ (i.e., Eq. (2b)), Eq. (43) may be written as:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \mathbf{\bar{M}} \mathbf{X}^{R}.$$
(44)

Utilizing Eq. (29), the above relation may also be expressed as:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix} \mathbf{\bar{K}} \mathbf{X}^{R} \mathbf{\bar{\Lambda}}^{-1}, \qquad (45)$$

or alternatively as:

$$\mathbf{X}^{L} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{B}^{T} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1}^{R}\,\bar{\mathbf{\Lambda}}^{-1} \\ \mathbf{X}_{2}^{R}\,\bar{\mathbf{\Lambda}}^{-1} \end{bmatrix}. (46)$$

Having in mind the partitioned form of \mathbf{X}^{L} as mentioned above, Eq. (46) reveals the following:

$$\mathbf{X}_{2}^{L} = \mathbf{X}_{2}^{R} \, \bar{\mathbf{\Lambda}}^{-1}. \tag{47}$$

Finally, a simple relation is obtained between the left and right modal matrices through utilizing Eqs. (41) and (47), which is written as:

$$\begin{bmatrix} \mathbf{X}_1^L \\ \mathbf{X}_2^L \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1^R \\ \mathbf{X}_2^R \bar{\mathbf{\Lambda}}^{-1} \end{bmatrix}.$$
(48)

3. Numerical example

An idealized symmetric model of Morrow Point arch dam-reservoir system is considered as an example, which is explained in this section.

3.1. Details of the model

The geometry of the dam can be found in [17]. The dam is discretized by 40 isoparametric 20-node solid finite elements, while impounded water domain is modeled by 360 isoparametric 20-node fluid finite elements (Figure 1). It should be mentioned that the length of reservoir is taken as two times the dam height or maximum water depth (i.e., L/H = 2.0). The length of the reservoir is measured in stream direction at dam mid-crest point.

The dam concrete is assumed to be homogeneous with isotropic linearly elastic behavior and the following main characteristics:



Figure 1. Discretization of dam-reservoir system.

- Elastic modulus $(E_d) = 27.5$ GPa;
- Poisson's ratio = 0.2;
- Unit weight = 24.8 kN/m^3 .

Also, the impounded water is taken as inviscid and compressible fluid with unit weight equal to 9.81 kN/m^3 and pressure wave velocity C = 1440 m/sec.

3.2. Results

The first 10 smallest eigen-values and right and left eigen-vectors of the dam-reservoir system are calculated by the pseudo symmetric subspace iteration method. These eigen-values and the corresponding natural frequencies are presented in Table 1.

Moreover, the first 2 right eigen-vectors are depicted in Figures 2 and 3. It should be mentioned that the part of eigen-vectors related to displacement degrees of freedom is shown as the deformed shape of dam body, while pressure degrees of freedom are presented with contour values in the reservoir domain. Furthermore, the y-displacement values (i.e., stream

 Table 1. Eigen-values and natural frequencies of the dam-reservoir system.

| Mode number (i) | $\lambda_i \; ({ m Rad}^2/{ m Sec}^2)$ | $f_i~({ m Hz})$ |
|-------------------|--|-----------------|
| 1 | 300.9 | 2.76094 |
| 2 | 337.0 | 2.92179 |
| 3 | 438.3 | 3.33198 |
| 4 | 622.2 | 3.96993 |
| 5 | 898.9 | 4.77185 |
| 6 | 1145.7 | 5.38715 |
| 7 | 1362.3 | 5.87431 |
| 8 | 1655.8 | 6.47635 |
| 9 | 1668.4 | 6.50078 |
| 10 | 2117.4 | 7.32348 |

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Figure 2. The first right mode shape of the dam-reservoir system: (a) Dam body and (b) impounded water domain.



Figure 3. The second right mode shape of the dam-reservoir system: (a) Dam body and (b) impounded water domain.

direction component) are shown as contours on the deformed shape of dam body in each case.

It is worthwhile to mention that the mode shapes of the system considered are either exactly symmetric or anti-symmetric. This is due to the fact that the problem is completely symmetric. This can easily be noticed in the first 2 mode shapes through the deformed shapes, y-displacement, and pressure contour patterns and values illustrated.

The left eigen-vectors are not shown here, since they are exactly same as right eigen-vectors with the exception that the pressure values are multiplied by a scaled value (i.e., λ_i^{-1}), as it was proven.

Additionally, the above example is also solved by an unsymmetric eigen-solution routine to make sure about the validity of the results presented. However, the results for this case are not given here, since they are exactly same as the above-mentioned results. Certainly, this is expected due to the fact that the finite element mesh, material properties, and boundary conditions are the same. In other words, matrices of the eigen-value problem are identical and the only difference is the method employed.

4. Efficiency of the method

It is also interesting to get a general feeling of the efficiency of the pseudo symmetric subspace iteration method. For this purpose, the Morrow Point arch dam-reservoir system is considered once again with the material properties, basic assumptions, and dam body discretization mentioned above. However, 6 different cases are defined in this section among which the only difference is the length of the reservoir and the finite element mesh utilized in the fluid domain. The third case (i.e., L/H=2.0) is same as that discussed in the previous section. The first 10 eigen-values and the corresponding right and left eigen-vectors are obtained in each case. The execution times for these cases are compared in Table 2 along with the total number of degrees of freedom employed in each case. It should be mentioned that all these cases are executed on a Pentium-4 personal computer with CPU speed of 2.8 GHz.

It is noteworthy that Pseudo symmetric subspace iteration method is extremely efficient and the execution time for the lower numbers of degrees of freedom (e.g., less than 2000) is less than 10 seconds. Although it increases for the higher numbers of degrees of freedom, it is still less than 60 seconds for the largest problem considered in this study (i.e., case 6 with NDOF = 3775).

5. Concluding remarks

This study proposed an efficient technique for calculation of coupled modes of fluid-structure interaction systems, which was referred to as pseudo symmetric subspace iteration method.

Although it was developed for achieving a so-

| Case | L/H | Number of solid elem. | Number of fluid elem. | Total NDOF | Execution time (sec.) |
|------|-----|--------------------------|--------------------------|------------|--------------------------|
| 1 | 0.2 | 40 | 80 | 1165 | 2.4 |
| 2 | 1.0 | 40 | 200 | 1600 | 5.3 |
| 3 | 2.0 | 40 | 360 | 2180 | 11.3 |
| 4 | 3.0 | 40 | 520 | 2760 | 27.3 |
| 5 | 4.0 | 40 | 680 | 3340 | 37.5 |
| 6 | 5.0 | 40 | 800 | 3775 | 53.2 |

Table 2. Total execution time in seconds for different cases.

lution to an unsymmetrical eigen-value problem of fluid-structure systems, the algorithm was presented with symmetric operations mentality. Therefore, the problem seemed to be symmetric and additional calculations were minimal.

Furthermore, it was proven that each left eigenvector was related to the corresponding right eigenvector through a simple relation. Therefore, the subsequent transient analysis could readily be performed.

Overall, it is believed that the method is very efficient and it is ideal to employ in general-purpose finite element programs for solving the above-mentioned eigen-problems.

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Appendix

The following special matrix multiplication notation are defined which were used in Section 2.3 extensively. There are six different types of this matrix multiplication. In each case, one or several parts of the first matrix are not participating in the multiplication process as observed:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{4} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} , (A.1)$$
$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{2} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} , (A.2)$$
$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{3} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} , (A.3)$$

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{4} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix}, (A.4)$$

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{5} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} , (A.5)$$
$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{6} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1} \\ \mathbf{X}_{2} \end{bmatrix} . (A.6)$$

Biographies

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