Consequences of activation energy and chemical reaction in radiative flow of tangent hyperbolic nanoliquid

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Abstract:
Mixed convection flow of tangent hyperbolic liquid over stretching sheet is explored. Joule heating, double stratification, non-linear thermal radiation, Brownian motion and thermophoresis are present. Phenomenon of mass transfer is examined by activation energy along with binary chemical. Computations of convergent solutions are carried out for the nonlinear mathematical system. Graphical representation is employed for outcome of sundry variables on velocity, temperature and concentration of nanoparticles. Moreover, Nusselt number, coefficient of drag force and mass transfer rate are examined. It is observed that velocity decays for larger Weissenberg number. Concentration of fluid enhances for higher activation energy parameter.

Keywords: Tangent hyperbolic nanofluid; Double stratification; Joule heating; Mixed convection; Non-linear thermal radiation; Activation energy.

Introduction
Thermal radiation involves electromagnetic radiations transmitted due to the thermal motion of the fluid particles almost in entire directions. Process of thermal radiation is observed in heating of the bodies by diverse means like solar light, fire and radiator. Heat transfer with radiation has extensive industrial applications including atomic reactor security, boiler design, heat exchangers, power stations and much propulsion equipment's for missiles, aircraft, in space automobiles and satellites. Radiative heat change in energy equation is measured by Rosseland calculation. Cortell [1] investigated non-linear radiative heat transfer in flow by stretched surface. Properties of non-linear thermal radiation and cubic autocatalysis in nanofluid flow with rotational effects are described by Kumar et al. [2]. Influence of non-linear radiative flow on magneto-Burgers fluid with gyrotactic microorganisms is studied by Khan et al. [3]. Bhatti et al. [4] developed numerical solution for entropy generation minimization for radiative non-linear flow over a stretching sheet. Few recent efforts under this aspect can be observed through the refs. [5-7]
Examination of heat transfer via mixed convection has attained substantial interest in numerous fields of engineering and technology. It is due to its applications for heat conversion in nuclear reactors, temperature variant atmospheric flow, electrical devices and the flow path owing to density variation in a stream across the vertical direction through seasonal variation. Mixed convection is normally entitled through buoyancy force created by density and temperature variations. By Boussinesq's theory, equations of energy and momentum containing mixed convection term are highly coupled. Fluid flow with mixed convection generates a boundary layer nearby vertical plate. A sizeable literature exists now for mixed convective flow. Some representative contributions in this regard can be seen by the studies [8-14].
Mass transfers effects in fluid mixture are observed due to concentration variation of species exist in fluid. This variation occurs in a fluid when species moves from a higher concentration region to a lower concentration region.

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Further the energy obtained by reactants before a chemical reaction can occur is referred to as activation energy. This is also called as a minimum required energy to initiate a reaction. Mass transfer phenomenon escorted by chemical reaction along activation energy is frequently encountered in several applications including chemical and geothermal engineering, mechanism comprised of (oil, water) suspension and food making. Recent contributions in this field are made by few researchers [15-18].

Stratification is a process of developing layers through temperature and concentration variation or influence of different fluids. Through simultaneous effects of heat and mass transfer, it is important to analyze the effectiveness of thermal and solutal stratification concerned with stratification of fluid medium in view of heat convection and mass transfer because of convective motion of nanofluids. Study of mixed convection with double stratification is observed in diverse industrial and engineering applications. This phenomenon has dominant role on polymer extrusion, in hydraulic flow of thermal fluids, geothermal reservoirs, volcanic flows and geological systems. A number of studies have been attempted yet in this direction [19-22].

Here we discuss the influence of activation energy on double stratified flow of tangent hyperbolic nanofluid over a stretched surface. Modified Arrhenius function is used. Buongiorno model is considered which emphasized the novel features of thermophoresis and Brownian movement. Additionally mixed convection disclose the impact of buoyancy forces. Joule heating and nonlinear radiation are examined. Homotopy analysis method (HAM) is utilized for development of convergent solutions [23-35]. Homotopic method seems better in following aspects. This method is not directly influenced by small or large parameters. Thus, HAM can easily employed to weaker as well as strongly nonlinear problems. HAM is a unification of some other analytical methods including, delta approximation, Adomian decomposition method, homotopy perturbation technique and Lyapunov artificial parameter method. Explicit solutions of highly nonlinear problems can be calculated through HAM. It gives freedom to choose initial guesses and operators. HAM provides a simplest way to ensure the convergence of series solution unlike other methods.

1. Modeling
Unsteady flow of tangent hyperbolic nanofluid is examined here. Impermeable stretched sheet (at $y = 0$) is accountable for fluid motion. Material density is assumed constant. Fluid confined the space $y > 0$. Sheet is stretched with velocity $U_w(x) = ax$ in $x$-direction. The sheet has constant temperature ($T_w$) and constant concentration ($C_w$) whereas ($C_\infty, T_\infty$) are fluid's ambient concentration and temperature respectively. Brownian diffusion and thermophoresis are present. Magnetic fluid is in transverse direction to flow (Fig.1). Effects of electric field are assumed negligible. Moreover influence of activation energy with binary chemical reaction is accounted. Thermal radiation is taken non-linear. Viscous dissipation and heat generation/absorption are absent. Under these considerations, the velocity, temperature and concentration expressions are [4, 17, 35]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]  

(1)
\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu(1 - \tilde{n}) \frac{\partial^2 u}{\partial y^2} + \sqrt{2} \nu \Gamma \tilde{n} \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} - \sigma \frac{B_0^2}{\rho_f} u + g[\beta_r(T - T_\infty) + \beta_c(C - C_\infty)] ,
\]
(2)

\[
\left( \frac{u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}{\frac{\partial T}{\partial y}} \right) = \frac{k}{(\rho c_p)_f} \left( \frac{\partial^2 T}{\partial y^2} \right) + \tau D_B \left( \frac{\partial C}{\partial y} \right) +
\]
(3)

\[
u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \left( \frac{\partial^2 C}{\partial y^2} \right) + \frac{D_r}{T_\infty} \left( \frac{\partial^2 T}{\partial y^2} \right) - K_r^2 (C - C_\infty) \left[ \frac{T}{T_\infty} \right]^n \exp \left[ \frac{-E_a}{k_r T} \right].
\]
(4)

The relevant boundary conditions are

\[
u = U_w(x) = ax, \quad v = 0, \quad T = T_w = T_0 + \epsilon x, \quad C = C_w = C_0 + \epsilon x \quad \text{at} \quad y = 0,
\]
(5)

\[
u \to 0, \quad T \to T_\infty = T_0 + \epsilon x, \quad C \to C_\infty = C_0 + \epsilon x \quad \text{at} \quad y \to \infty.
\]
(6)

Here \( u \) and \( v \) depict the velocity components parallel to \( x \) and \( y \)-axes, \( \Gamma \) material constant, \( \tilde{n} \) power law index, \( \nu \) kinematic viscosity, \( \rho \) fluid density, \( c_p \) specific heat, \( \mu \) the dynamic viscosity, \( D_r \) the thermophoresis coefficient, \( U_w \) the stretching velocity, \( D_B \) the Brownian coefficient, \( \tau \) the heat capacity of nanoparticles, \( \sigma \) the electrical conductivity, \( T_0 \) the reference temperature, \( k \) the thermal conductivity, \( \beta_r \) thermal expansion coefficient, \( \beta_c \) concentration expansion coefficient and \( (\tilde{a}, \tilde{b}, \tilde{e}, \tilde{d}) \) the dimensional constants. Further \( T_w \) and \( C_w \) are the constant temperature and solute concentration near the surface while \( T_\infty \) and \( C_\infty \) represent ambient temperature and concentration. Term \( \left[ \frac{T}{T_\infty} \right]^n \exp \left[ \frac{-E_a}{k_r T} \right] \) in Eq. (4) is referred to as the modified Arrhenius function. Here \( k_r = 8.61 \times 10^{-5} \text{eV/K} \) represents the Boltzmann constant, \( n \) is the dimensionless constant or rate constant having range \(-1 < n < 1\), \( K_r \) the chemical reaction parameter and \( E_a \) the activation energy.

Considering,

\[
\psi = \sqrt{avxf}(\eta), \quad \eta = \sqrt{av}\gamma
\]
(7)

\[
\theta(\eta) = \frac{T - T_\infty}{T_w - T_0}, \quad \varphi(\eta) = \frac{C - C_\infty}{C_w - C_0}.
\]

We define velocity components in terms of stream function \( \psi \) by \( v = -\frac{\psi}{\partial \psi} \) and \( u = \frac{\partial \psi}{\partial \xi} \). Now

\[
u = a\psi'\eta, \quad v = -\sqrt{av}\psi(\eta),
\]
(8)
Radiative heat flux through Rosseland approximation yields

\[
\frac{\partial q_r}{\partial y} = \frac{-4\sigma^*}{3k^*} \frac{\partial}{\partial y} \left( \frac{\partial T^4}{\partial y} \right) = \frac{-16\sigma^*}{3k^*} \left[ T^3 \frac{\partial^2 T}{\partial y^2} + 3T^2 \left( \frac{\partial T}{\partial y} \right)^2 \right],
\]

(9)

where \( \sigma^* \) and \( k^* \) represent Stefan-Boltzmann constant and mean absorption coefficient. From Equation (3), (9) becomes

\[
\left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{k}{(\rho c_p)_f} \left( \frac{\partial^2 T}{\partial y^2} \right) + \tau D_b \left( \frac{\partial T}{\partial y} \right) \left( \frac{\partial C}{\partial y} \right) + \tau \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \\
+ \sigma \frac{B_0^2}{\rho} u^2 + \frac{16\sigma^*}{3k^*} \left[ T^3 \frac{\partial^2 T}{\partial y^2} + 3T^2 \left( \frac{\partial T}{\partial y} \right)^2 \right],
\]

(10)

Continuity expression verified and equations (2) to (10) yield

\[
(1 - \bar{n}) f'''' + ff''' - (f')^2 + \bar{n} \frac{\partial}{\partial y} f'''' - M^2 f' + \lambda [\theta + N^* \varphi] = 0,
\]

(11)

\[
\theta'' = Pr \left[ S_\theta f' + Pr(f \theta' - f' \theta) + Pr N_\theta \theta' \psi + Pr N_\psi (\theta')^2 + MEc^2(f')^2 + 4R_d [\theta' + 3(\theta')^2 (\theta' + 1) + 1] \right] \left[ \theta (\theta' + 1) + 1 \right] = 0,
\]

(12)

\[
\varphi'' + \frac{N_\psi}{N_b} \varphi'' + \frac{(f \varphi' - f' \varphi)}{2} + \frac{Sc}{S_c} f' = \frac{Sc}{S_c} [A + \theta (\theta' + 1)] \exp \left[ -\frac{E}{1 + \theta (\theta' + 1)} \right] = 0,
\]

(13)

\[
f'(\eta) = 1, \quad f(\eta) = 0 \quad \theta(\eta) = 1 - S_\theta, \quad \varphi(\eta) = 1 - S_\psi \quad \text{at} \quad \eta = 0,
\]

\[
f'(\eta) \to 0, \quad \theta(\eta) \to 0, \quad \varphi(\eta) \to 0 \quad \text{at} \quad \eta \to \infty.
\]

(14)

where \( We \) denotes the Weissenberg number, \( M \) magnetic parameter, \( N^* \) the local buoyancy parameter, \( \lambda \) the mixed convection parameter, \( Pr \) the Prandtl number, \( S_\theta \) the thermal stratified parameter, \( S_\psi \) the solutal stratified parameter, \( Sc \) the Schmidt number, \( E \) dimensionless activation energy and \( A' \) the dimensionless chemical reaction parameter. Furthermore, \( Ec, \ R_d, \ (\theta_\psi, \theta_\theta), \ N_b, \ N_r, \ (Gr, Gr^*) \) are the Eckert number, radiation parameter, temperature ratio parameter, Brownian motion parameter, thermophoresis parameter and the Grashof number (temperature and concentration). Definition of these parameters are
Fig 3

\[ \text{We} = \sqrt{\frac{2\rho^3}{\nu}} \Gamma x, \quad M^2 = \frac{\sigma B_w^2}{\rho a}, \quad Gr = \frac{g \beta_c (T_w - T_0) x^3}{\nu^2}, \quad \lambda = \frac{Gr}{Re^2}, \]

\[ Gr^* = \frac{g \beta_c (C_w - C_0) x^3}{\nu^2}, \quad N^* = \frac{Gr^*}{Gr}, \quad R_d = \frac{4 \sigma^2 T^3}{k_k}, \quad S_p = \frac{b}{a}, \]

\[ Ec = \frac{U_w^2}{(T_w - T_0) c_p}, \quad \theta_w = \frac{T_w}{T_n}, \quad \theta_0 = \frac{T_0}{T_n}, \quad Sc = \frac{\nu}{D_B}, \quad Pr = \frac{\nu}{\alpha}, \]

\[ N_b = \frac{\tau D_B}{\nu} \tilde{e} x, \quad N_i = \frac{\tau D_T}{\nu T_w} \tilde{a} x, \quad S_0 = \frac{\tilde{d}}{e}, \quad \tilde{E} = \frac{E_a}{k_1 T_w}, \quad A' = K^2. \]

Coefficient of drag force \( (C_{fx}) \), heat transfer rate \( (Nu_x) \) and mass transfer rate \( (Sh_x) \) are described as [30]:

\[ C_{fx} = \frac{\tilde{r}_w}{\frac{1}{2} \rho U_w^2 x}, \quad Nu_x = \frac{\chi \tilde{q}_w}{k (T_w - T_0)}, \quad Sh_x = \frac{\chi \tilde{q}_m}{D_B (C_w - C_0)}. \tag{16} \]

where,

\[ \tilde{r}_w = \mu \left[ (1 - \tilde{n}) \frac{\partial u}{\partial y} + \frac{\Gamma \tilde{n}}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right]_{y=0}, \]

\[ \tilde{q}_w = -\tilde{k} \left[ \left( 1 + T^3 \frac{16 \sigma^2}{3 k_k} \right) \left( \frac{\partial T}{\partial y} \right) \right]_{y=0}, \tag{17} \]

\[ \tilde{q}_m = -D_B \left( \frac{\partial C}{\partial y} \right)_{y=0}. \]

In non-dimensional form we obtain

\[ \text{Re}_x^{1/4} C_{fx} = (1 - \tilde{n}) f''(0) + \text{We} \tilde{n} \left( f''(0) \right)^2, \]

\[ \text{Re}_x^{1/4} Nu_x = -\tilde{\theta}'(0) \{ 1 + R_d [ 1 + (\theta_w - \theta_0) \theta(0)] \}, \tag{18} \]

\[ \text{Re}_x^{1/4} Sh_x = -\varphi'(0). \]

in which \( \text{Re}_x = \frac{U_x}{\nu} \) is Reynold number.

2. Solution methodology

Series solution of above mentioned system is obtained via homotopic technique. Initial guesses \((f_0, \theta_0, \varphi_0)\) and associated linear operators \((\mathbf{L}_f, \mathbf{L}_\theta, \mathbf{L}_\varphi)\) are
Fig 3

\[ f_0(\eta) = 1 - e^{-\eta}, \]
\[ \theta_0(\eta) = (1 - S_\theta) e^{-\eta}, \]
\[ \varphi_0(\eta) = (1 - S_\varphi) e^{-\eta}, \]

(19)

\[ L_f(\eta) = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta}, \quad L_\theta(\eta) = \frac{d^3 \theta}{d\eta^3} - \theta, \quad L_\varphi(\eta) = \frac{d^2 \varphi}{d\eta^2} - \varphi, \]

(20)

\[ L_f \left[ \tilde{X}_i e^{\eta} + \tilde{X}_\varepsilon e^{-\eta} \right] = 0, \]

(21)

\[ L_\theta \left[ \tilde{X}_\varepsilon e^{\eta} + \tilde{X}_\varepsilon e^{-\eta} \right] = 0, \]

(22)

\[ L_\varphi \left[ \tilde{X}_\varepsilon e^{\eta} + \tilde{X}_\varepsilon e^{-\eta} \right] = 0. \]

(23)

with \( \tilde{X}_i \) \( (i = 1 - 7) \) as the arbitrary constants.

3. Convergence analysis

Homotopy analysis technique is helpful to find the convergent series solutions and provides a chance to sketch the region of convergence. This region can be controlled by setting appropriate values of \( h_f, h_\theta \) and \( h_\varphi \). Fig. 2 displays the h-curves. It is analyzed that the ranges of parameters \( h_f, h_\theta \) and \( h_\varphi \) are \( -1.5 \leq h_f \leq -0.5, -1.3 \leq h_\theta \leq -0.4 \) and \( -1.4 \leq h_\varphi \leq -0.7 \). Numerically obtained solution convergence is presented in Table 1. Here 20\(^{th}\) order of approximation are enough for momentum equation while energy and concentration equations converge at 25\(^{th}\) order of approximation.

Figure 2

Table 1

4. Discussion

Here velocity, temperature and concentration are discussed through Figs. 3-16. Fig. 3 predicts Weissenberg number \( W_e \) variation on \( f'(\eta) \). Clearly velocity and momentum layer are less through \( W_e \). In fact the Weissenberg number is ratio of relaxation time to particular time of fluid. Hence for more relaxation time the fluid thickness increases and consequently there is reduction in fluid velocity. Fig. 4 indicates variation of velocity \( f'(\eta) \) by increasing local buoyancy parameter \( N^* \). Clearly velocity and related layer thickness are enhanced through larger \( N^* \). Fig. 5 illustrates impact of magnetic parameter \( M \) on \( f'(\eta) \). A rise in \( M \) leads to a decrease of velocity field \( f'(\eta) \). Obviously larger magnetic parameter \( M \) increase Lorentz force (also termed as resistive force) which opposes fluid movement and so velocity reduces. Influence of mixed convection parameter \( \lambda \) is pictured in Fig. 6. Velocity is enhanced through \( \lambda \). Higher values of mixed convection enhances buoyancy forces and thus velocity and corresponding momentum layer increases.
Fig. 7 explores variation of temperature field for magnetic parameter $M$. Here temperature is increased for larger magnetic parameter. Fig. 8 witnesses that temperature and thermal layer are enhanced for higher Eckert number $Ec$. Graphical illustration of $N_b$ for $\theta(\eta)$ is presented in Fig. 9. Temperature rises in response to larger variation of $N_b$. Since motion of fluid particle is unsystematic in reaction of higher Brownian motion parameter that leads to increase in heat production. Fig. 10 explores variation of temperature for radiation parameter $R_d$. Larger $R_d$ give rise to rate of heat transfer since heat production enhanced the radiation process. Impact of thermal stratified variable $S_\theta$ on temperature $\theta(\eta)$ is reflected in Fig. 11. Thermal field decays with an increase in $S_\theta$ as the difference in temperature is decreased gradually between the ambient temperature $T_\infty$ and surface temperature $T_w$.

Fig. 12 clarifies outcome of $A'$ on concentration of nanoparticles. As predicted, the decline in nanoparticle concentration is noticed for larger $A'$. Variation of activation energy $E$ on nanoparticles concentration is reflected in Fig. 13. As expected concentration is enhanced through $E$. Outcome of Schmidt number $Sc$ on $\phi(\eta)$ is displayed in Fig. 14. Here concentration decayed for higher $Sc$. Analysis of concentration stratification parameter $S_\phi$ on $\phi(\eta)$ is displayed in Fig. 15. It seems that larger values of $S_\phi$ corresponds to lesser nanoparticles concentration $\phi(\eta)$. Fig. 16 elucidates marginal increase in nanoparticles concentration when thermophoresis parameter $N_i$ varies from $N_i = 0.2$ to $N_i = 0.35$.

Influences of various embedded parameters on skin-friction coefficient, Nusselt number and Sherwood number are demonstrated in Tables 2-4. Table 2 declares that coefficient of drag force is affected by mixed convection parameter $\tilde{\nabla}$ and non-dimensional activation energy $\tilde{E}$. Coefficient of drag force is decreased by these parameters. Skin-friction coefficient has increased behavior for higher $\tilde{n}$, $We$, $M$ and $N'$. Table 3 displays variation in heat transfer rate against some parameters of interest. Tabulated values witness that heat transfer rate decreases for Eckert number $Ec$, dimensionless thermally stratified parameter $S_\theta$, Brownian diffusion parameter $N_b$. 
and thermophoresis parameter $N_t$ while it enhances for Prandtl number $Pr$ and $R_d$. Further mass transfer rate is decreased via higher Schmidt number $Sc$ and Brownian diffusion coefficient $N_b$, however it increases for other parameters mentioned in Table 4.

| Table 2 | Table 3 | Table 4 |

5. Conclusions

Thermal and concentration stratifications under the effects of non-linear thermal radiation and activation energy are explored. We have main results as under:

- Velocity field decays via larger Weissenberg number $We$.
- Concentration distribution reduces for an increment in solutal stratification parameter $S_p$.
- Temperature distribution significantly reduces for stronger thermal stratified parameter $S_\theta$.
- Stronger chemical reaction parameter $A'$ results in an increase of concentration distribution.
- Velocity has direct relation with mixed convection parameter $\lambda$.
- Concentration against activation energy variable $E$ is increased.
- Heat transfer rate enhances for radiation parameter $R_d$.
- Higher mass transfer rate is noted for thermophoresis parameter $N_t$.

References


Captions List:
Fig.1: Physical model
Fig.2: $h$– plots in view of $f''(0), \theta'(0)$, and $\phi'(0)$
Fig. 3: $f'(\eta)$ via $We$
Fig. 4: $f'(\eta)$ via $N^*$
Fig. 5: $f'(\eta)$ via $M$
Fig. 6: $f'(\eta)$ via $\lambda$
Fig. 7: $M$ impact on $\theta(\eta)$
Fig. 8: $Ec$ impact on $\theta(\eta)$
Fig. 9: $N_b$ impact on $\theta(\eta)$
Fig. 10: $R_d$ impact on $\theta(\eta)$
Fig. 11: $S_{\theta}$ impact on $\theta(\eta)$
Fig. 12: $A'$ effect on $\phi(\eta)$
Fig. 13: $\dot{E}$ effect on $\phi(\eta)$
Fig. 14: $Sc$ effect on $\phi(\eta)$
Fig. 15: $S_{\phi}$ effect on $\phi(\eta)$
Fig. 16: $\phi(\eta)$ via $N_i$
Table 1: HAM solutions convergence when
\( \tilde{n} = A' = 0.1, \lambda = 0.3 = M = Sc, S_\theta = 0.3 = S_\varphi, Pr = 1.5, R_d = 0.5, \)

\( Ec = 1.3 = E, Sc = 0.8, N^* = 1, N_b = 0.1 = N, \theta_1 = 1.1, \theta_0 = 1.2. \)

Table 2: Numerical description of skin friction coefficient.
Table 3: Numerical description of Nusselt number.
Table 4: Numerical description of Sherwood number.

**Figures and Tables:**

Fig 1

Fig 2
Fig 3

\[ n = 0.1, \ Pr = 1.5, \lambda = 0.3, \tilde{N} = 0.5 = M = \phi, \]
\[ S_p = 0.7, \ N_b = 0.3, \ N_c = 0.2 = \phi_b, \ Ec = 0.5 = \tilde{E} \]
\[ R_d = 0.6, \lambda = 1.1, \ S_p = 0.2, \ Sc = 0.4. \]

\[ f' \]
\[ \eta \]

\[ \text{We} = 1, 1.5, 2, 2.2 \]

Fig 4

\[ n = 0.1, \ Pr = 1.5, \lambda = 0.3, Ec = 0.5 = M = \phi, \]
\[ S_p = 0.7, \ N_b = 0.3, \ N_c = 0.2 = \phi_b, \ R_d = 0.5 = \tilde{E} \]
\[ \text{We} = 0.8, \lambda = 1.1, \ S_p = 0.2, \ Sc = 0.4. \]

\[ \tilde{N} = 1, 2, 3, 4 \]

Fig 5

\[ n = 0.1, \ Pr = 1.5, \lambda = 0.3, \tilde{N} = 0.5 = R_d = \phi, \]
\[ S_p = 0.7, \ N_b = 0.3, \ N_c = 0.2 = \phi_b, \ Ec = 0.5 = \tilde{E} \]
\[ \text{We} = 0.8, \lambda = 1.1, \ S_p = 0.2, \ Sc = 0.4. \]

\[ M = 0.1, 0.5, 0.8, 1 \]
Fig 3

Fig 16

Table 1

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### Table 4

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<tr>
<td><strong>Sumaira Jabeen</strong> is Ph.D student of mathematics at Quaid-i-Azam university, Pakistan. She received his master’s degree from Quaid-i-Azam university. Her research interests are fluid mechanics, non-linear flow problems and heat transfer.</td>
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<td><strong>Tasawar Hayat</strong> is a Pakistani mathematician who has made pioneering research contributions to the area of mathematical fluid mechanics. He is considered one of the leading mathematicians working in Pakistan and currently is a Professor of Mathematics at the Quaid-i-Azam University</td>
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<td><strong>Ahmad Alsaedi</strong> is professor of department of mathematics at King Abdulaziz University, Jeddah, Saudi Arabia. He belongs to Nonlinear Analysis and Applied Mathematics (NAAM) research group. His area of interests includes fluid dynamics, non linear flow analysis and flow problem in nanosystems.</td>
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<td><strong>M. Sh. Alhodaly</strong> is Assistant professor at King Abdulaziz University. His research interests are fluid dynamics, heat transfer and combustion.</td>
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