T-spherical Fuzzy Soft Sets and its Aggregation Operators with Application in Decision Making

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Abstract

In the present manuscript, we introduce a novel concept of T-spherical fuzzy soft set with various important operations and properties. In the field of information theory, an aggregation operator is a structured mathematical function which aggregates all the information received as input and provides a single output entity, which are found to be applicable for various important decision making applications. Some averaging aggregation operators and geometric aggregation operators (weighted, ordered and hybrid) for T-spherical fuzzy soft numbers have been proposed with their various properties. Further, utilizing the proposed aggregation operators of various types along with the properly defined score function/accuracy function, an algorithm for solving a decision making problem has been provided. The proposed methodology has also been well illustrated with the help of a numerical example. Some comparative remarks and advantages of the introduced notion of T-spherical fuzzy soft set and the proposed methodology have been listed for a better motivation and readability.

Keywords: Soft set, T-spherical fuzzy set, Aggregation operators, Score and Accuracy function, Decision making.

1 Introduction

The researchers in the field of fuzzy sets (FSs) and information are well aware of various generalized extensions of the notion of FSs [1] and intuitionistic fuzzy sets (IFSs) [2] [3] which have been taken place to model the uncertainties and the hesitancy inherent in many practical circumstances to have a wider coverage for flexibility. Yager [4] revealed that the existing structures of fuzzy set and intuitionistic fuzzy set is not capable enough to depict the human opinion in more practical sense and introduced Pythagorean fuzzy set (PyFS) which effectively enlarged the span of information by introducing the new conditional constraint. The concept of membership/belongingness (yes), nonmembership/nonbelongingness (no) and indeterminacy/neutral (abstain) have been well described by the definition of IFSs as well as by the PyFSs. Consider an example of voting system where voters can be categorized into four different classes: one who votes for (yes), one who votes against (no), one who neither vote for nor against (abstain), one who refused for voting (refusal). It may be noted that the concept of refusal has not been taken into consideration by any of the sets stated above. Cuong [5] introduced the concept of picture fuzzy set (PFS) to handle such circumstances which would be sufficiently close to humans nature of flexibility, where all the four parameters, that is, degree of membership, indeterminacy (neutral), nonmembership and refusal have been taken into account. For the sake of better understanding of the available literature, the various generalizations of fuzzy set are being presented by a road map given in Figure 1.

Recently, Mahmood et al. [6] introduced the notion of spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) which give additional strength to the picture fuzzy set by broadening the span for the membership of all the four parameters stated above. Next, Kifayat et al. [7] studied the geometrical comparison of FSs, IFSs, PyFSs, PFS with SFSs and TSFSs. In addition, the limitations of the existing similarity measures for IFSs and PFSs have been provided in view of the broader setup of the spherical fuzzy environment. Further, they proposed various types of similarity measures for TSFSs with their useful applications in various fields. Recently, Garg et al. [8] presented a new improved interactive aggregation operators for TSFSs with application in decision making. Also, in recent years, researchers have gone through watchful deliberations to the existing theories and applied them to various application fields [9] [10] [7] [6].

2 Literature Review

Many researchers in the field of soft computing have widely utilized the concept of the aggregation operators in which they have featured the concerns of each criteria/attribute or its ordered position. However, in real world circumstances where the interrelationships or connections among the different criteria, for example, priority, support feature, and inter–impact, reflects a significant role in the process of aggregation, have not been addressed successfully.

Some geometric aggregated operators for intuitionistic fuzzy numbers (IFNs) have been proposed by Xu and Yager [11]. further, Yager [12] introduced the idea of power average aggregation operator which takes the argument values in order to support each other in the process of aggregation. For intuitionistic fuzzy numbers, Xu & Yager [13] and Yu [14] studied the prioritized averaging/geometric aggregations. In addition to this, the new operator called Bonferroni mean, which was able to handle the interrelationship between the input values, was proposed by Yager [15]. Subsequently, Xu and Yager [16] extended the existing Bonferroni mean for aggregating the IFNs. Liu and Li [17] incorporated power Bonferroni mean operators for the interval-valued IFNs and defined some standard operations with their operational laws. They proposed different kinds of important operators such as power, weighted power, power geometric & weighted power geometric Bonferroni mean operators with various properties and applications in detail. Wang and Liu [18] using the Einstein norm operators extended the aggregation operators in intuitionistic fuzzy environment.

Tao et al. [19] implemented the Archimedean copulas and associated co-copulas on the intuitionistic fuzzy numbers with some basic operations. They also proposed the copula weighted averaging aggregation operators and studied their basic properties. Further, they provided the modified maximizing deviation decision algorithm and illustrated its applicability in the decision making problem. Garg [20] extended the generalized geometric as well as averaging operators over the Pythagorean fuzzy set.

In literature, there are many theories which are used to deal with vagueness and uncertainties of many problems arise in engineering, economics, social science etc. But all the theories have their own limitations intuitively because of the parametrization tool involved in it. In order to overcome these difficulties, Molodtsov [21] introduced the notion of soft set to deal with the uncertainties, which is free from the inadequacy of parametrization and established various results based on this. Maji et al. [22] studied the theory of soft sets and defined soft binary operations such as AND, OR, Union, Intersection, Equality, Complement of the soft sets. Various researchers have utilized the notion of fuzzy soft sets in the field of investment decision making problems [23] [24], optimization modelling [25], ranking in decision making problem [26], etc.

Further, Maji et al. [27] [28] successfully extended the soft set to fuzzy soft set (FSS)/intuitionistic fuzzy soft set (IFSS) and studied their applications in decision-making problems. Peng et al. [29] introduced the Pythagorean fuzzy soft set along with various binary operations and also proposed an algorithm for decision making. Next, Peng and Yang [30] proposed two novel approaches to solve stochastic MCDM based on regret and prospect theory for interval valued fuzzy soft sets. Also, Peng and Garg [31] devised three algorithms to solve a decision making problem by utilizing WDBA, CODAS and similarity measures. Further, algorithms for neutrosophic soft decision making based on EDAS [32] and for hesitant fuzzy soft decision making problem based on WDBA/CODAS [33] have been studied in recent literature. Cuong [5] proposed the combination of picture fuzzy set and soft set as picture fuzzy soft set (PFSS) and various operations and properties have also been discussed.

Arora and Garg [9] & Garg et al. [10] studied the aggregated operators for IFSSs with their applications in decision making. Wei [34] proposed some picture fuzzy aggregated operators viz., weighted, ordered-weighted & hybrid average operator/weighted, ordered-weighted & hybrid geometric operator by utilizing arithmetic/geometric operators. Liu and Zhang [35] introduced the picture fuzzy linguistic numbers with some basic operations and operational laws. They also proposed the Archimedean picture fuzzy weighted averaging operator and studied their basic properties. Further, they provided an algorithm for decision making process by utilizing the proposed averaging operator and illustrated the applicability with the help of a numerical example.

The discussions on the existing literature reveal that we can enhance the flexibility of human opinions with revised conditional/spherical constraints by proposing a notion of T-spherical fuzzy soft set as a new paradigm so that the parametrization may also be taken care to deal with the decision making problems. Additionally, the study of various aggregation operators (avergaing/geometric/hybrid) for T-spherical fuzzy soft to be carried to propose the methodologies.

In the present work, we extend the concept of picture fuzzy soft set by proposing the T-spherical fuzzy soft set (TSFSS) along with various aggregation operators and applications. The organization of the paper is as follows. In Section 3, basic definitions and preliminaries have been provided for a better understanding of the proposed extension and application. In Section 4, we have formally introduced T-spherical fuzzy soft set, their basic set theoretic operations, properties and score/accuracy function. Various types of aggregation operators (avergaing/geometric/hybrid) for T-spherical fuzzy soft numbers with their properties and results have been studied in Section 5. In Section 6, a decision making problem has been formulated and an algorithm implementing the proposed aggregation operators has been provided. A numerical example has also been provided in Section 7 for illustrating all the necessary steps of the proposed methodology. Some comparative remarks and advantages of the proposed work have been listed in Section 8. Finally, the paper has been concluded by indicating the direction for some future work in Section 9.

3 Preliminaries

Some fundamental concepts in connection with SFS and TSFS, which are well known in literature, are being presented in this section. The following notions explain the generalization process from IFSs to TSFSs:

Let U be the universe of discourse with $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ being the degree of membership and degree of non-membership respectively. The set $A = \{< u, \mu_A(u), \nu_A(u) >| \, u \in U\}$ is called

- ***Intuitionistic Fuzzy Set*** [2] A in U if it satisfies the condition $0 \leq \mu_A(u) + \nu_A(u) \leq 1$ with the degree of indeterminacy given by $\pi_A(u) = 1 - \mu_A(u) - \nu_A(u)$.
- ***Pythagorean Fuzzy Set*** [4] or ***Intuitionistic Fuzzy Set of second type*** [36] A in U if it satisfies the condition $0 \leq \mu_A^2(u) + \nu_A^2(u) \leq 1$ with the degree of indeterminacy given by $\pi_A(u) = \sqrt{1 - \mu_A^2(u) - \nu_A^2(u)}$.

In order to have further generalization, we consider the universe of discourse U with $\mu_A : U \to [0,1]$, $\eta_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ being the degree of membership, degree of neutral membership (abstain) and degree of non-membership respectively. The set $A = \{< u, \mu_A(u), \eta_A(u), \nu_A(u) >| \, u \in U\}$ is called as

- ***Picture Fuzzy Set*** [5] A in U if it satisfies the condition $\mu_A(u)+\eta_A(u)+\nu_A(u) \leq 1$ with the degree of refusal given by $r_A(u) = 1 - (\mu_A(u) + \eta_A(u) + \nu_A(u))$.
- ***Spherical Fuzzy Set*** [6] A in U if it satisfies the condition $\mu_A^2(u)+\eta_A^2(u)+\nu_A^2(u) \leq 1$ with the degree of refusal given by
$$r_A(u) = \sqrt{1 - (\mu_A^2(u) + \eta_A^2(u) + \nu_A^2(u))}.$$
- ***T-spherical Fuzzy Set*** [6] Let n is any natural number. A set A in U if it satisfies the condition $\mu_A^n(u) + \eta_A^n(u) + \nu_A^n(u) \leq 1$ with the degree of refusal given by
$$r_A(u) = \sqrt[n]{1 - (\mu_A^n(u) + \eta_A^n(u) + \nu_A^n(u))}.$$

Particular Cases:

- For $n = 2$, TSFS becomes SFS.
- For $n = 1$, TSFS becomes PFS.
- For $n = 2$ and $r_A = 0$, TSFS becomes PyFS or IFS of second type.
- For $n = 1$ and $r_A = 0$, TSFS becomes IFS.

Similarly, the generalization in the field of Soft Sets to Picture fuzzy soft sets with explanatory examples are available in literature and are being listed below:
Let $U = \{u_1, u_2, \ldots, u_m\}$ be the universe of discourse and $P = \{p_1, p_2, \ldots, p_n\}$ be the set of parameters. The pair (Φ, P) is called

- ***Soft Set*** [21] over U iff $\Phi : P \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ is the power set of U.
- ***Fuzzy Soft Set*** [37] over $\Phi(U)$, where Φ is a mapping given by $\Phi : P \to (F(U))$ and $F(U)$ denotes the set of all fuzzy sets of U.
- ***Intuitionistic Fuzzy Soft Set*** [22] over U if $\Phi : P \to IFS(U)$ and can be represented as

  $$(\Phi, P) = \{(p, \Phi(p)) : p \in P, \Phi(p) \in IFS(U)\},$$

  where $IFS(U)$ represents the set of all IFSs of U.
- ***Pythagorean Fuzzy Soft Set*** [29] over U if $\Phi : P \to PYFS(U)$ and can be represented as

  $$(\Phi, P) = \{(p, \Phi(p)) : p \in P, \Phi(p) \in PYFS(U)\},$$

  where $PYFS(U)$ represents the set of all PyFSs of U.
- ***Picture Fuzzy Soft Set*** [5] over U if $\Phi : P \to PFS(U)$ and can be represented as

  $$(\Phi, P) = \{(p, \Phi(p)) : p \in P, \Phi(p) \in PFS(U)\},$$

  where $PFS(U)$ represents the set of all PFSs of U.

4 T-spherical Fuzzy Soft Set

In this section, we introduce T-spherical fuzzy soft set as an extension to Picture fuzzy soft set [5]. Further, the score and accuracy function for the defined T-spherical fuzzy soft set have been proposed along with various operations and different properties.

Definition 1 (T-spherical Fuzzy Soft Set) *Suppose U be the domain of discourse and $TSFS(U)$ denotes the collection of all T-spherical fuzzy sets over U. Let P be the set of parameters. The pair (Φ, P) is a T-spherical fuzzy soft set over U iff $\Phi : P \to TSFS(U)$. For any parameter $p_k \in P$, Φ_{p_k} is a T-spherical fuzzy soft set given by $\Phi_{p_k} = \{< u_i, \mu_k(u_i), \eta_k(u_i), \nu_k(u_i) >| \ u_i \in U\}$; where $\mu_k(u_i), \eta_k(u_i)$, and $\nu_k(u_i)$ are the degree of membership, neutral membership (abstain) and non-membership respectively, with the condition*

$$0 \leq \mu_k^n(u_i) + \eta_k^n(u_i) + \nu_k^n(u_i) \leq 1$$

and the degree of refusal

$$r_k(u_i) = \sqrt[n]{1 - (\mu_k^n(u_i) + \eta_k^n(u_i) + \nu_k^n(u_i))};$$

where n is a natural number.

Example 1 *Consider the set of four houses, say, $H = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and the set of parameters under consideration, say, $P = \{expensive(p_1), wooden(p_2), cheap(p_3), beautiful(p_4), good\ location(p_5)\}$. Then the perception for the attractiveness of the houses may be described as a T-spherical fuzzy soft set given by*

$$(\Phi, P) = \{\Phi_{p_1}, \Phi_{p_2}, \Phi_{p_3}, \Phi_{p_4}, \Phi_{p_5}\};$$

where

$$\begin{aligned}
\Phi_{p_1} &= \{< \gamma_1, 0.2, 0.5, 0.6 >, < \gamma_2, 0.3, 0.3, 0.5 >\};\\
\Phi_{p_2} &= \{< \gamma_3, 0.4, 0.3, 0.4 >, < \gamma_4, 0.2, 0.2, 0.7 >, < \gamma_5, 0.9, 0.1, 0.1 >\};\\
\Phi_{p_3} &= \{< \gamma_1, 0.3, 0.5, 0.2 >, < \gamma_4, 0.5, 0.3, 0.2 >\};\\
\Phi_{p_4} &= \{< \gamma_1, 0.6, 0.3, 0.2 >, < \gamma_4, 0.9, 0.2, 0.1 >, < \gamma_5, 0.5, 0.4, 0.2 >\};\\
\Phi_{p_5} &= \{< \gamma_2, 0.2, 0.1, 0.7 >, < \gamma_4, 0.6, 0.2, 0.2 >\}.
\end{aligned}$$

In particular, if we take $n = 2$, the Definition 1 leads to the definition of ***Spherical Fuzzy Soft Set***.

Next, we propose some basic operations on TSFSs.

Operations on T-spherical Fuzzy Soft Sets:
Let (Φ, Q) and (Ψ, R) are two T-spherical fuzzy soft sets on the same universe of discourse U. Let $Q, R \subseteq P$ be the set of parameters, then

- ***Complement*** $(\Phi, Q)^c = (\Phi^c, Q)$ where $\Phi^c : Q \to TSFS(U)$ is a mapping given by $\Phi^c(p) = (\Phi(p))^c$, for all $p \in Q$.
- ***Subsethood:*** $(\Phi, Q) \subseteq (\Psi, R)$, iff $Q \subseteq R$ and for all $p \in Q,\ \Phi(p) \subseteq \Psi(p)$.
- ***Equality:*** $(\Phi, Q) = (\Psi, R)$, if $(\Phi, Q) \subseteq (\Psi, R)$ and $(\Psi, R) \subseteq (\Phi, Q)$.
- ***Union:*** $(\Phi, Q) \cup (\Psi, R) = (H, S)$; where $S = Q \cup R$ for all $\xi \in S$ and

$$H(\xi) = \begin{cases} \Phi(\xi) & \xi \in Q - R, \\ \Psi(\xi) & \xi \in R - Q, \\ \Phi(\xi) \cup \Psi(\xi) & \xi \in Q \cap R. \end{cases}$$

  In other words, for all $\xi \in Q \cap R$,

$$H(\xi) = \left\{ \left(u, \max(\mu_{\Phi(\xi)}(u), \mu_{\Psi(\xi)}(u)), \min(\eta_{\Phi(\xi)}(u), \eta_{\Psi(\xi)}(u)), \min(\nu_{\Phi(\xi)}(u), \nu_{\Psi(\xi)}(u))\right) \right\}.$$

- ***Intersection:*** $(\Phi, Q) \cap (\Psi, R) = (H, S)$; where $S = Q \cap R$ for all $\xi \in S$ and

$$H(\xi) = \begin{cases} \Phi(\xi) & \xi \in Q - R, \\ \Psi(\xi) & \xi \in R - Q, \\ \Phi(\xi) \cap \Psi(\xi) & \xi \in Q \cap R. \end{cases}$$

  In other words, for all $\xi \in Q \cap R$,

$$H(\xi) = \left\{ \left(u, \min(\mu_{\Phi(\xi)}(u), \mu_{\Psi(\xi)}(u)), \min(\eta_{\Phi(\xi)}(u), \eta_{\Psi(\xi)}(u)), \max(\nu_{\Phi(\xi)}(u), \nu_{\Psi(\xi)}(u))\right) \right\}.$$

Proposition 1 *Suppose (Φ, Q) and (Ψ, R) are two T-spherical fuzzy soft sets on the universal set U. Let $Q, R \subseteq P$ be two subsets of the set of parameters, then as per their definitions, the following properties clearly hold:*

(i) $((\Phi, Q)^c)^c = (\Phi, Q)$.

(ii) $((\Phi, Q) \cap (\Psi, R))^c = (\Phi, Q)^c \cup (\Psi, R)^c$.

(iii) $((\Phi, Q) \cup (\Psi, R))^c = (\Phi, Q)^c \cap (\Psi, R)^c$.

Further, for simplicity and necessary computations, TSFSS can also be regarded as $T_u = (\mu_u, \eta_u, \nu_u)$ and called as T-spherical fuzzy soft number (TSFSN), where u is referential subscript used for establishing a connection between alternatives and parameters in computational examples. For application purposes to rank these numbers, we propose the score and accuracy functions for the T-spherical fuzzy soft numbers as follows:

Definition 2 *Let $T_u = (\mu_u, \eta_u, \nu_u)$ be the T-spherical fuzzy soft number, then*

- *the score function is given as $S(T_u) = \mu_u^n - \eta_u^n - \nu_u^n;\ \ S(T_u) \in [-1, 1]$.*
- *the accuracy function is given as $H(T_u) = \mu_u^n + \eta_u^n + \nu_u^n;\ \ H(T_u) \in [0, 1]$.*

Based on these two functions above defined, the ordering of two T-spherical fuzzy soft numbers can be determined as follows:
Let $T_u = (\mu_u, \eta_u, \nu_u)$ and $T_v = (\mu_v, \eta_v, \nu_v)$ be two T-spherical fuzzy soft numbers then

- $T_u \geq T_v$ if $S(T_u) \geq S(T_v)$.
- $T_u \leq T_v$ if $S(T_u) \leq S(T_v)$.

  In case, if $S(T_u) = S(T_v)$ for any two TSFSN, then
- $T_u \geq T_v$ if $H(T_u) \geq H(T_v)$.
- $T_u \leq T_v$ if $H(T_u) \leq H(T_v)$.
- $T_u \sim T_v$ if $H(T_u) = H(T_v)$.

5 Aggregation Operators for T-spherical Fuzzy Soft Numbers

In information fusion process, aggregation operators mathematically aggregate all the input information such as significance of criteria/attribute, prioritization, support and interrelationships of the individual data into single one. The aggregation operators have been widely implemented by the various researchers in decision making problems. In this section, we broadly propose two types of operators for T-spherical Fuzzy Soft Numbers - *averaging aggregation operators* and *geometric aggregation operators.*

5.1 Averaging Aggregation Operators

Here, we propose some averaging aggregation operators such as weighted averaging operator, ordered weighted/hybrid averaging operator with their properties for the proposed T-spherical fuzzy soft numbers. Some standard operations for the T-spherical fuzzy soft numbers are being defined as follows:

Let $T_u = (\mu_u, \eta_u, \nu_u)$ and $T_v = (\mu_v, \eta_v, \nu_v)$ be two T-spherical fuzzy soft numbers and $\lambda > 0$ be any real number. Then the following operations are defined over the two T-spherical fuzzy soft numbers as:

(a) $T_u \oplus T_v = (\sqrt[n]{\mu_{T_u}^n + \mu_{T_v}^n - \mu_{T_u}^n \mu_{T_v}^n},\ \eta_{T_u}\eta_{T_v},\ \nu_{T_u}\nu_{T_v})$.

(b) $T_u \otimes T_v = (\mu_{T_u}\mu_{T_v},\ \sqrt[n]{\eta_{T_u}^n + \eta_{T_v}^n - \eta_{T_u}^n \eta_{T_v}^n},\ \sqrt[n]{\nu_{T_u}^n + \nu_{T_v}^n - \nu_{T_u}^n \nu_{T_v}^n})$.

(c) $\lambda T_u = (\sqrt[n]{1-(1-\mu_{T_u}^n)^\lambda}, \eta_{T_u}^\lambda, \nu_{T_u}^\lambda)$.

(d) $T_u^\lambda = \big((\mu_{T_u})^\lambda,\ \sqrt[n]{1-(1-\eta_{T_u}^n)^\lambda},\ \sqrt[n]{1-(1-\nu_{T_u}^n)^\lambda}\big)$.

(e) $T_u^c = (\nu_u, \eta_u, \mu_u)$.

Theorem 1 *Suppose $T_u = (\mu_u, \eta_u, \nu_u)$ and $T_v = (\mu_v, \eta_v, \nu_v)$ be two T-spherical fuzzy soft numbers and $\lambda, \lambda_1, \lambda_2 > 0$ be the real numbers. Then the following operational laws hold:*

(i) $T_u \oplus T_v = T_v \oplus T_u$

(ii) $T_u \otimes T_v = T_v \otimes T_u$

(iii) $\lambda(T_u \oplus T_v) = \lambda T_v \oplus \lambda T_u$

(iv) $(T_u \otimes T_v)^\lambda = T_v^\lambda \otimes T_u^\lambda$

(v) $\lambda_1 T_u \oplus \lambda_2 T_u = (\lambda_1 + \lambda_2) T_u$

(vi) $T_u^{\lambda_1} \otimes T_u^{\lambda_2} = T_u^{(\lambda_1+\lambda_2)}$

(vii) $(T_u^{\lambda_1})^{\lambda_2} = T_u^{\lambda_1\lambda_2}$.

Proof: The proof of these operational laws immediately follows from the definitions stated in (a) - (e).

Definition 3 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSWA_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is said to be T-spherical fuzzy soft weighted averaging operator, if*

$$TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \oplus_{j=1}^m (\omega_j T_{u_j}); \tag{1}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ *is the weight vector corresponding to* $(T_{u_j})_{j=1}^m$ *such that* $\omega_j \geq 0,$ *for all* $j; \sum_{j=1}^{m} \omega_j = 1.$

Theorem 2 *The* T*-spherical fuzzy soft weighted averaging operator* $TSFSWA_\omega$ *aggregates all the input values and yields a TSFSN given by*

$$TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\sqrt[n]{1 - \prod_{j=1}^{m}(1 - \mu_{T_{u_j}}^n)^{\omega_j}}, \prod_{j=1}^{m}(\eta_{T_{u_j}})^{\omega_j}, \prod_{j=1}^{m}(\nu_{T_{u_j}})^{\omega_j} \right); \tag{2}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ *is the weight vector corresponding to* $(T_{u_j})_{j=1}^m$ *such that* $\omega_j \geq 0,$ *for all* $j; \sum_{j=1}^{m} \omega_j = 1.$

Proof: Using the *principle of mathematical induction* on m, the result in the Equation (2) given in the above stated Theorem 2 can be proved as follows:
For $m = 2$, we have

$$TSFSWA_\omega(T_{u_1}, T_{u_2}) = \omega_1 T_{u_1} \oplus \omega_2 T_{u_2}.$$

It may be observed that both $\omega_1 T_{u_1}$ and $\omega_2 T_{u_2}$ are T-spherical fuzzy soft numbers. From the operation (c) defined in Subsection 5.1 for T-spherical fuzzy soft numbers, we have

$$\omega_1 T_{u_1} = (\sqrt[n]{1 - (1 - \mu_{T_{u_1}}^n)^{\omega_1}}, \eta_{T_{u_1}}^{\omega_1}, \nu_{T_{u_1}}^{\omega_1});$$

and

$$\omega_2 T_{u_2} = (\sqrt[n]{1 - (1 - \mu_{T_{u_2}}^n)^{\omega_2}}, \eta_{T_{u_2}}^{\omega_2}, \nu_{T_{u_2}}^{\omega_2}).$$

Then,

$$\begin{aligned} TSFSWA_\omega(T_{u_1}, T_{u_2}) &= \omega_1 T_{u_1} \oplus \omega_2 T_{u_2} \\ &= \Bigg(\sqrt[n]{(2 - (1 - \mu_{T_{u_1}}^n)^{\omega_1} - (1 - \mu_{T_{u_2}}^n)^{\omega_2}) - \Big((1 - (1 - \mu_{T_{u_1}}^n)^{\omega_1})(1 - (1 - \mu_{T_{u_2}}^n)^{\omega_2})\Big)}, \\ &\qquad (\eta_{T_{u_1}}^n)^{\omega_1} (\eta_{T_{u_2}}^n)^{\omega_2}, (\nu_{T_{u_1}}^n)^{\omega_1} (\nu_{T_{u_2}}^n)^{\omega_2} \Bigg) \\ &= \left(\sqrt[n]{1 - \Big((1 - \mu_{T_{u_1}}^n)^{\omega_1} (1 - \mu_{T_{u_2}}^n)^{\omega_2}\Big)}, (\eta_{T_{u_1}}^n)^{\omega_1} (\eta_{T_{u_2}}^n)^{\omega_2}, (\nu_{T_{u_1}}^n)^{\omega_1} (\nu_{T_{u_2}}^n)^{\omega_2} \right). \end{aligned}$$

Hence, the result holds for $m = 2$. Next, we suppose that the Equation (2) holds for $m = k$, that is,

$$\begin{aligned} TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_k}) &= \omega_1 T_{u_1} \oplus \omega_2 T_{u_2} \oplus \ldots \oplus \omega_k T_{u_k} \\ &= \left(\sqrt[n]{1 - \prod_{j=1}^{k}(1 - \mu_{T_{u_j}}^n)^{\omega_j}}, \prod_{j=1}^{k}(\eta_{T_{u_j}}^n)^{\omega_j}, \prod_{j=1}^{k}(\nu_{T_{u_j}}^n)^{\omega_j} \right). \end{aligned}$$

Next, the result for $m = k+1$ to be proved, that is, $TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_{k+1}})$ is also a TSFSN. By operational laws of T-spherical fuzzy soft numbers, we have

$$
\begin{aligned}
&TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_{k+1}}) \\
&= \omega_1 T_{u_1} \oplus \omega_2 T_{u_2} \oplus \ldots \oplus \omega_k T_{u_k} \oplus \omega_{k+1} T_{u_{k+1}} \\
&= \Bigg(\sqrt[n]{1 - \prod_{j=1}^{k} (1 - \mu^n_{T_{u_j}})^{\omega_j} + 1 - (1 - \mu^n_{T_{u_{k+1}}})^{\omega_{k+1}} - \big((1 - \prod_{j=1}^{k} (1 - \mu^n_{T_{u_j}})^{\omega_j})(1 - (1 - \mu^n_{T_{u_{k+1}}})^{\omega_{k+1}})\big)}, \\
&\qquad\qquad \prod_{j=1}^{k+1} (\eta^n_{T_{u_j}})^{\omega_j}, \prod_{j=1}^{k+1} (\nu^n_{T_{u_j}})^{\omega_j} \Bigg) \\
&= \Bigg(\sqrt[n]{1 - \prod_{j=1}^{k+1} (1 - \mu^n_{T_{u_j}})^{\omega_j}}, \prod_{j=1}^{k+1} (\eta^n_{T_{u_j}})^{\omega_j}, \prod_{j=1}^{k+1} (\nu^n_{T_{u_j}})^{\omega_j} \Bigg).
\end{aligned}
$$

This clearly shows that the aggregated value is also a TSFSN. Therefore, the Equation (2) holds for all $m = k+1$. This finally proves the result.

Next, we state the following properties related to the T-spherical fuzzy soft weighted averaging operator which can easily be proved in view of the definitions:

(i) ***Idempotency*:** If $T_{u_j} = T_u$; for all $j = 1, 2, \ldots, m$, then

$$TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) = T_u.$$

(ii) ***Boundedness*:** If T_{u_j} $(j = 1, 2, \ldots, m)$ be the collection of TSFSNs, then

$$\min_j T_{u_j} \leq TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) \leq \max_j T_{u_j}.$$

(iii) ***Monotonicity*:** Let T_{u_j} and $T_{u'_j}$ $(j = 1, 2, \ldots, m)$ be the two collections of T-spherical fuzzy soft numbers. If $T_{u_j} \leq T_{u'_j}$ then

$$TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) \leq TSFSWA_\omega(T_{u'_1}, T_{u'_2}, \ldots T_{u'_m}).$$

Further, we propose definition of an ordered weighted averaging operator for T-spherical fuzzy soft numbers as follows:

Definition 4 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSOWA_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is called T-spherical fuzzy soft ordered weighted averaging operator, if*

$$TSFSOWA_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \oplus_{j=1}^{m} (\omega_j T_{u_{\sigma(j)}}); \tag{3}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ *is the weight vector corresponding to* $(T_{u_j})_{j=1}^m$ *such that* $\omega_j \geq 0,$ *for all* $j;$ $\sum\limits_{j=1}^{m} \omega_j = 1$ *and* $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a possible permutation of* $(1, 2, \ldots, m)$, *s.t.* $T_{u_{\sigma(j+1)}} \leq T_{u_{\sigma(j)}}$ *for all* $j = 1, 2, \ldots, m-1$.

Theorem 3 *The* T*-spherical fuzzy soft ordered weighted averaging operator aggregates all the input values and yields a TSFSN given by*

$$TSFSOWA_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} (1 - \mu^n_{T_{u_{\sigma(j)}}})^{\omega_j}}, \prod_{j=1}^{m} (\eta_{T_{u_{\sigma(j)}}})^{\omega_j}, \prod_{j=1}^{m} (\nu_{T_{u_{\sigma(j)}}})^{\omega_j} \right); \quad (4)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ *is the weight vector corresponding to* $(T_{u_j})_{j=1}^m$ *such that* $\omega_j \geq 0,$ *for all* $j;$ $\sum\limits_{j=1}^{m} \omega_j = 1$ *and* $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ *is a possible permutation of* $(1, 2, \ldots, m)$, *s.t.* $T_{u_{\sigma(j+1)}} \leq T_{u_{\sigma(j)}}$ *for all* $j = 1, 2, \ldots, m-1$.

Proof: The proof is similar to the proof of Theorem 2 and can easily be carried out.

Next, the following properties related to the T-spherical fuzzy soft ordered weighted averaging operator which can easily be proved in view of the definitions:

(i) ***Idempotency*:** If $T_{u_j} = T_u;$ for all $j = 1, 2, \ldots, m$, then

$$TSFSOWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) = T_u.$$

(ii) ***Boundedness*:** Let T_{u_j} $(j = 1, 2, \ldots, m)$ be the collection of TSFSNs, then

$$\min_j T_{u_j} \leq TSFSOWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) \leq \max_j T_{u_j}.$$

(iii) ***Monotonicity*:** Let T_{u_j} and $T_{u'_j}$ $(j = 1, 2, \ldots, m)$ be the two collections of T-spherical fuzzy soft numbers. If $T_{u_j} \leq T_{u'_j}$ then

$$TSFSOWA_\omega(T_{u_1}, T_{u_2}, \ldots T_{u_m}) \leq TSFSOWA_\omega(T_{u'_1}, T_{u'_2}, \ldots T_{u'_m});$$

where $T_{u'_j}$ is any permutation of T_{u_j} for $j = 1, 2, \ldots, m$.

It may be noted that T-spherical fuzzy soft weighted averaging operator takes weights of TSFSNs into account, while T-spherical fuzzy soft ordered weighted averaging operator takes only the weights of the ordered positions of TSFSNs into account. This shows that both the operators take only one aspect into account. In order to overcome this issue and to incorporate both the aspects together, we introduce T-spherical fuzzy soft hybrid averaging (TSFSHA) operator and define as follows:

Definition 5 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSHA_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is called T-spherical fuzzy soft hybrid averaging operator, if*

$$TSFSHA_{\omega,\gamma}(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \oplus_{j=1}^{m}(\gamma_j \widetilde{T}_{u_{\sigma(j)}}); \tag{5}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)^T$ is the weight vector corresponding to $(\widetilde{T}_{u_{\sigma(j)}})_{j=1}^m$ s.t. $\gamma_j \geq 0$, for all j; $\sum_{j=1}^{m} \gamma_j = 1$. $\widetilde{T}_{u_{\sigma(j)}}$ is the j^{th} largest of the weighted TSFSNs $\widetilde{T}_{u_j}$; where $\widetilde{T}_{u_j} = (m\omega_j)T_{u_j}$ and m is the balancing coefficient with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ being the weight vector of T_{u_j} with $\omega_j \geq 0$, for all j; $\sum_{j=1}^{m} \omega_j = 1$.

Remarks:

- In case, we take uniformly distributed weights as $\gamma = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then T-spherical fuzzy soft hybrid averaging operator gives T-spherical fuzzy soft weighted averaging operator.
- However, if we take $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then T-spherical fuzzy soft hybrid averaging operator gives T-spherical fuzzy soft ordered weighted averaging operator.

Theorem 4 *The T-spherical fuzzy soft hybrid averaging operator $TSFSHA_{\omega,\gamma}$ aggregates all the input values and yields a TSFSN given by*

$$TSFSWA_{\omega,\gamma}(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\sqrt[n]{1 - \prod_{j=1}^{m}(1 - \mu^n_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j}}, \prod_{j=1}^{m}(\eta_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j}, \prod_{j=1}^{m}(\nu_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j} \right) \tag{6}$$

Proof: The proof is similar to the proof of Theorem 2 and can easily be carried out. Similarly, the properties of Idempotency, Boundedness and Monotonicity related to the T-spherical fuzzy soft hybrid averaging operator can easily be listed and proved in view of the definitions.

5.2 Geometric Aggregation Operators

Here, we propose geometric aggregation operators (weighted, ordered and hybrid) with their properties for the proposed T-spherical fuzzy soft numbers.

Definition 6 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSWG_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is said to be T-spherical fuzzy soft weighted geometric operator, if*

$$TSFSWG_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \otimes_{j=1}^{m}(T_{u_j})^{\omega_j}; \tag{7}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the weight vector corresponding to $(T_{u_j})_{j=1}^m$ such that $\omega_j \geq 0$, for all j; $\sum\limits_{j=1}^{m} \omega_j = 1$.

Theorem 5 *The T-spherical fuzzy soft weighted geometric operator $TSFSWG_\omega$ aggregates all the input values and yields a TSFSN given by*

$$TSFSWA_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\prod_{j=1}^{m} (\mu_{T_{u_j}})^{\omega_j}, \sqrt[n]{1 - \prod_{j=1}^{m}(1 - \eta^n_{T_{u_j}})^{\omega_j}}, \sqrt[n]{1 - \prod_{j=1}^{m}(1 - \nu^n_{T_{u_j}})^{\omega_j}} \right); \tag{8}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the weight vector corresponding to $(T_{u_j})_{j=1}^m$ such that $\omega_j \geq 0$, for all j; $\sum\limits_{j=1}^{m} \omega_j = 1$.

Proof: Using the *principle of mathematical induction* on m, the result in the Equation (8) given in the above stated Theorem 5 can be proved as follows:
For $m = 2$, we have

$$TSFSWG_\omega(T_{u_1}, T_{u_2}) = T_{u_1}^{\omega_1} \oplus T_{u_2}^{\omega_2}.$$

It may be observed that both $T_{u_1}^{\omega_1}$ and $T_{u_2^{\omega_2}}$ are T-spherical fuzzy soft numbers. From the operation (c) defined in Subsection 5.1 above for T-spherical fuzzy soft numbers, we have

$$(T_{u_1})^{\omega_1} = (\mu^{\omega_1}_{T_{u_1}}, \sqrt[n]{1 - (1 - \eta^n_{T_{u_1}})^{\omega_1}}, \sqrt[n]{1 - (1 - \nu^n_{T_{u_1}})^{\omega_1}});$$

and

$$(T_{u_2})^{\omega_2} = (\mu^{\omega_2}_{T_{u_2}}, \sqrt[n]{1 - (1 - \eta^n_{T_{u_2}})^{\omega_2}}, \sqrt[n]{1 - (1 - \nu^n_{T_{u_2}})^{\omega_2}}).$$

Then,

$$\begin{aligned}
&TSFSWG_\omega(T_{u_1}, T_{u_2}) = (T_{u_1})^{\omega_1} \otimes (T_{u_2})^{\omega_2} \\
&= \Bigg((\mu^n_{T_{u_1}})^{\omega_1} (\mu^n_{T_{u_2}})^{\omega_2}, \sqrt[n]{2 - (1 - \eta^n_{T_{u_1}})^{\omega_1} - (1 - \eta^n_{T_{u_2}})^{\omega_2} - ((1 - (1 - \eta^n_{T_{u_1}})^{\omega_1})(1 - (1 - \eta^n_{T_{u_2}})^{\omega_2}))}, \\
&\qquad \sqrt[n]{(2 - (1 - \nu^n_{T_{u_1}})^{\omega_1} - (1 - \nu^n_{T_{u_2}})^{\omega_2}) - ((1 - (1 - \nu^n_{T_{u_1}})^{\omega_1})(1 - (1 - \nu^n_{T_{u_2}})^{\omega_2}))} \Bigg) \\
&= \Bigg((\mu^n_{T_{u_1}})^{\omega_1} (\mu^n_{T_{u_2}})^{\omega_2}, \sqrt[n]{1 - \Big((1 - \eta^n_{T_{u_1}})^{\omega_1} (1 - \eta^n_{T_{u_2}})^{\omega_2} \Big)}, \\
&\qquad \sqrt[n]{1 - \Big((1 - \nu^n_{T_{u_1}})^{\omega_1} (1 - \nu^n_{T_{u_2}})^{\omega_2} \Big)} \Bigg).
\end{aligned}$$

Hence, the result holds for $m = 2$. Next, we suppose that the Equation (8) holds for $m = k$, that is,

$$\begin{aligned} TSFSGA_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_k}) &= (T_{u_1})^{\omega_1} \otimes (T_{u_2})^{\omega_2} \otimes \dots \otimes (T_{u_k})^{\omega_k} \\ &= \Bigg(\prod_{j=1}^{k} (\mu^n_{T_{u_j}})^{\omega_j}, \sqrt[n]{1 - \prod_{j=1}^{k} (1 - \eta^n_{T_{u_j}})^{\omega_j}}, \sqrt[n]{1 - \prod_{j=1}^{k} (1 - \eta^n_{T_{u_j}})^{\omega_j}} \Bigg). \end{aligned}$$

Next, the result for $m = k + 1$ is to be proved, that is, $TSFSGA_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_{k+1}})$ is also a TSFSN. By operational laws of T-spherical fuzzy soft numbers, we have

$$\begin{aligned} & TSFSWG_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_{k+1}}) = (T_{u_1})^{\omega_1} \otimes (T_{u_2})^{\omega_2} \otimes \dots \otimes (T_{u_k})^{\omega_k} \otimes (T_{u_{k+1}})^{\omega_{k+1}} \\ &= \Big(\prod_{j=1}^{k+1} (\mu^n_{T_{u_j}})^{\omega_j}, \\ &\quad \sqrt[n]{1 - \prod_{j=1}^{k} (1 - \eta^n_{T_{u_j}})^{\omega_k} + 1 - (1 - \eta^n_{T_{u_{k+1}}})^{\omega_{k+1}} - \big((1 - \prod_{j=1}^{k} (1 - \eta^n_{T_{u_j}})^{\omega_k})(1 - (1 - \eta^n_{T_{u_{k+1}}})^{\omega_{k+1}})\big)}, \\ &\quad \sqrt[n]{1 - \prod_{j=1}^{k} (1 - \nu^n_{T_{u_j}})^{\omega_k} + 1 - (1 - \nu^n_{T_{u_{k+1}}})^{\omega_{k+1}} - \big((1 - \prod_{j=1}^{k} (1 - \nu^n_{T_{u_j}})^{\omega_k})(1 - (1 - \nu^n_{T_{u_{k+1}}})^{\omega_{k+1}})\big)} \Big) \\ &= \Big(\prod_{j=1}^{k+1} (\mu^n_{T_{u_j}})^{\omega_j}, \sqrt[n]{1 - \prod_{j=1}^{k+1} (1 - \eta^n_{T_{u_j}})^{\omega_j}}, \sqrt[n]{1 - \prod_{j=1}^{k+1} (1 - \nu^n_{T_{u_j}})^{\omega_j}} \Big). \end{aligned}$$

This clearly shows that the aggregated value is also a TSFSN. Therefore, the Equation (8) holds for all $m = k+1$. This finally proves the result using the technique of mathematical induction.

Next, we list the following properties related to the T-spherical fuzzy soft weighted geometric operator which can easily be proved in view of the definitions:

(i) ***Idempotency*:** If $T_{u_j} = T_u$; for all $j = 1, 2, \dots, m$, then

$$TSFSWG_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_m}) = T_u.$$

(ii) ***Boundedness*:** If T_{u_j} $(j = 1, 2, \dots, m)$ be the collection of TSFSNs, then

$$\min_j T_{u_j} \leq TSFSWG_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_m}) \leq \max_j T_{u_j}.$$

(iii) ***Monotonicity*:** Let T_{u_j} and $T_{u'_j}$ $(j = 1, 2, \dots, m)$ be the two collections of T-spherical fuzzy soft numbers. If $T_{u_j} \leq T_{u'_j}$ then

$$TSFSWG_{\omega}(T_{u_1}, T_{u_2}, \dots T_{u_m}) \leq TSFSWG_{\omega}(T_{u'_1}, T_{u'_2}, \dots T_{u'_m}).$$

Further, we propose definition of an ordered weighted geometric operator for T-spherical fuzzy soft numbers as follows:

Definition 7 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSOWG_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is said to be T-spherical fuzzy soft ordered weighted geometric operator, if*

$$TSFSOWG_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_n}) = \otimes_{j=1}^{n} (T_{u_{\sigma(j)}})^{\omega_j}; \tag{9}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the weight vector corresponding to $(T_{u_j})_{j=1}^m$ such that $\omega_j \geq 0$, for all j; $\sum_{j=1}^{m} \omega_j = 1$ and $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ is a possible permutation of $(1, 2, \ldots, m)$, s.t. $T_{u_{\sigma(j+1)}} \leq T_{u_{\sigma(j)}}$ for all $j = 1, 2, \ldots, m-1$.

Theorem 6 *The T-spherical fuzzy soft ordered weighted geometric operator aggregates all the input values and yields a TSFSN given by*

$$TSFSWG_\omega(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\prod_{j=1}^{m} (\mu_{T_{u_{\sigma(j)}}})^{\omega_j}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \eta^n_{T_{u_{\sigma(j)}}})^{\omega_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \eta^n_{T_{u_{\sigma(j)}}})^{\omega_j}} \right); \tag{10}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the weight vector corresponding to $(T_{u_j})_{j=1}^m$ such that $\omega_j \geq 0$, for all j; $\sum_{j=1}^{m} \omega_j = 1$ and $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ is a permutation of $(1, 2, \ldots, m)$, such that $T_{u_{\sigma(j+1)}} \leq T_{u_{\sigma(j)}}$ for all $j = 1, 2, \ldots, m-1$.

Proof: The proof is similar to the proof of Theorem 5 and can easily be carried out.

Similarly, the properties of *Idempotency, Boundedness and Monotonicity* related to the T-spherical fuzzy soft ordered weighted geometric operator can easily be listed and proved in view of the definitions. It may be noted that T-spherical fuzzy soft weighted geometric operator takes only the weights of TSFSNs into account, while T-spherical fuzzy soft ordered weighted geometric operator takes only the weights of the ordered positions of TSFSNs into account. This shows that both the operators take only one aspect into account. In order to overcome this issue and to incorporate both the aspects together, we introduce T-spherical fuzzy soft hybrid geometric (TSFSHG) operator and define as follows:

Definition 8 *Suppose $\mathcal{P}$ is a collection of all T-spherical fuzzy soft numbers. Let $(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) \in \mathcal{P}^m$. A mapping $TSFSHG_\omega : \mathcal{P}^m \longrightarrow \mathcal{P}$ is said to be T-spherical fuzzy soft hybrid geometric operator, if*

$$TSFSHG_{\omega,\gamma}(T_{u_1}, T_{u_2}, \ldots, T_{u_n}) = \otimes_{j=1}^{n} (\widetilde{T}_{u_{\sigma(j)}})^{\gamma_j}; \tag{11}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)^T$ is the weight vector corresponding to $(\widetilde{T}_{u_{\sigma(j)}})_{j=1}^m$ s.t. $\gamma_j \geq 0$, for all j; $\sum_{j=1}^{m} \gamma_j = 1$. $\widetilde{T}_{u_{\sigma(j)}}$ is the j^{th} largest of the weighted TSFSNs $\widetilde{T}_{u_j}$; where $\widetilde{T}_{u_j} =$

$(m\omega_j)T_{u_j}$ *and* m *is the balancing coefficient with* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *being the weight vector of* T_{u_j} *with* $\omega_j \geq 0,$ *for all* $j;$ $\sum\limits_{j=1}^{m} \omega_j = 1.$

Remarks:

- In case, we take uniformly distributed weights as $\gamma = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then T-spherical fuzzy soft hybrid geometric operator gives T-spherical fuzzy soft weighted geometric operator.
- However, if we take $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then T-spherical fuzzy soft hybrid geometric operator gives T-spherical fuzzy soft ordered weighted geometric operator.

Theorem 7 *The* T*-spherical fuzzy soft hybrid geometric operator* $TSFSHG_{\omega,\gamma}$ *aggregates all the input values and yields a TSFSN given by*

$$TSFSHG_{\omega,\gamma}(T_{u_1}, T_{u_2}, \ldots, T_{u_m}) = \left(\prod_{j=1}^{m} (\mu_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \eta^n_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \eta^n_{\widetilde{T}_{u_{\sigma(j)}}})^{\gamma_j}} \right) \tag{12}$$

Proof: The proof is similar to the proof of Theorem 5 and can easily be carried out.

Similarly, the properties of *Idempotency, Boundedness and Monotonicity* related to the T-spherical fuzzy soft hybrid geometric operator can easily be listed and proved in view of the definitions.

6 Decision Making Model with T-spherical Fuzzy Soft Information

We formulate a decision making problem for our consideration in view of T-spherical fuzzy soft information and propose a new methodology for its solution using the proposed averaging operators and geometric operators.

Problem Formulation:
Let $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^q\}$ be the collection of q criterions and $\mathcal{A} = \{\mathcal{A}^1, \mathcal{A}^2, \ldots, \mathcal{A}^p\}$ be the collection of p alternatives. Based on the decision maker's perception, we consider the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_q)$ corresponding to each criterion $\mathcal{C}^j$ $(j = 1, 2, \ldots, q)$, where $\omega_j \in [0, 1]$, for all $j;$ $\sum\limits_{j=1}^{q} \omega_j = 1$. Further, on the basis of information received from the decision makers in the form of T-spherical fuzzy soft numbers, we construct a decision

matrix $R = [r_{ij}]_{p\times q} = [(\mu_{ij}, \eta_{ij}, \nu_{ij})]_{p\times q}$, where μ_{ij} represents the grade of membership of $\mathcal{A}^i$ positively satisfying the criterion $\mathcal{C}^j$, η_{ij} represents the degree of neutral/abstain membership of the alternative $\mathcal{A}^i$ for the criterion $\mathcal{C}^j$ and ν_{ij} represents non-membership degree of the $\mathcal{A}^i$ not satisfying the criterion $\mathcal{C}^j$, such that $\mu_{ij}^n + \eta_{ij}^n + \nu_{ij}^n \leq 1$; for all $i = 1, 2, \ldots, p$; for all $j = 1, 2, \ldots, q$ and n is a natural number (value to be chosen suitably). In view of the laid down criteria, we choose the best alternative from the available alternatives by using the flowchart of the proposed algorithm given in Figure 2.

Methodology:

For the implementation of the proposed algorithm outlined above, we comprehensively present the necessary steps of the methodology below:

Step 1: Collect the information as T-spherical fuzzy soft number related to each alternative satisfying the different criterions and arrange them as T-spherical fuzzy soft decision matrix $R = [(\mu_{ij}, \eta_{ij}, \nu_{ij})]_{p\times q}$, given by

$$R_{p\times q} = \begin{array}{c} \\ \mathcal{A}^1 \\ \mathcal{A}^2 \\ \vdots \\ \mathcal{A}^p \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{C}^1 & \mathcal{C}^2 & \cdots & \mathcal{C}^q \end{array} \\ \begin{pmatrix} (\mu_{11}, \eta_{11}, \nu_{11}) & (\mu_{12}, \eta_{12}, \nu_{12}) & \cdots & (\mu_{1q}, \eta_{1q}, \nu_{1q}) \\ (\mu_{21}, \eta_{21}, \nu_{21}) & (\mu_{22}, \eta_{22}, \nu_{22}) & \cdots & (\mu_{2q}, \eta_{2q}, \nu_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{p1}, \eta_{p1}, \nu_{p1}) & (\mu_{p2}, \eta_{p2}, \nu_{p2}) & \cdots & (\mu_{pq}, \eta_{pq}, \nu_{pq}) \end{pmatrix} \end{array}$$

Step 2: In case, the criteria are homogenous in nature, we go to step 3. Otherwise if the criteria are heterogenous, say, *cost criteria* and *benefit criteria*, we normalize the decision matrix using the equation:

$$r_{ij} = \begin{cases} T_{u_{ij}}; & \mathcal{C}^j \text{ is } \textit{cost criteria}, \\ T_{u_{ij}}^c; & \mathcal{C}^j \text{ is } \textit{benefit criteria}. \end{cases} \tag{13}$$

Step 3: Using either of the proposed aggregation operator (averaging operators or geometric operators), obtain the aggregated values (TSFSN) of all the row-entries corresponding to each alternative $\mathcal{A}^i$ $(i = 1, 2, \ldots, p)$ of the normalized decision matrix.

Step 4: By applying Definition (2), calculate the score value for each aggregated value obtained in step 3 corresponding to each alternative $\mathcal{A}^i$ $(i = 1, 2, \ldots, p)$.

Step 5: Ranking of the alternatives is finally done based on the score values obtained.

7 Numerical Example

The proposed methodology for solving the multi-criteria decision making problem is being illustrated with a numerical example as follows:

Example: Assume that an Indian multi-national company is planning some financial strategy for the upcoming year as per the group strategy objective. Four well defined investment alternatives have been taken into consideration and labeled as $\mathcal{A}^1$: investment in "*South Indian Markets*"; $\mathcal{A}^2$: investment in "*East Indian Markets*"; $\mathcal{A}^3$: investment in "*North Indian Markets*"; and $\mathcal{A}^4$: investment in "*West Indian markets*". After a preliminary screening for evaluation purpose, it has been decided to proceed by taking four criteria, namely as $\mathcal{C}^1$: "*growth*"; $\mathcal{C}^2$: "*risk analysis*"; $\mathcal{C}^3$: "*the socio-political impact*" and $\mathcal{C}^4$: "*the environmental and other factors*". Suppose that based on the financial strategies adopted for the welfare of the company, the weight vector is $\omega = (0.2, 0.3, 0.1, 0.4)^T$.

Here, for the simplicity of the computation for the example under consideration, we take the value of n as 2 in the definitions. The computational steps for the the above stated problem using the proposed algorithm are below.

Step 1. First we construct the following spherical fuzzy soft decision matrix $R = [(r_{ij})] = [(\mu_{ij}, \eta_{ij}, \nu_{ij})], (i, j = 1, 2, 3, 4)$ for the four alternatives $\mathcal{A}^i$ $(i = 1, 2, 3, 4)$ and the four criteria $\mathcal{C}^j$ $(j = 1, 2, 3, 4)$ based on the information provided by the experts:

$$R = \begin{array}{c} \\ \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \\ \mathcal{A}^4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{C}^1\quad & \quad\mathcal{C}^2\quad & \quad\mathcal{C}^3\quad & \quad\mathcal{C}^4\quad \end{array} \\ \begin{pmatrix} (0.2, 0.2, 0.6) & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.4, 0.3, 0.2) \\ (0.3, 0.4, 0.4) & (0.6, 0.3, 0.1) & (0.5, 0.3, 0.2) & (0.2, 0.1, 0.7) \\ (0.4, 0.5, 0.2) & (0.6, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.3, 0.3, 0.5) \\ (0.3, 0.2, 0.6) & (0.2, 0.2, 0.6) & (0.2, 0.3, 0.6) & (0.4, 0.2, 0.4) \end{pmatrix} \end{array}$$

Step 2. Since $\mathcal{C}^2$ and $\mathcal{C}^3$ are the cost criterions whereas $\mathcal{C}^1$ and $\mathcal{C}^4$ are the benefit criterions, therefore, we have to normalize the decision matrix by using Equation (13). Hence, we obtain the normalized decision matrix is as follows:

$$R = \begin{array}{c} \\ \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \\ \mathcal{A}^4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{C}^1\quad & \quad\mathcal{C}^2\quad & \quad\mathcal{C}^3\quad & \quad\mathcal{C}^4\quad \end{array} \\ \begin{pmatrix} (0.6, 0.2, 0.2) & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.2, 0.3, 0.4) \\ (0.4, 0.4, 0.3) & (0.6, 0.3, 0.1) & (0.5, 0.3, 0.2) & (0.7, 0.1, 0.2) \\ (0.2, 0.5, 0.4) & (0.6, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.5, 0.3, 0.3) \\ (0.6, 0.2, 0.3) & (0.2, 0.2, 0.6) & (0.2, 0.3, 0.6) & (0.4, 0.2, 0.4) \end{pmatrix} \end{array}$$

Step 3. Using the T-spherical fuzzy soft weighted average aggregating operator for the normalized decision matrix calculated in step 2, the aggregated value for each alternative is presented below:

$$\begin{array}{c} \text{Aggregated Value: } TSFSWA_{\omega} \\ \begin{array}{c} \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \\ \mathcal{A}^4 \end{array} \begin{pmatrix} T_{u_1} : (0.444983, 0.26564, 0.274822) \\ T_{u_2} : (0.610765, 0.204767, 0.176173) \\ T_{u_3} : (0.526385, 0.319067, 0.270192) \\ T_{u_4} : (0.401021, 0.208276, 0.436105) \end{pmatrix} \end{array}$$

Step 4. Calculate the score value for each aggregated value of corresponding alternative by using Definition (2). The computed values are given as:

$$S(T_{u_1}) = 0.051918,\ S(T_{u_2}) = 0.300067,\ S(T_{u_3}) = 0.102274,\ S(T_{u_4}) = -0.07275.$$

Step 5. On the basis of values obtained in the last step, we observe that $S(T_{u_2}) > S(T_{u_3}) > S(T_{u_1}) > S(T_{u_4})$ and the ranking of the alternatives is done.

Thus it has been found that the alternative $\mathcal{A}^2$ is the best one. Therefore, the best alternative strategy for the company is to invest in the East Indian Market.

8 Comparative Remarks and Advantages

The proposed notion of T-spherical fuzzy soft set is a novel concept and an advanced extension of the classical fuzzy set. The T-spherical fuzzy sets have an added advantage to deal with a wider sense of applicability in uncertain situations. In detail, some important comparative remarks and advantages of utilizing T-spherical fuzzy set are listed below:

- The existing fuzzy sets, intuitionistic fuzzy sets and picture fuzzy sets have their own limitations that they are not capable to capture the full information specification, that is, there is a missing additional component of degree of refusal which is addressed by the spherical fuzzy sets/soft sets.
- When the uncertain or imprecise information takes the form of a fuzzy relation then to ensure a kind of parametrization in the relation, we utilize the concept of T-spherical fuzzy soft sets in natural sciences for therapeutic recommendations.
- The drawback in the existing literature of the fuzzy sets is that the condition does not allow the experts/decision makers to allocate the membership values of their own choice (Refer Table 1). Somehow, this makes the decision makers bounded for

providing their input in a particular domain. However, the proposed T-spherical fuzzy soft sets provide a generalization feature which make a strong impact in an application/decision making process.

- The discussion over implementing the proposed T-spherical fuzzy soft sets and the various aggregation operators for a financial strategic multi-criteria decision making model/problem in Section 6 and Section 7 shows that the proposed work handled the generalized framework in an effective and consistent way.

9 Conclusions and Future Research Directions

The novel concept of T-spherical fuzzy soft set has been successfully introduced along with various operations. Some important properties and the notion of score function/accuracy function for T-spherical fuzzy soft set have also been studied in brief. Averaging aggregation operators and geometric aggregation operators (weighted, ordered and hybrid) for T-spherical fuzzy soft numbers have been proposed and well utilized along with their different properties in multi-criteria decision making problems. Further, the proposed algorithm, that utilizes the aggregation operators, has been well implemented for solving a decision making problem. A numerical example well presents the outline of the methodology. The proposed notion and the algorithms using the aggregation operators may further be utilized and extended in future with the following possibilities:

- In literature, a variety of extensions of soft sets [21] to imprecise and incomplete information have been proposed. In view of the generalizations and extensions of fuzzy sets shown by Figure 1 in the introduction section, we may further propose to extend the notion of T-spherical fuzzy soft matrices based on [38], [39], [40] along with their various matrix operations, properties and engineering applications.
- Since there is kind of parametrization tool involved in the soft sets and consequently in soft matrices, therefore based on this, various related applications, for example, stock management [41], medical diagnosis [42], dimensionality reduction [43] have been studied recently. Hence, introducing the concept of T-spherical fuzzy soft matrices can lead to a new dimension in soft set theory and related applications.
- The extended notion of T-spherical complex fuzzy soft sets and their aggregation operators may also be introduced based on the extension outlines discussed in [44] [45].

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Figure Captions

- Figure 1: Extensions of Fuzzy Sets
- Figure 2: Flow Chart of the Proposed Algorithm

Table Caption

- Table 1: Need to address the problem arises in IFSs, PyFSs and PFSs

Table 1: Need to address the problem arises in IFSs, PyFSs and PFSs

R	C_1	C_2	C_3	C_4
C_1	$(1.0 + 0.0 + 0.0 = 1)$	$(0.40 + 0.20 + 0.69 > 1)$	$(0.36 + 0.19 + 0.79 > 1)$	$(0.56 + 0.17 + 0.62 > 1)$
C_2	$(0.68 + 0.20 + 0.44 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.40 + 0.24 + 0.56 > 1)$	$(0.51 + 0.29 + 0.61 > 1)$
C_3	$(0.76 + 0.20 + 0.42 > 1)$	$(0.54 + 0.24 + 0.42 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.48 + 0.14 + 0.77 > 1)$
C_4	$(0.49 + 0.17 + 0.68 > 1)$	$(0.59 + 0.29 + 0.53 > 1)$	$(0.77 + 0.17 + 0.38 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$

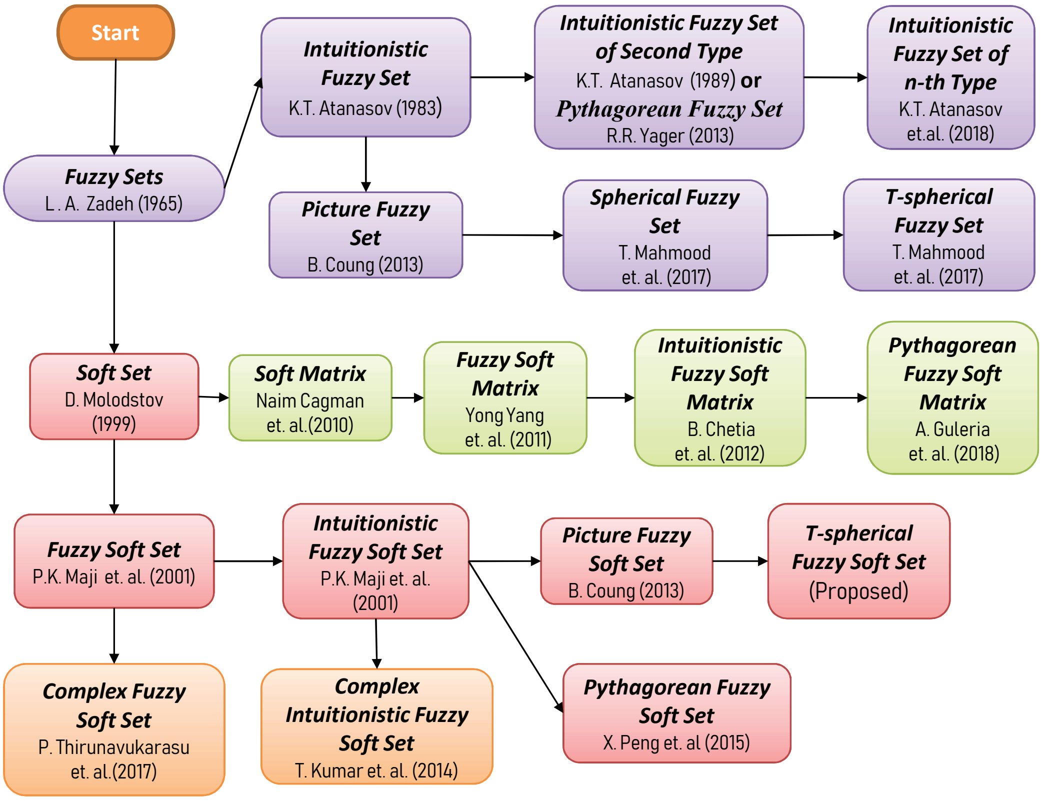

Figure 1: Extensions of Fuzzy Sets

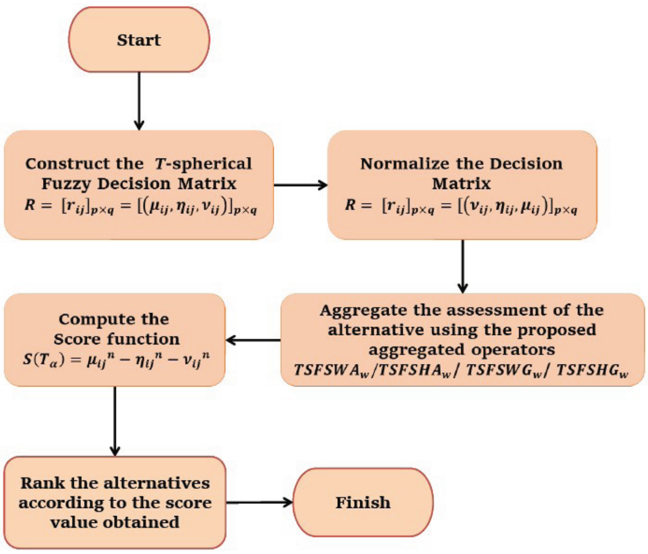

Figure 2: Flow Chart of the Proposed Algorithm