Scientia Iranica E (2020) 27(5), 2555–2571

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
http://scientiairanica.sharif.edu

On a new family of Kies Burr III distribution: Development, properties, characterizations, and applications

F.A. Bhatti* **and M. Ahmad**

National College of Business Administration and Economics, Lahore, Pakistan.

Received 28 January 2018; received in revised form 30 January 2019; accepted 22 April 2019

KEYWORDS
Moments;
Reliability;
Characterizations;
Maximum likelihood estimation.

Abstract. In this paper, a new lifetime family of distributions called 'New Family of Kies Burr III (NFKBIII) distribution' was developed by using T-X family technique. The NFKBIII distribution is very flexible and its hazard rate function accommodates various shapes such as increasing, decreasing, increasing-decreasing-increasing, and bathtub. The density function of the NFKBIII was arc, J, reverse-J, U, bimodal, left-skewed, right-skewed and symmetrical shaped. Some structural and mathematical properties including quantiles, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions, and reliability measures were derived. Two characterizations for the NFKBIII distribution were studied. The Maximum Likelihood Estimates (MLEs) for unknown parameters of NFKBIII distribution were obtained. A simulation study was performed to evaluate the behavior of the maximum likelihood estimators. The NFKBIII distribution was applied to two real data sets to illustrate its potentiality and utility. The adequacy of the NFKBIII distribution was tested via different goodness of fit statistics.

1. Introduction

Although numerous univariate continuous distributions have been established in recent decades, many datasets composed of reliability, life testing, risk analysis, finance, ecology, climatology, geology, hydrology, and other fields do not fit these distributions. Therefore, the application of the modified distributions to problems in these fields is a vibrant necessity, today.

The modified, generalized, and extended distributions are attained by adding one or more parameters, or introducing some transformation, to the parent distribution. Therefore, the new proposed distributions provide best fit among the sub and competing models.

Burr [1] proposed a family of 12 distributions by fitting cumulative frequency functions to frequency data called Burr family. Burr distributions III, VI, X, and XII may enjoy applications. Burr III (BIII) distribution is commonly applied to model risk data in business and finance, crop rice in market, failure time data in life testing, and reliability and ozone data in environmental sciences.

Many modified, generalized, and extended types of BIII distribution are presented in statistical literature such as two-parameter family of distributions [2], inverse Burr [3], BIII type [4], extended Burr III [5],

*. *Corresponding author.*
E-mail addresses: fiazahmad72@gmail.com (F.A. Bhatti); munirahmaddr@yahoo.co.uk (M. Ahmad)

doi: 10.24200/sci.2019.50355.1655

Dagum [6], modified BIII [7], McDonald BIII [8], interpolating family [9], mixture of two BIII [10], generalized gamma BIII [11], four-parameter gamma BIII [12], odd BIII family [13], Kumaraswamy odd Burr G family [14], and generalized BIII [15].

Marshall and Olkin [16] presented a new technique to add a parameter to a family of distribution. Cordeiro and Castro [17] established Kumaraswamy generalized family with its distributional properties. Alizadeh et al. [18] studied Burr generalized family with various properties. Cordeiro et al. [19] developed the generalizability of odd log-logistic family with properties. Haghbin et al. [20] presented a new generalized odd log-logistic family of distributions. Korkmaz and Genç [21,22] studied a generalized two-sided class of distributions along with applications. Cordeiro et al. [23] studied a new family based on the Burr XII density with detailed properties. Alizadeh et al. [24] studied the odd log-logistic logarithmic class of continuous distributions. Yousof et al. [25] developed Burr Hatke-G family of distributions. Korkmaz et al. [26] presented the Weibull Marshall-Olkin family along with its properties.

The main concern of this article is to develop and study a flexible lifetime family of BIII-type distribution with two extra shape parameters and two location parameters called the NFKBIII distribution. The shapes of NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed, and symmetrical shapes. The hazard rate function for the NFKBIII distribution is characterized by various shapes such as increasing, decreasing, increasing-decreasing-increasing, and bathtub. The NFKBIII distribution is the best model for modeling data such as time to failures of items in life testing, maximum annual flood discharges in hydrology, and other various fields. The NFKBIII distribution offers better fits than sub and competing models.

This paper is organized as follows. In Section 2, the NFKBIII distribution is derived from T-X family technique, transformation, and compounding mixture of distributions. Structural properties, quantile function, sub-models, and various plots of density and hazard rate functions are discussed. In Section 3, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, residual life functions, and reliability measures are derived. The characterization of the NFKBIII distribution is studied in Section 4. In Section 5, the Maximum Likelihood Estimates (MLEs) for unknown parameters of the NFKBIII distribution are obtained. In Section 6, a simulation study is performed to assess the behavior of the maximum likelihood estimators. In Section 7, the potentiality and utility of the NFKBIII distribution is illustrated via its application to two real data sets: times to failures of devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution is tested via different goodness of fit statistics. The ultimate comments are given in Section 8.

2. Development of NFKBIII distribution

The cumulative distribution function (cdf) of the generalized uniform distribution is given by:

$$G(x;a,b,\kappa)=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-a^{\kappa}},$$

$$x\in[a,b],\quad a>0,\quad b>0,\quad \kappa>0. \tag{1}$$

The odds ratio for the generalized uniform random variable X is given below:

$$W\left(G(x)\right)=\frac{G(x;a,b,\kappa)}{\bar{G}(x;a,b,\kappa)}=\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}. \tag{2}$$

Gurvich et al. [27] replaced 'x' with odds ratio of the Weibull distribution for the development of a class of extended Weibull distributions. Alzaatreh et al. [28] developed the cdf of the T-X family of distributions as follows:

$$F\left(x\right)=\int_{a}^{W(G(x))} r\left(t\right)dt, \tag{3}$$

where $W(G(x))$ is a function of $G(x)$ and $r(t)$ is the pdf of a non-negative random variable.

Bourguignon et al. [29] inserted the odds ratio of a baseline distribution in place of 'x' in the cdf of the Weibull distribution to develop a new family of distributions.

The NFKBIII was developed by inserting the odds ratio for the generalized uniform in place of 'x' in the cdf of MBIII distribution. The cdf for the NFKBIII distribution is obtained as follows:

$$F(x)=\int_{0}^{W(G(x))}\alpha\beta t^{-\beta-1}\left(1+\gamma t^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

or:

$$F(x;\alpha,\beta,\gamma,\lambda)=\int_{0}^{\frac{x^{\kappa}-a^{\kappa}}{b^{\kappa}-x^{\kappa}}}\alpha\beta t^{-\beta-1}\left(1+\gamma t^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

or:

$$F\left(x\right)=\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}},\quad a\leq x<b, \tag{4}$$

where $a, b,\ \alpha,\ \beta,\ \kappa$, and γ are the positive parameters, among which a, b are location parameters and $\alpha,\ \beta,\ \kappa$ and γ are shape parameters. Clearly, $F(x)$ is a strictly increasing and differential cdf on (a, b).

The pdf of the NFKBIII distribution is given below:

$$f(x) = \alpha\beta\left(b^{\kappa} - a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa} - x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa} - a^{\kappa}\right)^{\beta+1}}$$

$$\left[1 + \gamma\left(\frac{b^{\kappa} - x^{\kappa}}{x^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}, \quad x > a. \tag{5}$$

2.1. Transformation and compounding

The NFKBIII model was also developed by (i) transformation between the ratio of exponential and gamma random variables and (ii) compounding generalized inverse Kies (GNIK) and gamma distributions.

(i) Let Z_1 be a random variable having exponential distribution with parameter value 1 and Z_2 be a random variable with gamma, i.e., $Z_2 \sim gamma\left(\frac{\alpha}{\gamma}, 1\right)$; then, using the relationship $Z_1 = \gamma\left(\frac{b^{\kappa}-X^{\kappa}}{X^{\kappa}-a^{\kappa}}\right)^{\beta} Z_2$, we have:

$$X = \left\{\left[a^{\kappa} + b^{\kappa}\left(\frac{Z_1}{\gamma Z_2}\right)^{-\frac{1}{\beta}}\right]\right.$$

$$\left.\left[1 + \left(\frac{Z_1}{\gamma Z_2}\right)^{-\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{\kappa}}$$

$$\sim NFKBIII\left(a, b, \alpha, \beta, \gamma, \kappa\right).$$

(ii) Let X be a random variable with GNIK distribution, i.e., $X \sim GNIK\left(x; a, b, \beta, \kappa, \gamma, \theta\right)$, and θ be a random variable with gamma distribution, i.e., $\theta \sim gamma\left(\theta; \alpha, \gamma\right)$. Then, after simplifying the integral:

$$f\left(x; a, b, \alpha, \beta, \gamma, \kappa\right) = \int_0^{\infty} GNIK$$

$$\left(x/a, b, \beta, \kappa, \gamma, \theta\right) g\left(\theta/\alpha, \gamma\right) d\theta$$

we have $X \sim NFKBIII\left(a, b, \alpha, \beta, \gamma, \kappa\right)$.

2.2. Structural properties

The survival, hazard, cumulative hazard, reverse hazard functions, and the Mills ratio of a random variable X with the NFKBIII distribution are given respectively below:

$$S(x) = 1 - \left[1 + \gamma\left(\frac{b^{\kappa} - x^{\kappa}}{x^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}, \ x \geq a, \tag{6}$$

$$h(x) = \alpha\beta\kappa\left(b^{\kappa} - a^{\kappa}\right)x^{\kappa-1}$$

$$\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\frac{\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}}{\left\{1-\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right\}}, \quad x > a, \tag{7}$$

$$H(x) = -\ln\left\{1 - \left[1 + \gamma\left(\frac{b^{\kappa} - x^{\kappa}}{x^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right\},$$

$$x \geq a, \tag{8}$$

$$r(x) = \frac{f(x)}{F(x)} = \alpha\beta\left(b^{\kappa} - a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa} - x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa} - a^{\kappa}\right)^{\beta+1}}$$

$$\left[1 + \gamma\left(\frac{b^{\kappa} - x^{\kappa}}{x^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-1}, \quad x > a, \tag{9}$$

and:

$$m(x) = \frac{1 - F(x)}{f(x)} =$$

$$\frac{1 - \left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}}{\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}\left[1 + \gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}}. \tag{10}$$

The elasticity $e(x) = xr(x) = \frac{dlnF(x)}{dlnx}$ for the NFKBIII distribution is:

$$e(x) = \alpha\beta\left(b^{\kappa} - a^{\kappa}\right)\kappa x^{\kappa}\frac{\left(b^{\kappa} - x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa} - a^{\kappa}\right)^{\beta+1}}$$

$$\left[1 + \gamma\left(\frac{b^{\kappa} - x^{\kappa}}{x^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-1}. \tag{11}$$

The quantile function of NFKBIII distribution is:

$$x_q = \left[\frac{a^{\kappa}\gamma^{-\frac{1}{\beta}}\left(q^{-\frac{\gamma}{\alpha}} - 1\right)^{\frac{1}{\beta}} + b^{\kappa}}{\gamma^{-\frac{1}{\beta}}\left(q^{-\frac{\gamma}{\alpha}} - 1\right)^{\frac{1}{\beta}} - 1}\right]^{\frac{1}{\kappa}},$$

and its random number generator is:

$$X = \left[\frac{a^{\kappa}\gamma^{-\frac{1}{\beta}}\left(Z^{-\frac{\gamma}{\alpha}} - 1\right)^{\frac{1}{\beta}} + b^{\kappa}}{\gamma^{-\frac{1}{\beta}}\left(Z^{-\frac{\gamma}{\alpha}} - 1\right)^{\frac{1}{\beta}} - 1}\right]^{\frac{1}{\kappa}},$$

where the random variable Z has the uniform distribution on (0,1).

2.3. Sub-models

The NFKBIII distribution is widely applicable to life testing, reliability concept, survival analysis, and hydrology. The NFKBIII distribution has the subsequent nested models (Table 1).

2.4. Plots for the NFKBIII density and hazard rate functions

Figure 1 shows that the shapes of the NFKBIII density are arc, J, reverse-J, U, bimodal, left-skewed, right-skewed, and symmetrical (Figure 1). The shapes of failure rate function for the NFKBIII distribution are increasing, decreasing, increasing-decreasing-increasing, and bathtub (Figure 2).

Table 1. Sub-models of the New Family of Kies Burr III (NFKBIII) distribution.

1	X	a	b	α	β	γ	κ	New Family of Kies Burr III (NFKBIII)
2	X	a	b	α	β	γ	1	Kies Modified Burr III (KMBIII)
3	X	a	b	α	β	1	κ	New Kies Burr III (NKBIII)
4	X	a	b	α	β	1	1	Kies Burr III (KBIII)
5	X	0	1	α	β	1	1	Reduced Kies Burr III (RKBIII)
6	X	0	1	α	β	γ	κ	Reduced New Kies Burr III (RNKBIII)
7	X	a	b	α	1	γ	κ	New Kies Modified Inverse Lomax (NKMIL)
8	X	a	b	α	1	γ	1	Kies Modified Inverse Lomax (KMIL)
9	X	a	b	α	1	1	1	Kies Inverse Lomax(KIL)
10	X	0	1	α	1	1	1	Reduced Kies Inverse Lomax (RKIL)
11	X	a	b	α	β	γ	κ	Reduced Kies Burr III (RKBIII)
12	X	0	1	α	β	γ	κ	Reduced New Kies Burr III (RNKBIII)
13	X	a	b	α	β	$\gamma \to 0$	κ	New Inverse Kies (NIK)
14	X	0	1	α	β	$\gamma \to 0$	κ	New Reduced Inverse Kies (NRK)
15	X	a	b	α	β	$\gamma \to 0$	κ	New Modified Inverse Kies (NMIK)
16	X	0	1	α	β	$\gamma \to 0$	1	Reduced New Inverse Kies(RNIK)
17	X	0	1	α	β	$\gamma \to 0$	1	Reduced Inverse Kies (RIK)
18	X	a	b	α	β	$\gamma \to 0$	1	New Modified Inverse Kies (NMIK)
19	X	a	b	α	β	$\gamma \to 0$	κ	Generalized Inverse Kies
20	$\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)$	a	b	α	β	$\gamma \to 0$	1	Kies [30,31]
21	X	a	b	α	β	$\gamma \to 0$	1	Inverse Kies
22	X	0	b	α	β	γ	κ	Modified Burr III Power (MBIII-Power)
23		a	0	α	β	γ	κ	Modified Burr III Pareto (MBIII-Pareto)
24	X	0	b	α	β	1	κ	Burr III Power (BIII-Power)
25	X	a	0	α	β	1	κ	Burr III Pareto (BIII-Pareto)
26	X	0	b	α	β	$\gamma \to 0$	κ	Inverse Weibull- Power
27	X	a	0	α	β	$\gamma \to 0$	κ	Inverse Weibull Pareto

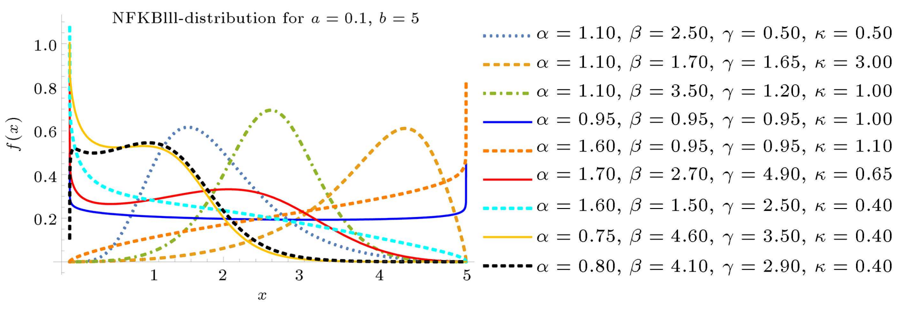

Figure 1. Plots of pdf of New Family of Kies Burr III (NFKBIII) distribution.

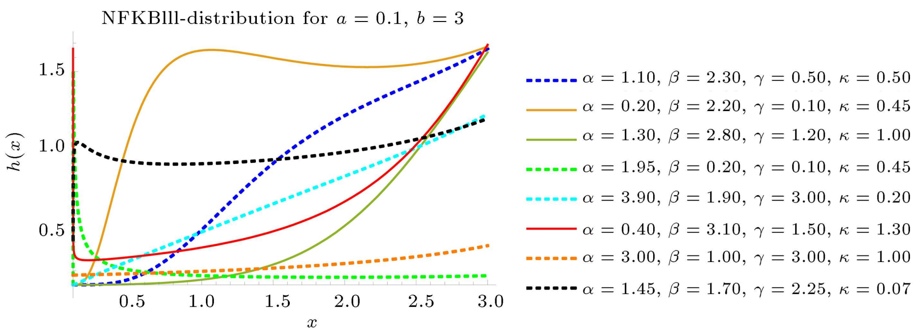

Figure 2. Plots of hrf of New Family of Kies Burr III (NFKBIII) distribution.

3. Mathematical properties

Some descriptive measures for the NFKBIII distribution such as ordinary and incomplete moments, inequality curves, mean deviations, residual life functions, and reliability measures are established in this section.

3.1. Moments of the NFKBIII distribution

The rth moment about origin of X with the NFKBIII distribution is:

$$\mu_r' = E(X^r) = \int_a^b x^r f(x) dx,$$

$$E(X^r) = \alpha\beta\left(b^\kappa - a^\kappa\right)\kappa \int_a^b x^r x^{\kappa-1} \frac{\left(b^\kappa - x^\kappa\right)^{\beta-1}}{\left(x^\kappa - a^\kappa\right)^{\beta+1}}$$

$$\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} dx.$$

Letting:

$$\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta = w$$

and:

$$x = \left\{\frac{\left(a^\kappa - b^\kappa\right)}{\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]} + b^\kappa\right\}^{\frac{1}{\kappa}},$$

we arrive at:

$$E(X^r) = \frac{\alpha}{\gamma}\int_0^\infty \left\{\frac{\left(a^\kappa - b^\kappa\right)}{\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]} + b^\kappa\right\}^{\frac{r}{\kappa}}$$

$$[1+w]^{-\frac{\alpha}{\gamma}-1} dw,$$

$$E(X^r) = \frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}} \binom{\frac{r}{\kappa}}{\ell} \left(a^\kappa - b^\kappa\right)^\ell b^{(r-\kappa\ell)}$$

$$\sum_{\nu=0}^\infty \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!}\left[\int_0^\infty w^{-\frac{\nu}{\beta}}[1+w]^{-\frac{\alpha}{\gamma}-1} dw\right],$$

and observe that:

$$\mu_r' = \frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}} \binom{\frac{r}{\kappa}}{\ell} \left(a^\kappa - b^\kappa\right)^\ell b^{(r-\kappa\ell)}$$

$$\sum_{\nu=0}^\infty \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!} B\left(1-\frac{\nu}{\beta}, \frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right),$$

$$r = 1, 2, 3, \ldots \tag{12}$$

where $(\ell)_k = \frac{\Gamma(\ell+k)}{\Gamma(\ell)}$ is the Pochhammer symbol.

The factorial moments $E[X]_n = \sum_{r=1}^n \vartheta_r E(X^r)$ for the NFKBIII distribution are:

$$E[X]_n = \sum_{r=1}^n \left[\vartheta_r \frac{\alpha}{\gamma}\sum_{\ell=0}^r \binom{r}{\ell}(a-b)^\ell b^{r-\ell}\right.$$

$$\left.\sum_{k=0}^\infty \frac{(-1)^k (\ell)_k \gamma^{\frac{k}{\beta}}}{k!} B\left(1-\frac{k}{\beta}, \frac{\alpha}{\gamma}+\frac{k}{\beta}\right)\right], \tag{13}$$

where $[Z]_i = Z(Z+1)(Z+2), \ldots, (Z+i-1)$ and ϑ_r is Stirling number of the first type.

The Mellin transform was used to obtain moments of a probability distribution. By definition, the Mellin transform is:

$$\mathrm{M}\{f(x); s\} = f^*(s) = \int_0^\infty f(x)\, x^{s-1} dx.$$

The Mellin transform of X with the NFKBIII distribution is:

$$\mathrm{M}\{f(x); s\} = \alpha\sum_{\ell=0}^{s-1}\sum_{k=0}^\infty \binom{s-1}{\ell}\frac{(-1)^k (\ell)_k}{k!}\gamma^{\frac{k}{\beta}-1} b^{s-1}$$

$$\left(\frac{a}{b}-1\right)^\ell B\left(1-\frac{k}{\beta}, \frac{\alpha}{\gamma}+\frac{k}{\beta}\right). \tag{14}$$

The rth moment about means, Pearson's measures for skewness and kurtosis, moment generating function, and cumulants of X for the NFKBIII distribution were obtained through the following relations:

$$\mu_r = \sum_{i=1}^r \binom{r}{i}(-1)^i \mu'_i\, \mu'_{i-r},$$

$$\gamma_1 = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}}, \qquad \beta_2 = \frac{\mu_4}{\left(\mu_2\right)^2}$$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{r=1}^\infty \frac{t^r}{r!}E(X)^r,$$

$$k_r = \mu_r' - \sum_{c=1}^{r-1}\binom{r-1}{c-1} k_c\, \mu_{r-c}'.$$

Table 2 displays the numerical descriptive measures such as median, mean, standard deviation, skewness, and kurtosis of the NFKBIII distribution for the carefully chosen parameter values to describe their effect on these descriptive measures.

3.2. Moments of order statistics

Moments of order statistics were applied to life testing and reliability. Moments of order statistics were aimed at anticipating the possible failure of future items after few initial failures.

The pdf for the mth order statistic $X_{m:n}$ is as follows:

Table 2. Median, mean, standard deviation, skewness, and Kurtosis of the New Family of Kies Burr III (NFKBIII) distribution.

Parameters $\alpha, \beta, \gamma, \kappa, a = 0.1, b = 5$	Median	Mean	Standard deviation	Skewness	Kurtosis
0.5,0.5,0.5,0.5	0.4897	1.4750	1.7079	0.9765	2.3803
0.5,1.5,1.5,0.5	0.6763	1.1133	1.1156	1.2380	3.7107
1,0.5,0.5,0.5	2.1126	2.3588	1.8197	0.1678	1.4319
1,1,1,2	3.5333	3.3328	1.1766	–0.5611	2.3903
0.5,1,1,1,	1.3210	1.7305	1.4595	0.6413	2.1471
1,1,1,1	2.5460	2.5475	1.4138	0.0019	1.8006
2,1,1,1	3.3649	3.3649	1.1546	–0.5638	2.3977
1.5,1,1,1	3.1834	3.0379	1.2827	–0.3377	2.0496
1,1,1,0.5	1.6268	1.9349	1.4408	0.4984	2.0057
1.5,1.5,1.5,1	2.8765	2.8088	1.1137	–0.2179	2.2432
1.5,1.5,1.5,1.5	3.4298	3.2977	0.9797	–0.5674	2.7456
1.5,1.5,1.5,2.5	3.9849	3.8405	0.7525	–0.9578	3.7872
1.5,1.5,1.5,3	4.1375	4.0009	0.6698	–1.0776	4.2189
1.5,1.5,1.5,0.5	1.9744	2.0897	1.1797	0.3375	2.2034
2.5,1.5,1.5,0.5	2.5717	2.5902	1.0939	0.0430	2.1806
2,2,2,2	3.8261	3.7287	0.6639	–0.8429	3.8245
2.5,1.5,2.5,2.5	4.2032	4.0413	0.6924	–1.2421	4.6981
2.5,1.5,2.5,2.5	4.0501	3.9854	0.4675	–0.9543	4.6041
5,2.5,2.5,2.5	4.2659	4.2409	0.3136	–0.5782	3.6917
3.25,2.4,0.65,1.5	3.7515	3.7577	0.4326	0.0016	2.7084
4.5,2.4,0.65,1.5	3.8735	3.8788	0.3956	0.0010	2.6739
5,2.5,2.5,2.5	4.2659	4.2409	0.3136	–0.5782	3.6917
5,2.5,2.5,0.5	3.2661	3.2192	0.9057	–0.2531	2.4020
6,2,1.5,0.5	3.0029	3.0224	0.7507	0.0597	2.5103
6,2,1.5,0.5	4.3075	4.2356	0.4691	–0.8193	3.6134
5,1.5,1.5,1.5	4.2268	4.1482	0.5200	–0.8090	3.5801
5,0.5,1.5,1.5	4.9222	4.6458	0.6553	–3.0134	13.2449
5,1,1.5,1.5	4.5453	4.3729	0.5804	–1.5137	5.6080

$$f\left(x_{m:n}\right) = \frac{1}{B\left(m,\, n-m+1\right)}\left[F\left(x\right)\right]^{m-1} \left[1-F\left(x\right)\right]^{n-m} f\left(x\right). \tag{15}$$

The pdf of $X_{m:n}$ for the NFKBIII distribution is given below:

$$f_{X_{m\,:\,n}}\left(x\right) = \left\{\frac{1}{B\left(m,\, n-m+1\right)}\sum_{i=0}^{n-m}\left(-1\right)^{i}\binom{n-m}{i} \times\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}} \left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}(m+i)-1}\right\}. \tag{16}$$

Moments about the origin of $X_{m:n}$ for the NFKBIII distribution are:

$$E\left(X^{r}{}_{m:n}\right) = \int_{a}^{b} x^{r} f\left(x_{m:n}\right)dx. \tag{17}$$

Eq. (18) is shown in Box I.

3.3. Incomplete moments

Bonferroni and Lorenz curves can be easily computed

$$E\left(X^r{}_{m:n}\right)=\frac{\alpha}{\gamma}\frac{1}{B\left(m,\,n-m+1\right)}\sum_{i=0}^{n-m}\sum_{\ell=0}^{\frac{r}{\kappa}}\sum_{\nu=0}^{\infty}\binom{\frac{r}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell}b^{(r-\kappa\ell)}\binom{n-m}{i}\frac{(-1)^{i+\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}\left(m+i\right)+\frac{\nu}{\beta}\right),$$

$$E\left(X^r{}_{m:n}\right)=\frac{\alpha}{\gamma}\frac{\sum\limits_{i=0}^{n-m}\sum\limits_{\ell=0}^{\frac{r}{\kappa}}\sum\limits_{\nu=0}^{\infty}\binom{\frac{r}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell}b^{(r-\kappa\ell)}\binom{n-m}{i}\frac{(-1)^{i+\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}\left(m+i\right)+\frac{\nu}{\beta}\right)}{B\left(m,\,n-m+1\right)},$$

$$r=1,2,3.... \tag{18}$$

Box I

using first incomplete moment. The life testing features such as residual life and mean inactivity life functions can be obtained from incomplete moments. The lower incomplete moments for the random variable X with the NFKBIII distribution are given below:

$$M_r'\left(z\right)=E_{X\le z}\left(X^r\right)=\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa$$

$$\int\limits_a^z x^r x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}$$

$$\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}dx.$$

Letting:

$$\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}=w$$

and:

$$x=\left\{\left(a^{\kappa}-b^{\kappa}\right)\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]^{-1}+b^{\kappa}\right\}^{\frac{1}{\kappa}},$$

we arrive at:

$$E(X^r)=\frac{\alpha}{\gamma}\int\limits_{\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta}}^{\infty}$$

$$\left\{\left(a^{\kappa}-b^{\kappa}\right)\left[1+\left(\gamma^{-1}w\right)^{-\frac{1}{\beta}}\right]^{-1}+b^{\kappa}\right\}^{\frac{1}{\kappa}}$$

$$[1+w]^{-\frac{\alpha}{\gamma}-1}dw,$$

$$E_{X\le z}\left(X^r\right)=\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell}b^{(r-\kappa\ell)}$$

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$\left[\int\limits_{\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta}}^{\infty}w^{-\frac{\nu}{\beta}}[1+w]^{-\frac{\alpha}{\gamma}-1}dw\right],$$

and observe that:

$$M_r'\left(z\right)=\left(\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell}b^{(r-\kappa\ell)}\right.$$

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}\times\left\{B\left(1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right)\right.$$

$$\left.\left.-B\left[\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]\right\}\right),$$

$$r=1,2,3,..., \tag{19}$$

where $B\left(z\;;.,.\right)$ is the incomplete beta function.

The upper incomplete moments for the random variable X with the NFKBIII distribution are:

$$E_{X\ge z}\left(X^r\right)=\int\limits_z^b x^r\,\alpha\beta\left(b^{\kappa}-a^{\kappa}\right)\kappa x^{\kappa-1}\frac{\left(b^{\kappa}-x^{\kappa}\right)^{\beta-1}}{\left(x^{\kappa}-a^{\kappa}\right)^{\beta+1}}$$

$$\left[1+\gamma\left(\frac{b^{\kappa}-x^{\kappa}}{x^{\kappa}-a^{\kappa}}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$E_{X\ge z}\left(X^r\right)=\frac{\alpha}{\gamma}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left(a^{\kappa}-b^{\kappa}\right)^{\ell}b^{(r-\kappa\ell)}$$

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}(\ell)_{\nu}\gamma^{\frac{\nu}{\beta}}}{\nu!}B\left[\gamma\left(\frac{b^{\kappa}-z^{\kappa}}{z^{\kappa}-a^{\kappa}}\right)^{\beta};1\right.$$

$$\left.-\frac{\nu}{\beta},\frac{\alpha}{\gamma}+\frac{\nu}{\beta}\right]. \tag{20}$$

The mean deviation about the mean is $MD_{\bar{X}} = E\left|X-\mu_1^1\right| = 2\mu_1^1 F\left(\mu_1^1\right) - 2\mu_1^1 M_1'\left(\mu_1^1\right)$ and mean deviation about the median is $MD_M = E\left|X-M\right| = 2MF(M) - 2MM_1'(M)$, where $\mu_1' = E(X)$ and $M = Q(0.5)$. Bonferroni and Lorenz curves for a specified probability p are computed by $B(p) = \frac{M'_1(q)}{p\mu'}$ and $L(p) = \frac{M'_1(q)}{\mu'}$, where $q = Q(p)$.

3.4. Residual life functions

The nth moment $m_n(z)$ of residual life for X with the NFKBIII distribution is given below:

$$m_n(z) = E\left[(X-z)^n | X > z\right]$$

$$= \frac{1}{S(z)} \int_z^{\infty} (x-z)^s f(x)dx,$$

$$m_n(z) = \frac{1}{S(z)} \sum_{s=0}^{n} \binom{n}{s} (-z)^{n-s} E_{X>z}(X^s),$$

$$m_n(z) = \frac{1}{S(z)} \sum_{s=0}^{n} \binom{n}{s} (-z)^{n-s} \frac{\alpha}{\gamma}$$

$$\sum_{\ell=0}^{\frac{s}{\kappa}} \binom{\frac{s}{\kappa}}{\ell} (a^\kappa - b^\kappa)^\ell b^{(r-\kappa\ell)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$B\left[\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta; 1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right]. \tag{21}$$

The residual life (MRL) function $m_1(z)$ of a component at time z, or the average remaining lifetime, is also called life expectancy given by:

$$m_1(z) = \frac{1}{S(z)} \sum_{s=0}^{1} \binom{1}{s} (-z)^{1-s} \frac{\alpha}{\gamma}$$

$$\sum_{\ell=0}^{\frac{s}{\kappa}} \binom{\frac{s}{\kappa}}{\ell} (a^\kappa - b^\kappa)^\ell b^{(r-\kappa\ell)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$B\left[\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta; 1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right]. \tag{22}$$

The nth moment of reverse residual life $M_n(z)$ for X with the NFKBIII distribution is:

$$M_n(z) = E\left[(z-X)^n / X \le z\right]$$

$$= \frac{1}{F(z)} \int_a^{z} (z-x)^n f(x)dx,$$

$$M_n(z) = \frac{1}{F(z)} \sum_{s=0}^{n} (-1)^s \binom{n}{s} z^{n-s} E_{X\le z}(X^s),$$

$$M_n(z) = \frac{1}{F(z)} \sum_{s=0}^{n} (-1)^s \binom{n}{s} z^{n-s} \left(\frac{\alpha}{\gamma} \sum_{\ell=0}^{\frac{s}{\kappa}} \binom{\frac{s}{\kappa}}{\ell}\right)$$

$$(a^\kappa - b^\kappa)^\ell b^{(r-\kappa\ell)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$\times\left\{B\left(1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right)\right.$$

$$\left.-B\left[\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta; 1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right]\right\}\Bigg). \tag{23}$$

The waiting time z for the failure of a component with a condition that this failure would occur at the interval $[0, z]$ is called mean waiting time or mean inactivity time. The waiting time z for the failure of a component X with the NFKBIII distribution is defined by:

$$M_1(z) = \frac{1}{F(z)} \sum_{s=0}^{1} (-1)^s \binom{1}{s} z^{1-s} \left(\frac{\alpha}{\gamma} \sum_{\ell=0}^{\frac{s}{\kappa}} \binom{\frac{s}{\kappa}}{\ell}\right)$$

$$(a^\kappa - b^\kappa)^\ell b^{(r-\kappa\ell)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\ell)_\nu \gamma^{\frac{\nu}{\beta}}}{\nu!}$$

$$\times\left\{B\left(1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right)\right.$$

$$\left.-B\left[\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta; 1 - \frac{\nu}{\beta}, \frac{\alpha}{\gamma} + \frac{\nu}{\beta}\right]\right\}\Bigg). \tag{24}$$

3.5. Stress-strength reliability for the NFKBIII distribution

Let X_1 be strength and X_2 be stress and X_1 follow NFKBIII distribution $(\alpha_1, \beta, \gamma, \kappa, a, b)$ and X_2 follow NFKBIII distribution $(\alpha_2, \beta, \gamma, \kappa, a, b)$. Then $R = Pr(X_2 < X_1) = \int_a^b f_{x_1}(x) F_{x_2}(x)\,dx$ is the reliability parameter [32]. The reliability of the component is computed as follows:

$$R = \int_a^b \alpha_1 \beta \left(b^\kappa - a^\kappa 2\right) \kappa x^\kappa \frac{(b^\kappa - x^\kappa)^{\beta-1}}{(x^\kappa - a^\kappa)^{\beta+1}}$$

$$\left[1 + \gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}-1}$$

$$\left[1 + \gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}} dx,$$

$$R = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{25}$$

Therefore, R is independent of a, b, β, κ and γ.

3.6. Estimation of multicomponent stress-strength system reliability with NFKBIII distribution

Consider a system that has m identical components out of which s components are functioning. The strengths of m components are $X_i, i = 1, 2...m$ with common cdf F, while the stress Y imposed on the components has cdf G. The strengths $X_i, i = 1, 2...m$ and stress Y are i.i.d. distributed. The probability that system operates properly is reliability of the system, i.e.:

$$R_{s,m} = P[strengths\,(X_i, i = 1, 2...m) > stress\,(Y)]$$

$$R_{s,m} = P[\text{at the minimum "}s\text{" } of\,(X_i, i = 1, 2...m)$$

$$exceed\ Y]. \tag{26}$$

$$R_{s,m} = \sum_{l=s}^{m} \binom{m}{l} \int_{-\infty}^{\infty} [1 - F(y)]^l [F(y)]^{m-l}$$

$$dG(y).\ \ [33] \tag{27}$$

Let $X \sim NFKBIII\,(\alpha_1, \beta, \gamma, \kappa, a, b)$ and $Y \sim NFKBIII\,(\alpha_2, \beta, \gamma, \kappa, a, b)$ such that α_1, and α_2 be unknown shape parameters and a, b be common location parameters. X and Y are independently distributed. The reliability that system operates properly with respect to the multicomponent stress strength for the NFKBIII distribution is given below:

$$R_{s,m} = \sum_{\ell=s}^{m} \binom{m}{\ell} \int_{a}^{b} \left(1 - \left[1 + \gamma\left(\frac{b^\kappa - y^\kappa}{y^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right)^\ell$$

$$\left(\left[1 + \gamma\left(\frac{b^\kappa - y^\kappa}{y^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_1}{\gamma}}\right)^{(m-\ell)}$$

$$\alpha_2 \beta\,(b^\kappa - a^\kappa)\,\kappa y^\kappa \frac{(b^\kappa - y^\kappa)^{\beta-1}}{(y^\kappa - a^\kappa)^{\beta+1}}$$

$$\left[1 + \gamma\left(\frac{b^\kappa - y^\kappa}{y^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma} - 1} dy.$$

Letting:

$$\left[1 + \gamma\left(\frac{b^\kappa - y^\kappa}{y^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha_2}{\gamma}} = u,$$

we obtain:

$$R_{s,m} = \sum_{\ell=s}^{m} \binom{m}{\ell} \int_{0}^{1} \left(1 - u^\Upsilon\right)^\ell u^{\Upsilon(m-\ell)} du,$$

where $\Upsilon = \frac{\alpha_2}{\alpha_1}$. Again letting $u^\Upsilon = w$, we reach:

$$R_{s,m} = \sum_{\ell=s}^{m} \binom{m}{\ell} \int_{0}^{1} (1 - u)^\ell w^{(m-\ell)} \frac{1}{\Upsilon} w^{\frac{1}{\Upsilon} - 1} dw.$$

$$R_{s,m} = \frac{1}{\Upsilon} \sum_{\ell=s}^{m} \binom{m}{\ell} B\left[1 + \ell,\ m + \frac{1}{\Upsilon} - \ell\right], \tag{28}$$

is the reliability of multicomponent stress-strength model [33].

4. Characterizations

In this section, two essential characterizations of the NFKBIII distribution are planned via (i) conditional expectation and (ii) ratio of truncated moments.

4.1. Characterization based on conditional expectation

Here, the NFKBIII distribution is characterized via conditional expectation.

Proposition 4.1.1. Let $X : \Omega \to (a, b)$ be a continuous random variable with cdf $F(x)$, $0 < F(x) < 1$ for $x \geq a$); then, for $\alpha > \gamma$, X has cdf (4) if and only if:

$$E\left[\left(\frac{X^\kappa - a^\kappa}{b^\kappa - X^\kappa}\right)^{-\beta} \middle| X < z\right] = \frac{1}{(\alpha - \gamma)}$$

$$\left[1 + \alpha\left(\frac{z^\kappa - a^\kappa}{b^\kappa - z^\kappa}\right)^{-\beta}\right], \quad \text{for z} > a. \tag{29}$$

Proof. If Eq. (5) is pdf of X, then:

$$E\left[\left(\frac{b^\kappa - X^\kappa}{X^\kappa - a^\kappa}\right)^\beta \middle| X < z\right]$$

$$= (F(z))^{-1} \int_a^z \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta f(x)\,dx$$

$$= (F(z))^{-1} \int_a^z \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta \times \frac{\alpha}{\gamma} \gamma \beta\,(b^\kappa - a^\kappa)$$

$$\kappa x^{\kappa-1} \frac{(b^\kappa - x^\kappa)^{\beta-1}}{(x^\kappa - a^\kappa)^{\beta+1}} \left[1 + \gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma} - 1} dx.$$

Integrating and simplifying, we arrive at:

$$E\left[\left(\frac{b^\kappa - X^\kappa}{X^\kappa - a^\kappa}\right)^\beta \middle| X < z\right] = \frac{1}{(\alpha - \gamma)}$$

$$\left[1 + \alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right], \quad \text{for } \ z > a.$$

Conversely, if Proposition 4.1.1 holds, then:

$$\int_a^z \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta f(x)\,dx = \frac{F(z)}{\gamma\left(\frac{\alpha}{\gamma}-1\right)} \left[1+\alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right]. \tag{30}$$

Differentiating Eq. (30) with respect to t, we achieve:

$$\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta f(z) = \frac{f(z)}{(\alpha-\gamma)}\left[1+\alpha\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right] - \frac{F(z)}{\gamma\left(\frac{\alpha}{\gamma}-1\right)}\left[\alpha\beta\left(b^\kappa - a^\kappa\right)\kappa x^{\kappa-1}\frac{(b^\kappa - z^\kappa)^{\beta-1}}{(z^\kappa - a^\kappa)^{\beta+1}}\right].$$

After simplification and integration, we work out:

$$F(z) = \left[1+\gamma\left(\frac{b^\kappa - z^\kappa}{z^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}}, \quad z \geq a.$$

4.2. *Characterization of the NFKBIII distribution through ratio of truncated moments*

The NFKBIII distribution is characterized using theorem G [34] from a simple relationship between two truncated moments of functions of X.

Proposition 4.2.1. Let $X : \Omega \to (a,b)$ be a continuous random variable. Let:

$$h_1(x) = \frac{1}{\alpha}\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{\frac{\alpha}{\gamma}+1}, \quad x > a,$$

and:

$$h_2(x) = 2\alpha^{-1}\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{\frac{\alpha}{\gamma}+1}, \quad x > a.$$

According to theorem G, the random variable X has pdf (5) if and only if that the function $p(x)$ has the form of $p(x) = \left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta, \quad x > a.$

Proof. For random variable X with pdf (5), we get:

$$(1-F(x))\,E\left(h_1(x)|\,X \geq x\right) = \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta, \quad x > a,$$

$$(1-F(x))\,E\left(h_2(x)|\,X \geq x\right) = \left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^{2\beta}, \quad x > a,$$

$$\frac{E[h_1(x)|\,X \geq x]}{E[h_2(x)|\,X \geq x]} = p(x) = \left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^\beta, \quad x > a,$$

and:

$$p'(x) = \beta\kappa\left(b^\kappa - a^\kappa\right)x^{\kappa-1}(x^\kappa - a^\kappa)^{\beta-1}(b^\kappa - x^\kappa)^{-\beta-1}, \quad x > a.$$

The differential equation $s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)} = 2\beta\left(b^\kappa - a^\kappa\right)\frac{\kappa x^{\kappa-1}}{(b^\kappa - x^\kappa)^2}\left[\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right]^{-1}$ has solution $s(x) = \ln\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^{2\beta}, \quad x > a.$ Thus, in the light of theorem G, X has pdf (5).

Corollary 4.2.1. Let $X : \Omega \to (a,b)$ be a continuous random variable and let:

$$h_2(x) = 2\alpha^{-1}\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{\frac{\alpha}{\gamma}+1}, \quad x > a.$$

The pdf of X is (5) if and only if there exist functions $p(x)$ and $h_1(X)$ (defined in theorem G), satisfying the differential equation:

$$\frac{p'(x)}{p(x)h_2(x) - h_1(x)} = \alpha\beta\left(b^\kappa - a^\kappa\right)\frac{\kappa x^{\kappa-1}}{(b^\kappa - x^\kappa)^2}\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^{\beta-1}\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1}. \tag{31}$$

Remarks 4.2.1. The solution of Eq. (31) is:

$$p(x) = \left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^{2\beta}\left[\int\left(-\alpha\beta\left(b^\kappa - a^\kappa\right)\frac{\kappa x^{\kappa-1}}{(b^\kappa - x^\kappa)^2}\left(\frac{x^\kappa - a^\kappa}{b^\kappa - x^\kappa}\right)^{-\beta-1}\left[1+\gamma\left(\frac{b^\kappa - x^\kappa}{x^\kappa - a^\kappa}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1} h_1(x)\right)dx\right] + D,$$

where D is constant.

5. Maximum likelihood estimation

In this section, estimates of parameters are derived using the maximum likelihood method. The log likelihood function for the NFKBIII distribution with the vector of parameters $\Phi = (a, b, \alpha, \beta, \gamma, \kappa)$ is:

$$\ln L(x_i, \Phi) = n\ln\alpha + n\ln\beta + n\ln\left(b^\kappa - a^\kappa\right) - (\beta+1)\sum_{i=1}^n \ln\left(x_i^\kappa - a^\kappa\right) + (\beta-1)$$

$$\sum_{i=1}^{n} \ln\left(b^{\kappa} - x_i^{\kappa}\right) - \left(\frac{\alpha}{\gamma} + 1\right)$$

$$\sum_{i=1}^{n} \ln\left[1 + \gamma\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}\right], \tag{32}$$

where a and b are assumed known, since the minimum and maximum likelihood values are equal to minimum and maximum order statistics. The MLEs of the parameters for the NFKBIII distribution can be computed using a solution to the following nonlinear equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\gamma}\sum_{i=1}^{n} \ln\left[1 + \gamma\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}\right] = 0, \tag{33}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \ln\left(x_i^{\kappa} - a^{\kappa}\right) + \sum_{i=1}^{n} \ln\left(b^{\kappa} - x_i^{\kappa}\right)$$

$$+ (\alpha + \gamma)\left[1 + \gamma\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-1}\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}$$

$$\ln\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right) = 0, \tag{34}$$

$$\frac{\partial \ln L}{\partial \gamma} = \alpha\gamma^{-2}\sum_{i=1}^{n} \ln\left[1 + \gamma\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}\right]$$

$$- \left(\frac{\alpha}{\gamma} + 1\right)\sum_{i=1}^{n}\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}$$

$$\left[1 + \gamma\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{\beta}\right]^{-1} = 0, \tag{35}$$

$$\frac{\partial \ln L}{\partial \kappa} = n\left[\frac{b^{\kappa}\ln b - a^{\kappa}\ln a}{b^{\kappa} - a^{\kappa}}\right]$$

$$- (\beta + 1)\sum_{i=1}^{n}\left[\frac{x_i^{\kappa}\ln x_i - a^{\kappa}\ln a}{x_i^{\kappa} - a^{\kappa}}\right]$$

$$+ (\beta - 1)\sum_{i=1}^{n}\left[\frac{b^{\kappa}\ln b - x_i^{\kappa}\ln x_i}{b^{\kappa} - x_i^{\kappa}}\right]$$

$$- (\alpha + \gamma)\,\beta\sum_{i=1}^{n}\left(a^{\kappa}\ln a + b^{\kappa}\ln b - 2x_i^{\kappa}\ln x_i\right)$$

$$\left[\left(\frac{b^{\kappa} - x_i^{\kappa}}{x_i^{\kappa} - a^{\kappa}}\right)^{-\beta} + \gamma\right]^{-1}. \tag{36}$$

Eqs. (33)–(36) can be solved either directly or using the R (optim and maxLik functions), SAS (PROC NLMIXED), Ox program (sub-routine Max BFGS), or using non-linear optimization approaches such as the quasi-Newton procedure.

6. Simulation study

In this section, the behavior of the MLEs of the NFKBIII parameters was assessed with respect to sample size n. The steps for simulation to assess the behavior are as follows. Generate 10000 samples of sizes n from the NFKBIII distribution using the inverse cdf method. Calculate the MLEs for 10000 samples, say $(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\kappa})$ for $i = 1, 2, ..., 10000$, using the non-linear optimization technique with constraints matching the range of parameters. Herein, (0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5), and (1, 6, 2, 0.8, 1.5, 1.75) are taken as the true parameter values $(a, b, \alpha, \beta, \gamma, \kappa)$. Calculate the means, biases, and Mean Squared Errors (MSEs) of MLEs.

For this purpose, we have chosen various arbitrary parameters and $n = 50, 100, 150$ sample sizes. All codes are written in R and the results are summarized in Table 3. The results clearly show that when the sample size n increases, the estimated MSE decreases and estimated biases drops to zero. MSE of estimated parameters increases as the shape parameter rises. This reveals that MLEs for NFKBIII distribution are reliable.

7. Applications

The potentiality and utility of using NFKBIII distribution were established by applying it to two datasets: failure times of devices [35] data and maximum annual flood discharges. The NFKBIII distribution was compared with KMBIII, NKBIII, KBIII, NIKL, KIL, Modified Burr XII (MBXII), Burr XII (BXII), Modified Burr III (MBIII), Burr III (BIII), Weibull distribution, and inverse Weibull distribution. R-package was applied to computing goodness of fit criteria such as "Cramer-von Mises (W^*), Anderson Darling (A^*), Kolmogorov-Smirnov statistics with p-values [KS(p-values], Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC)" and estimate of likelihood ratio statistics $(-\ell)$ values for times to failures of 50 components, and maximum annual flood discharges. Chen and Balakrishnan [36] described the statistics W^* and A^* in detail.

The best model is that for which the values of goodness of fit criteria are smaller. The MLEs for unknown parameters and goodness of fit criteria values for the NFKBIII, KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull and inverse Weibull models are computed.

Table 3. Means, bias, and MSEs of the New Family of Kies Burr III (NFKBIII) distribution (0.10, 4, 1.2, 0.4, 1.1, 1.2), (0.5, 5, 1.5, 0.5, 1.3, 1.5), and (1, 6, 2, 0.8, 1.5, 1.75).

Sample	Statistics	$a = 0.10$	$b = 4$	$\alpha = 1.2$	$\beta = 0.4$	$\gamma = 1.1$	$\kappa = 1.2$
$n = 50$	Means	0.1002	3.9997	1.2416	0.4132	1.2034	1.2803
	Bias	2e-04	–3e-04	0.0416	0.0132	0.1034	0.0803
	MSE	0	0	0.011	0.0015	0.027	0.0215
$n = 100$	Means	0.10	4	1.2191	0.4099	1.1843	1.2772
	Bias	0	0	0.0191	0.0099	0.0843	0.0772
	MSE	0	0	0.0046	7e-04	0.018	0.0193
$n = 150$	Means	0.1	4	1.2092	0.4083	1.1722	1.2714
	Bias	0	0	0.0092	0.0083	0.0722	0.0714
	MSE	0	0	0.0024	4e-04	0.0133	0.0165
Sample	**Statistics**	$a = 0.5$	$b = 5$	$\alpha = 1.5$	$\beta = 0.5$	$\gamma = 1.3$	$\kappa = 1.5$
$n = 50$	Means	0.5015	4.9989	1.5386	0.5069	1.4244	1.6339
	Bias	0.0015	–0.0011	0.0386	0.0069	0.1244	0.1339
	MSE	3e-04	0	0.0232	0.001	0.0456	0.0418
$n = 100$	Means	0.5003	4.9997	1.519	0.5011	1.4072	1.6386
	Bias	3e-04	–3e-04	0.019	0.0011	0.1072	0.1386
	MSE	0	0	0.0095	2e-04	0.0298	0.0402
$n = 150$	Means	0.5001	4.9999	1.5108	0.5	1.3944	1.6333
	Bias	1e-04	–1e-04	0.0108	0	0.0944	0.1333
	MSE	0	0	0.0049	0	0.0237	0.0357
Sample	**Statistics**	$a = 1.0$	$b = 6$	$\alpha = 2.0$	$\beta = 0.8$	$\gamma = 1.5$	$\kappa = 1.75$
$n = 50$	Means	1.0619	5.84	2.0538	0.8496	1.7006	1.8917
	Bias	0.0619	–0.16	0.0538	0.0496	0.2006	0.1417
	MSE	0.0119	0.0908	0.0365	0.0125	0.0835	0.0652
$n = 100$	Means	1.0177	5.9591	2.0144	0.8182	1.6366	1.8777
	Bias	0.0177	–0.0409	0.0144	0.0182	0.1366	0.1277
	MSE	0.0026	0.0218	0.0205	0.0041	0.0586	0.0542
$n = 150$	Means	1.0073	5.9861	2.0003	0.81	1.6132	1.8685
	Bias	0.0073	–0.0139	3e-04	0.01	0.1132	0.1185
	MSE	8e-04	0.0066	0.0116	0.0018	0.0417	0.0426

7.1. Times to failure

The times to failures of 50 components [35] are 0.10, 0.20, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. The Aarset dataset is recognized as bathtub shaped.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W^*, A^*, KS (p-values) are summarized in Table 4. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC, and $-\ell$ are written in Table 5.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII, MBIII, BIII, Weibull distribution, and inverse Weibull distribution because the values of all criteria are smaller

Table 4. MLEs and their standard errors (in parentheses) for times to failure of devices.

Model	α	β	γ	κ	a	b	W^*	$A*$	KS (p-value)
NFKBIII	115364.0 (379.4410)	4.956983 (0.3467780)	4889524 (16199.92)	4.722609 (0.7202355)	0.10	86	**0.0454792**	**0.414671**	**0.076 (0.9445)**
KMBIII	1.5122204 (0.9932218)	0.6942384 (0.2090516)	2.2315263 (2.7777485)	—	0.10	86	0.05448629	0.4984068	0.0777 (0.9343)
NKBIII	2.9425405 (2.26453756)	0.587427 (0.08992104)	—	0.0244289 (0.53930075)	0.10	86	0.05219844	0.4703996	0.0778 (0.9332)
KBIII	1.0520786 (0.1653031)	0.5805595 (0.0747764)	—	1	0.10	86	0.06114472	0.5322372	0.0744 (0.9531)
NKIL	0.7396103 (0.5740525)	1	—	1.0210726 (0.8272429)	0.10	86	0.07033046	0.5912904	0.2278 (0.0137)
KIL	0.7542468 (0.108866)	1	1	1	0.10	86	0.07026344	0.5906328	0.228 (0.01359)
MBXII	171.999510 (202.1714396)	5.275727 (0.9105317)	4243.399938 (5424.6188490)	—	—	—	1.298802	6.86492	0.3558 (6.342e-06)
BXII	0.2454656 (0.06939703)	1.2600795 (0.32079056)	1	—	—	—	1.09 33	5.850392	0.3336 (2.941e-05)
MBIII	455699.1 (23876.41)	3.224871 (0.08486702)	1959580 (13762.39)	—	—	—	0.3964946	2.474767	0.1614 (0.1478)
BIII	4.1810540 (0.63742201)	0.5766612 (0.05248543)	1	—	—	—	0.94 56 985	5.177504	0.2656 (0.001724)
Weibull	0.0272128 (0.39009785)	0.9476152 (0.04439031)	—	—	—	—	0.4949391	3.001556	0.1933 (0.04769)
Inverse Weibull	2.6499805 (0.39009785)	0.4634121 (0.04439031)	—	—	—	—	1.039875	5.565583	0.2856 (0.0005731)

Table 5. Goodness-of-fit statistics for times to failure of devices.

Model	AIC	CAIC	BIC	HQIC	$-\ell$
NFKBIII	**403.3062**	**404.2365**	**410.791**	**406.1348**	**197.6531**
KMBIII	408.9108	409.4563	414.5244	411.0322	201.4554
NKBIII	407.7712	408.3166	413.3848	409.8926	200.8856
KBIII	407.3655	407.6322	411.1079	408.7798	201.6828
NIKL	427.3011	427.5677	431.0435	428.7153	211.6505
KIL	425.3018	425.3887	427.173	426.0089	211.6509
MBXII	577.3329	577.8546	583.069	579.5172	285.6664
BXII	548.6714	548.9267	552.4954	550.1276	272.3357
MBIII	478.7943	479.316	484.5304	480.9786	236.3972
BIII	525.2932	525.5485	529.1172	526.7494	260.6466
Weibull	485.9593	486.2146	489.7833	487.4155	240.9796
Inverse Weibull	533.973	534.2283	537.797	535.4292	264.9865

for the NFKBIII distribution. We can identify that the NFKBIII distribution closely fits the empirical data (Figure 3).

7.2. Maximum annual flood discharges

The data of maximum annual flood discharges (1000 ft^3/sec) of the North Saskatchewan River (Edmonton) over a 47-year survey are: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300,

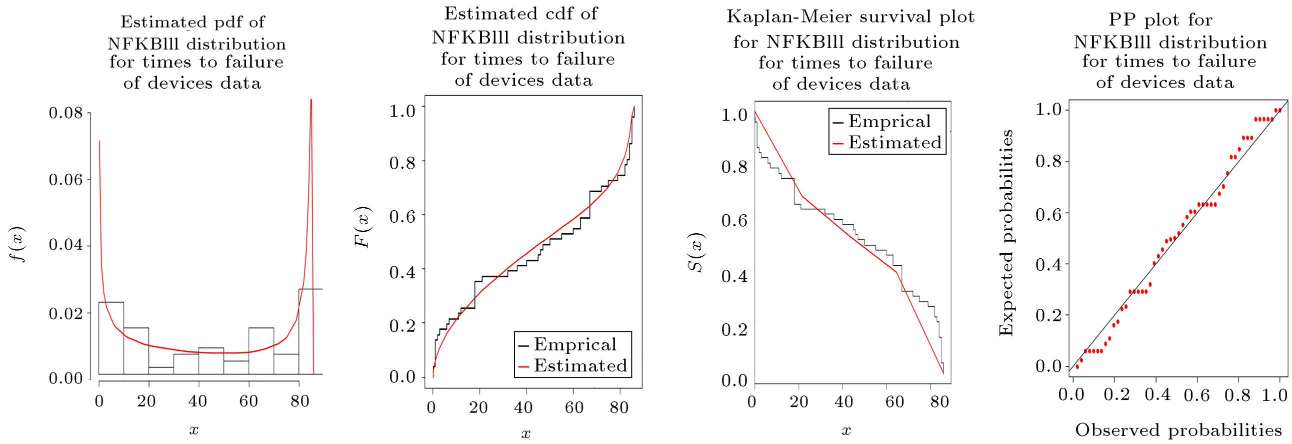

Figure 3. Fitted pdf, cdf, survival, and pp plots of the New Family of Kies Burr III (NFKBIII) distribution for device failure times.

Table 6. MLEs and their standard errors (in parentheses) for maximum annual flood discharges.

Model	α	β	γ	κ	a	b	W^*	A^*	K-S (p-value)
NFKBIII	0.30254078 (1.6306527)	1.70483835 (0.4027486)	0.35210310 (1.9110147)	0.02101155 (2.8260695)	19.885	185.560	0.01576999	0.125909	0.0571 (0.9982)
KMBIII	0.005113609 (0.002923934)	8.889992999 (NAN)	0.133291960 (0.041725229)	1	19.885	185.560	0.611651	3.719855	0.2657 (0.003024)
NKBIII	0.005549908 (0.003225763)	67.49297076 (40.584567979)	1	1.619899847 (0.042015909)	19.885	185.560	0.3781306	2.386018	0.3031 (0.0004277)
KBIII	0.1659637 (0.05820467)	3.0726999 (0.99621453)	1	1	19.885	185.560	0.1799505	1.224586	0.2379 (0.01094)
NKIL	0.7983516990 (0.1614468)	1	1	0.0000000001 (0.2410621)	19.885	185.560	0.01863201	0.1529237	0.239 (0.01046)
KIL	0.4679514 (0.06899535)	1	1	1	19.885	185.560	0.03613052	0.2757191	0.2982 (0.0005609)
MBXII	0.0113894 (0.01673093)	124.9109988 (760.63939777)	5.4099658 (37.86181308)	—	—	—	0.05350551	0.3691381	0.5461 (7.386e-13)
BXII	0.07560631 (0.3409857)	3.48218254 (15.7016695)	1	—	—	—	0.05340923	0.368504	0.5449 (8.376e-13)
MBIII	6107.715659 (12522.06)	2.447060 (0.5375678)	1.738394 (1616.620)	—	—	—	0.01920378	0.1367274	0.0701 (0.9725)
BIII	6106.072865 (6901.9841287)	2.447011 (0.3285967)	1	—	—	—	0.01920334	0.1367219	0.0701 (0.9724)
Weibull	0.002567956 (0.000497175)	1.489705016 (0.054448946)	—	—	—	—	0.2385166	1.511058	0.1981 (0.04618)
Inverse Weibull	6098.970551 (6310.531373)	2.446713 (0.301231)	—	—	—	—	0.0192021	0.1367111	0.0701 (0.9725)

50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

The MLEs along with standard errors (in parentheses) and goodness of fit criteria such as W^*, A^*, KS (p-values) are summarized in Table 6. The values of goodness-of-fit criteria such as AIC, CAIC, BIC, HQIC, and $-\ell$ are written in Table 7.

The NFKBIII distribution is best fitted than KMBIII, NKBIII, KBIII, NIKL, KIL, MBXII, BXII,

Table 7. Goodness-of-fit statistics for maximum annual flood discharges.

Model	AIC	CAIC	BIC	HQIC	$-\ell$
NFKBIII	**407.7306**	**408.7062**	**415.0452**	**410.4707**	**199.8653**
KMBIII	519.4343	520.0057	524.9202	521.4894	256.7171
NKBIII	434.2349	434.8064	439.7209	436.29	214.1175
KBIII	423.3242	423.6033	426.9815	424.6943	209.6621
NKIL	423.6303	423.9093	427.2875	425.0003	209.8151
KIL	437.3848	437.3848	437.4757	439.2134	217.6924
MBXII	595.1127	595.6582	600.7263	597.2341	294.5564
BXII	592.772	593.0386	596.5144	594.1862	294.386
MBIII	436.2281	436.7736	441.8417	438.3495	215.1141
BIII	434.2277	434.4944	437.9701	435.642	215.1139
Weibull	458.1291	458.3958	461.8715	459.5434	227.0646
Inverse Weibull	434.2272	434.4938	437.9696	435.6414	215.1136

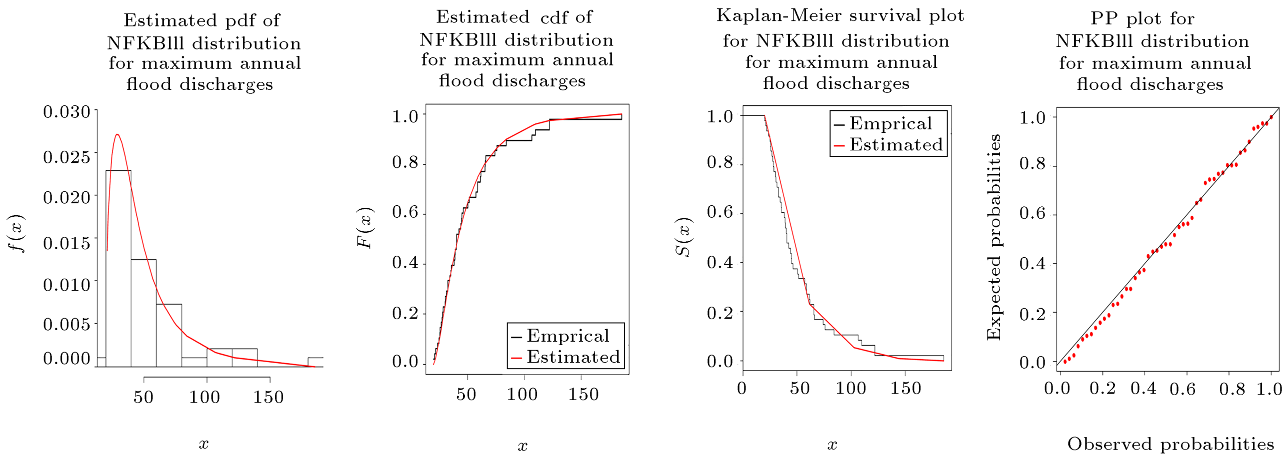

Figure 4. Fitted pdf, cdf, survival, and pp plots of the New Family of Kies Burr III (NFKBIII) distribution for maximum annual flood discharges.

MBIII, BIII, Weibull distribution, and inverse Weibull distributions as the values of all criteria are smaller for the NFKBIII distribution. We can identify that the NFKBIII distribution closely fits empirical data (Figure 4).

8. Conclusion remarks

This study derived the New Family of Kies Burr III (NFKBIII) distribution from the T-X family technique, transformation, and compounding mixture of distributions. The NFKBIII density had arc, J, reverse-J, U, bimodal, left-skewed, right-skewed, and symmetrical shapes. The hazard rate function for the NFKBIII distribution was characterized by increasing, decreasing, increasing-decreasing-increasing, and bathtub shapes. Different statistical properties including quantile function, sub-models, ordinary moments, moments of order statistics, incomplete moments, mean deviations, inequality curves, moments for residual life functions, and reliability measures were derived. Two characterizations of the NFKBIII distribution were studied. The Maximum Likelihood Estimates (MLE) for unknown parameters of NFKBIII distribution were computed. A simulation study was carried out to evaluate the behavior of the maximum likelihood estimators. The potentiality and utility of the NFKBIII distribution were demonstrated via its applications to times to failures of 50 devices and maximum annual flood discharges. The adequacy of the NFKBIII distribution was tested by different goodness-of-fit criteria. The goodness-of-fit statistics showed that the NFKBIII distribution was the best fit model. It was shown that the NFKBIII distribution was empirically the best for lifetime applications.

References

1. Burr, I.W. "Cumulative frequency distributions", *Ann. Math. Stat.*, **13**, pp. 215–232 (1942).
2. Mielke, P.W. "Another family of distributions for describing and analyzing precipitation data", *J. Appl. Meterol.*, **12**, pp. 275–280 (1973).
3. Kleiber, C. and Kotz, S., *Statistical Size Distribution in Economics and Actuarial Sciences*, New York, John Wiley & Sons (2003).
4. Gove, J.H., Ducey, M.J., Leak, W.B., and Zhang, L. "Rotated sigmoid structures in managed uneven-aged northern hardwork stands: a look at the Burr Type III distribution", *Foresty*, **81**(2), pp. 161–176 (2008).
5. Shao, Q., Chen, Y.D., and Zhang, L. "An extension of three-parameter Burr III distribution for low-flow frequency analysis", *Comput. Stat. Data Anal.*, **52**, pp. 1304–1314 (2008).
6. Benjamin, S.M., Humberto, V.H., and Arnold, B.C. "Use of the dagum distribution for modeling tropospheric ozone levels", *J. Env. Stat.*, **5**(6) pp. 1–11 (2013).
7. Ali, A., Husnain, S.A., and Ahmad, M. "Modified Burr III distribution, properties and applications", *Pak. J. Statist.*, **31**(6), pp. 697–708 (2015).
8. Gomes, A.E., da-Silva, C.Q., and Cordeiro, G.M. "Two extended Burr models: Theory and practice", *Communications in Statistics-Theory and Methods*, **44**(8), pp. 1706–1734 (2015).
9. Sinner, C., Dominicy, Y., Ley, C., Trufin, J., and Weber, P. "An interpolating family of size distributions", arXiv preprint arXiv: 2016; 1606.04430 (2016).
10. Moisheer, A.S.A. "A mixture of two Burr Type III distributions: Identifiability and estimation under type II censoring", *Mathematical Problems in Engineering*, **2016**, Article ID 7035279 (2016). http://dx.doi.org/10.1155/2016/7035279
11. Kehinde, O., Osebi, A., and Ganiyu, D. "A new class of generalized Burr III distribution for lifetime data", *International Journal of Statistical Distributions and Applications*, **4**(1), pp. 6–21 (2018).
12. Cordeiro, G.M., Gomes, A.E., da-Silva, C.Q., and Ortega E.M.M. "A useful extension of the Burr III distribution", *Journal of Statistical Distributions and Applications*, **4**(1), p. 24 (2017).
13. Jamal, F., Nasir, M.A., Tahir, M.H., and Montazeri, N.H. "The odd Burr-III family of distributions", *Journal of Statistics Applications and Probability*, **6**(1), pp. 105–122 (2017).
14. Nasir, A., Bakouch, H.S., and Jamal, F. "Kumaraswamy odd Burr G family of distributions with applications to reliability data", *Studia Scientiarum Mathematicarum Hungarica,* **55**(1), pp. 94–114 (2018).
15. Kehinde, O., Osebi, A., and Ganiyu, D. "A new class of generalized Burr III distribution for lifetime data", *International Journal of Statistical Distributions and Applications*, **4**(1), p. 6 (2018).
16. Marshall, A.W. and Olkin, I. "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families", *Biometrika*, **84**(3), pp. 641–652 (1997).
17. Cordeiro, G.M. and Castro, M. "A new family of generalized distributions", *Journal of Statistical Computation and Simulation*, **81**(7), pp. 883–898 (2011).
18. Alizadeh, M., Cordeiro, G.M., Nascimento, A.D., Lima, M.D.C.S., and Ortega, E.M. "Odd-Burr generalized family of distributions with some applications", *Journal of Statistical Computation and Simulation*, **87**(2), pp. 367–389 (2017).
19. Cordeiro, G.M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E.M.M., and Altun, E. "The generalized odd log-logistic family of distributions: properties, regression models and applications", *Journal of Statistical Computation and Simulation*, **87**(5), pp. 908–932 (2017).
20. Haghbin, H., Ozel, G., Alizadeh, M., and Hamedani, G.G. "A new generalized odd log-logistic family of distributions", *Communications in Statistics-Theory and Methods*, **46**(20), pp. 9897–9920 (2017).
21. Korkmaz, M.Ç. and Genç, A.İ. "Two-sided generalized exponential distribution", *Communications in Statistics-Theory and Methods*, **44**(23), pp. 5049–5070 (2015).
22. Korkmaz, M.Ç. and Genç, A.I. "A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution", *Communications in Statistics-Simulation and Computation*, **46**(2), pp. 1441–1460 (2017).
23. Cordeiro, G.M., Yousof, H.M., Ramires, T.G., and Ortega, E.M.M. "The Burr XII system of densities: properties, regression model and applications", *Journal of Statistical Computation and Simulation*, **88**(3), pp. 432–456 (2018).
24. Alizadeh, M., Korkmaz, M.Ç., Almamy, J.A., and Ahmed, A.A.E. "Another odd log-logistic logarithmic class of continuous distributions", *Journal of Statisticians: Statistics and Actuarial Sciences*, **11**(2), pp. 55–72 (2018).
25. Yousof, H.M., Altun, E., Ramires, T.G., Alizadeh, M., and Rasekhi, M. "A new family of distributions with properties, regression models and applications", *Journal of Statistics and Management Systems*, **21**(1), pp. 163–188 (2018).

26. Korkmaz, M.Ç., Cordeiro, G.M., Yousof, H.M., Pescim, R.R., Afify, A.Z., and Nadarajah, S. "The weibull marshall-olkin family: Regression model and application to censored data", *Communications in Statistics-Theory and Methods*, **48**(16), pp. 4171–4194 (2019).

27. Gurvich, M.R., Dibenedetto, A.T., and Ranade, S.V. "A new statistical distribution for characterizing the random strength of brittle materials", *Journal of Materials Science*, **32**(10), pp. 2559–2564 (1997).

28. Alzaatreh, A., Mansoor, M., Tahir, M.H., Zubair, M., and Ali, S. "The gamma half-cauchy distribution: properties and applications", *Hacettepe Journal of Mathematics and Statistics,* **45**(4), pp. 1143–1159 (2016).

29. Bourguignon, M., Silva, R.B., and Cordeiro, G.M. "The weibull-G family of probability distributions", *Journal of Data Science*, **12**, pp. 53–68 (2014).

30. Kies, J.A. "The strength of glass performance", Naval Research Lab Report No. 5093, Washington, D.C. (1958).

31. Kumar, C.S. and Dharmaja, S.H.S. "On some properties of Kies distribution", *Metron*, **72**(1), pp. 97–122 (2014).

32. Kotz, S., Lai, C.D., and Xie, M. "On the effect of redundancy for systems with dependent components", *IIE Trans*, **35**, pp. 1103–1110 (2003).

33. Bhattacharyya, G.K. and Johnson, R.A. "Estimation of reliability in a multicomponent stress-strength model", *Journal of the American Statistical Association*, **69**(348), pp. 966–970 (1974).

34. Glänzel, W.A. "Characterization theorem based on truncated moments and its application to some distribution families", *Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986)*, **B**, Reidel, Dordrecht, pp. 75–84 (1987).

35. Aarset, M.V. "How to identify a bathtub hazard rate", *IEEE Transactions on Reliability*, **36**(1), pp. 106–108 (1987).

36. Chen, G. and Balakrishnan N. "A general purpose approximate goodness-of-fit test", *J. Qual. Technol.*, **27**, pp. 154–161 (1995).

Biographies

Fiaz Ahmad Bhatti is currently working as an Assistant Professor at Govt. Postgraduate College, Satellite Town, Gujranwala, Pakistan. He completed his MSc from Quad-i-Azam University, Islamabad, Pakistan. He completed his MPhil from Allama Iqbal Open University, Islamabad; Pakistan. He received the PhD degree in Statistics from National College of Business Administration and Economics, Lahore, Pakistan. He has published 12 research articles. His research interests include distribution theory, reliability analysis, and Multivariate Analysis.

Munir Ahmad is currently working as the Rector of National College of Business Administration and Economics, Lahore, Pakistan. He is the Chief Editor of Pakistan Journal of Statistics. He received the PhD degree in Statistics from Iowa State University, USA. He has published over 150 research articles. His research interests include distribution theory, reliability analysis, rank set sampling, quality control, and statistical inference.