Finite Element Model and Size Dependent Stability Analysis of Boron Nitride and Silicon Carbide Nanowires/Nanotubes

Hayri Metin Numanoğlu¹, Kadir Mercan², Ömer Civalek³*

¹,²,³ Akdeniz University, Faculty of Engineering, Civil Engineering Dept., Division of Mechanics 07058 Antalya, TURKEY

*E-mail: metin_numanoglu@hotmail.com¹, mercankadir@akdeniz.edu.tr², civalek@yahoo.com³

*Phone: +902423106319¹, +902423104387², +902423106319³*

Abstract
In present paper, the stability analysis of boron nitride and silicon carbide nanotubes/nanowires is investigated using different size effective theories, finite element method, and computer software. Size effective theories used in paper are modified couple stress theory (MCST), modified strain gradient theory (MSGT), nonlocal elasticity theory (NET), surface elasticity theory (SET), nonlocal surface elasticity theory (NSET). As computer software, ANSYS and COMSOL multiphysics are used. Comparative results between theories and software and literature are given in result section. Comparative results are in good harmony. As results, it is clearly seen that nonlocal elasticity theory gives lowest results for every modes and structures while modified strain gradient theory gives the highest.

Keywords: Boron nitride, silicon carbide, nanotube, nanowire, buckling.
1. Introduction

Nanotubes and nanowires are being used in a very wide range of scientific area since their discovery. The using areas of nanotubes and nanowires can be divided to two main group as ‘current’ and ‘potential’ using areas. With its ultimate mechanical and adjustable geometric parameters, second usage group ‘potential using area’ can be described almost as ‘limitless’. On the other hand, the current usage areas of nanotubes and nanowires are in bulk state which means a mass of unorganized nanostructures [1-3]. Nanostructures in bulk form are widely used as composite fibers to advance the mechanical, electrical and thermal properties of polymers [4]. For example, one of the biggest company in sport equipment producing area which pay attention on investment for futuristic research and developments, has produced carbon nanotube (CNT) reinforced bicycle components. With the reinforcement, bicycle components reach to very high strain-stress resistance in addition to massive loss in weight [5]. Another current applications of nanostructures is to absorb gases by their active surface area [6]. Absorbing gases has vital importance for environmental monitoring and the future of planet. Elseways, with the improve in nanoscale technology, nanostructures promise limitless highly beneficial usage areas. Also new nanostructures are being studied such as three-dimensional nanoblocks. These 3D nanoblocks composed from nanostructures and have limited size up to 1mm in all dimensions. A new method has been researched and published by Lalwani et al. which uses single and multi-walled carbon nanotubes to eventually produce nanoblocks [7]. These developments can be described as promising for new supercapasitors, batteries with super energy storage capasity, transistors used for field emission, catalysis with high performance [8].

In literature, the history of nanotubes is widely based on the research published by Iijima about nanotubes in 1991 [9]. However, literature comprise older researches about
nanotubes. These researches have also pointed out nanotubes without identifying the structures as nanotube. The first research published was published in 1946 by Watson and Kaufmann [10]. Watson and Kaufmann have investigated the synthesis of carbon nanotube structure under the name of ‘tubular carbon’. The ‘tubular carbon’ was obtained with around 100 nm diameter by examining cuprene over fine copper oxide catalyst up to 300 °C. After six years of the research published by Watson and Kaufmann, Radushkevich and Lukyanovich published the primary images of carbon nanotubes with diameters range from 30 nm to 50 nm by using transmission electron microscopy in 1952 [11]. Moreover, the properties, structure, and growth methods of nanostructures by using arc discharge in 1960 by R. Bacon [12]. Another research has been published by Oberlin et al. in 1976 [13]. Oberlin et al. have observed carbon fibres by pyrolysing a mixture of hydrogen and benzene at about 1100 °C. The specimens obtained from this chemical vapour-growth method were carbon nanotubes with 2-50 nm diameter. These nanotubes have been named as ‘hollow tubes’. Hollow tubes obtained were actually multi-walled carbon nanotubes (MWCNTs). After that, in 1979, John Abrahamson presented a research at the 14th Biennial Conference of Carbon at Pennsylvania State University. The research was describing the carbon nanotubes obtained from arc discharge on carbon anodes. Later, in 1982, the first chirality model of carbon nanotubes have been suggested in two combination [14]. Kolesnik et al. have presented that ‘carbon multi-layer tubular crystals’ in other word multi-walled carbon nanotubes can be obtained by simply rolling graphene layers into cylinder. The circular rolling arrangement results in two different structures. These two structures were armchair and chiral nanostructures. On the other hand, first samples of nanotubes in history have been discovered in Damascus steel which has been made around 400 years ago. These samples are identified as first carbon nanotube samples found in history [15].
Recently, with the rise in the popularity of renewable energy for transportation and electronic, researches about carbon nanotube usage to improve the durability, lifespan, and capacity of batteries has been published [16-27]. Also, in last decade, researches has been published about the usage of CNT in one of the most effective using area, gas sensors [28-44]. The discovery of CNTs may be a revolutionary point for many application areas such as processor technology, biotechnology, gas/chemical sensors, aerospace technology, etc. [45-47]. Carbon nanotubes have attracted extreme attention due to its superior mechanical properties comparing to traditional materials. However, within the foundation of CNTs, scientist commenced to produce nano-sized materials with superior properties than CNTs. Later, novel nanotubes/nanowires have been produced based on different atomic structure than CNTs. Some of these novel nanostructures are boron nitride nanotube (BNNT) and silicon carbide nanotube (SiCNT). To illustrate, in case of mechanical properties, CNT has Young’s modulus around 1 TPa while BNNT has 1.8 TPa and 0.62 TPa for SiCNT. The poisson ratio used for analysis in current research is 0.37 for SiCNT and 0.25 for BNNT [48-50]. Moreover, CNTs can resist to thermal environment up to 600 °C while SiCNT can resist up to 1000°C in air without any damage [51, 52]. BNNT and SiCNT have not been investigated as CNT had in last decade. BNNT has very high potential to be used in drug delivery into blood flow using binding drugs to BNNT [53]. The drugs can then be delivered into the cells for curing cancer cells by killing cancer cells without damaging healthy cells. Due to the BNNTs are biocompatible and non-toxic, they can be used as nano sized drug delivery cargo vehicles of these anticancer drugs to deliver them directly to the cancer cells [54]. Recently, Ferreira et al. [55] have investigated the performance of BNNT into biomedical application area to deliver protein drugs and kill cancer cells by magnetohyperthermia therapy in 2018. The results obtained by Ferreira et al. demonstrated that BNNT nano sized structures carried magnetite nanoparticles and magnetic measurements
illustrated that well coercivity and magnetization observed after incorporation to the BNNT. Also, the boron nitride structure has been investigated in other forms than nanotube. Nanoribbons and nanowires structures have been investigated to be used in gas sensors [56]. Although these nanostructures seem analogous, obtaining nanowires is much more laborious than obtaining nanotubes in technical perspective. Nanotubes are both used in single-walled and multi-walled forms according to its application area and characteristic needed [57]. Furthermore, due to their limited resistance to thermal environment nano sized technology demanded novel nano structure with superior thermal resistance. Carbon nanotubes and carbon nanowires are nanostructures based on graphene. The thermal resistance of graphene is up to 600ºC in air. To address this issue a new base nanostructure have been obtained and developed. Also, by overcoming the thermal resistance the limited usage area of graphene based nanostructures has been expanded. The novel nanostructure has been based on Si atoms and named as ‘silicene’. Silicene, has superior thermal resistance which can stay stable until 1200ºC [58]. Silicene is a layer of hexagonally arranged silicon atoms [59]. On the contrary to superior thermal resistance silicene has lower Young’s modulus which conclude silicene to be to be mechanically weaker than graphene. To illustrate Si-Si bond length in silicene is 2.29 Å where the C-C bond length is 1.42 Å in graphene and 1.46 Å for boron nitride sheet (base material of BNNT), so that silicene performs higher chemical reactivity than graphene [60]. Longer bond length end up with lower mechanical properties which make silicene weaker than boron nitride sheet and graphene. Later, silicene and graphene have been composed to obtain a nanostructured material with superior mechanical properties. The novel composition of silicon and carbon atoms formed ‘silicon carbide sheet’. NASA Glenn Research Center has cooperated with Rensselaer Polytechnic Institute to produce silicon carbide sheets from carbon and silicon atoms. Many methods to obtain silicon carbide sheet have been developed by this cooperation. Finally, silicon carbide sheet has been produced by this cooperation.
Thermal resistance of silicon carbide sheet made the nanostructure capable to stay stable up to 1000 °C with superior mechanical properties than silicene [58]. Silicon carbide nanowires and nanotubes are widely used in gas sensors [61]. These sensors have been used for the detection of CO and HCN gases in the environment. CO and HCN gases can be absorbed on SiCNWs at Si lattice sites. With the absorption, significant waves in binding energy and charge transfer can easily be observed. The wave in electrical conductivity of SiCNWs is caused from the chemisorption of gas molecules on the surface of nanowire metal oxides. Main electro-transducers structures are field effect transistors (FET), resistive gas micro-sensors, and resistive gas sensors [62].

2. Continuum models of nanostructures

Because of astronomical high cost in micro and nano sized experiments mathematical and continuum models of these structures have always been a cost-effective choice for researchers and developers [63-65]. In literature many mathematical and continuum mechanic models have been used to model nano and micro sized structures. Nanowires have been mostly modeled to analysis using classic and size effective Euler-Bernoulli and Timoshenko beam theories [66-70]. Also, shell and plate theories have been used widely to make analysis possible without using any high-tech laboratory or real nanotubes [71-73]. Furthermore, these theories have been used for model nano and micro composite structures without need to any lab or real composite material also [74-78]. Various theories have been developed which are showing the importance of small scale effect such as strain gradient theory [79, 80], couple stress elasticity theory [80-83], modified couple stress theory [84-87], nonlocal elasticity theory [88, 89], and surface elasticity theory [90-94].
In last decade, many scientists have published researches on the topic of the stability and analyzes of micro-nanowires and micro-nanotubes. Ansari et al. [95] have investigated the buckling behavior of single-walled silicon carbide nanotubes using ANSYS commercial FE code in 2012. After that, in 2013, Arani et al. [96] have investigated the surface stress effects on dynamic stability of double-walled boron nitride nanotubes conveying viscose fluid using nonlocal shell theory. Later, in 2014, Saljooghi et al. [97] have investigated the vibration and buckling behavior of functionally graded beams [97]. Saljooghi et al. [97] have used reproducing kernel particle method with very good accuracy. In 2015, Darvizeh et al. [98] demonstrated the pre- and post-buckling analysis of functionally graded beams (FGMs), a mixture of ceramic and metal, subjected to statically mechanical and thermal loads. Nonlinear free vibration of symmetric circular fiber metal laminated hybrid plates has been published by Shooshtari and Dalir in 2015 also. Shooshtari and Dalir [99] have also demonstrated the effects of several parameters on linear and nonlinear frequencies and the free vibration response on circular fiber metal laminated plates [99]. After that, Ansari and Gholami [100] have considered the size effect using Eringen’s nonlocal elasticity theory on nonlinear first-order shear deformable beam model for post-buckling analysis of magneto-electro-thermo-elastic nanobeams [100]. Rouzegar and Sharifpoor [101] have investigated the finite element formulations for free vibration analysis of isotropic and orthotropic plates using two-variable refined plate theory that predicts parabolic variation of transverse shear stresses along the thickness of the plate, satisfies the zero traction condition on the plate surfaces, and does not need the shear correction factor [101]. Rouzegar and Sharifpoor have demonstrated the effects of orthotropy ratio, side-to-thickness ratio, and types of boundary conditions on the natural frequencies of plates. Later, in 2017, Rafeaimejad et al. have presented an analytical solution for bending, buckling, and free vibration of FG nanobeam [102]. Nanobeams have been modeled resting on double parameter Winkler-Pasternak elastic foundation and results have
been obtained using different nonlocal higher order shear deformation beam theories. Rafaeeinejad et al. showed clearly the effect of foundation, gradient index, aspect ratio, nonlocal parameter on stability and vibration analyzes. More recently, Jabbarian and Ahmadian presented the free vibration analysis of functionally graded stiffened micro-cylinder [103]. Jabbarian and Ahmadian have took the size effect into consideration using the modified couple stress theory (MCST). Results demonstrated that the stiffeners lead to an increase in natural frequencies due to an increase in stiffness of the micro-cylinder. Also, in 2018, Sahoo et al. have investigated the natural frequency and transient responses of carbon/epoxy layered composite plate structure using two higher-order mid-plane kinematic models [104].

In this paper, nanotubes and nanowires were modeled using classical Euler-Bernoulli beam theory (CT), nonlocal elasticity theory (NET), surface elasticity theory (SET), modified couple stress theory (MCST), modified strain gradient theory (MSGT), finite element model and COMSOL Multiphysics analysis software [105], and ANSYS software [106] to investigate critical and other buckling loads of simply supported boron nitride and silicon carbide nanotubes and nanowires. Comparative results are given in figures and tables.

The atomic structure of boron nitride and silicon carbide sheets is demonstrated in Fig. (1). The top structure is boron nitride sheet structure which compose from hexagonally arranged boron atoms (B) with nitrogen N atoms. The bottom structure consists of silicon (Si) and carbon (C) atoms. Also, the bond lengths of structures are demonstrated on the right side. Si-C bond length in silicene is 2.29 Å where the B-N bond length is 1.42 Å in graphene, so that silicene performs higher chemical reactivity than graphene which makes silicene a weaker material than graphene.
As it can be clearly seen from Fig. (2), to obtain nanotubes, simply rolling the nanosheet structured material is needed. Nanostructures can be gathered to three main groups up to the angle they are rolled-up. These three main group are armchair, zigzag and chiral [107].

To model nanotubes, classical Euler-Bernoulli beam theory is used with size effective theories also by using hollow cylindrical beam model. In Fig. (3) the transition from real BNNT (top) and SiCNT (middle) to its continuum mechanic model (bottom). Geometric parameters are also represented in Fig. (3). A comparative image of BNNW and SiCNW is demonstrated in Fig. (4). Furthermore, to model nanowires, the cylindrical beam model is used. Similarly, to demonstrate the transition from nanowire to cylindrical beam model Fig. (5) is depicted with geometric parameters. As it is seen, L and D symbolize the length and the diameter of nanowire respectively.

In current paper, nano structures are analyzed for both with and without the elastic foundation effect. Images of continuum models on double parameter elastic foundation are denoted in Fig. (6) for nanotubes and nanowires. Winkler foundation is demonstrated as springs as its behavior while Pasternak foundation is demonstrated as vertical rods.

3. Formulations

3.1. Modified couple stress and modified strain gradient theory

The total deformation (strain) energy $U$ based on MSGT can be written as follows [80]

$$U = \frac{1}{2} \int_{0}^{L} \left( (\sigma_{g} \varepsilon_{g} + p \gamma_{r} + \tau_{g}^{1} \eta_{1}^{1} + m_{0} \chi_{0}) \right) dA dx$$

(1)
\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \]  
(2)

\[ \gamma_i = \varepsilon_{nm,i} \]  
(3)

\[ \eta_{ijk}^{(1)} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) \]  
(4)

\[ -\frac{1}{15}[\delta_{ij}(\varepsilon_{nm,k} + 2\varepsilon_{mk,n}) + \delta_{jk}(\varepsilon_{mn,i} + 2\varepsilon_{mi,n}) + \delta_{ki}(\varepsilon_{mn,j} + 2\varepsilon_{mj,n})] \]

\[ \chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \]  
(5)

\[ \theta_i = \frac{1}{2}e_{ijk}u_{k,j} \]  
(6)

where the rotation vector is represented as \( \theta \), the strain tensor \( \varepsilon \), the dilatation gradient vector \( \gamma \), the deviatoric stretch gradient tensor \( \eta^{(1)} \) and the symmetric rotation gradient tensor \( \chi^s \). Furthermore, \( \delta \) is Kronecker delta symbol and \( e_{ijk} \) is the permutation symbol. On the other hand, the components of the classical stress tensor \( \sigma \) (combined with the strain tensor) and the higher-order stress tensors \( p \), \( \tau^{(1)} \) and \( m^s \) (combined with the higher-order deformation gradient tensors) can be expressed as

\[ \sigma_{ij} = \lambda \varepsilon_{nm} \delta_{ij} + 2G\varepsilon_{ij} \]  
(7)

\[ p_i = 2Gl_0^2 \gamma_i \]  
(8)

\[ \tau_{ijk}^{(1)} = 2Gl_1^2 \eta_{(1)jk} \tau_{ij}^{(1)} \]  
(9)

\[ m_{ij}^s = 2Gl_2^2 \chi_{ij}^s \]  
(10)
where \( l_0, l_1, l_2 \) are dilatation gradient length scale parameters, deviatoric stretch gradients and rotation gradients, respectively. Furthermore, \( \lambda \) and \( G \) represent the Lamé constants. These Lamé constants can be expressed as

\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}
\]  

(11)

According to Euler-Bernoulli beam theory, displacement components can be stated as follow

\[
u_i(x, z) = -z \frac{d\omega(x)}{dx}
\]

\[
u_2(x, z) = 0
\]

\[
u_3(x, z) = w(x)
\]

where \( \nu_i, i = 1, 2, 3 \) are the \( x, y \) and \( z \) components of the displacement vector respectively. The transverse displacement is expressed as \( w(x) \). By implementing Eq. (12) into Eq. (2), the non-zero strain component can be found as follows

\[
\varepsilon_{11} = -\varepsilon \frac{d^2\nu}{dx^2}
\]  

(13)

By using Eqs. (12) and (13) and implementing into Eqs. (3-5), the non-zero higher-order gradients can be obtained as follow
\gamma_1 = -z \frac{d^3 w}{dx^3}, \quad \gamma_3 = -\frac{d^2 w}{dx^2} \hfill (14)

\eta_{11}^{(i)} = -\frac{2}{5} z \frac{d^3 w}{dx^3}, \quad \eta_{13}^{(i)} = \eta_{31}^{(i)} = \eta_{33}^{(i)} = -\frac{4}{15} \frac{d^2 w}{dx^2}

\eta_{22}^{(i)} = \eta_{33}^{(i)} = \eta_{21}^{(i)} = \eta_{22}^{(i)} = \eta_{31}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = -\frac{1}{5} \frac{d^2 w}{dx^2} \hfill (15)

\eta_{22}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = \eta_{33}^{(i)} = \frac{1}{5} \frac{d^2 w}{dx^2}

\chi_{12} = \chi_{21} = \frac{1}{2} \frac{d^2 w}{dx^2} \hfill (16)

To obtain the non-zero components of classical stress tensor, Eq. (13) need to be substituted in Eq. (7),

\sigma_{11} = -\eta \frac{d^2 w}{dx^2}, \quad \sigma_{22} = \sigma_{33} = -\frac{Ev}{(1+v)(1-2v)} z \frac{d^2 w}{dx^2} \hfill (17)

Where

\eta = \frac{(1-v)}{(1+v)(1-2v)} \hfill (18)
By using above equations and implementing into Eqs. (8-10), the non-zero components of higher-order stress tensors can be found as

\[ p_1 = -2Gl_0^2z \frac{d^3w}{dx^3}, \quad p_3 = -2Gl_0^2 \frac{d^3w}{dx^2} \]  

(19)

\[ \tau_{111}^{(i)} = - \frac{4}{5} Gl_i^2 z \frac{d^3w}{dx^3}, \quad \tau_{113}^{(i)} = \tau_{131}^{(i)} = \tau_{311}^{(i)} = - \frac{8}{15} Gl_i^2 \frac{d^3w}{dx^2}, \]  

(20)

\[ \tau_{122}^{(i)} = \tau_{133}^{(i)} = \tau_{212}^{(i)} = \tau_{222}^{(i)} = \tau_{233}^{(i)} = \tau_{333}^{(i)} = \frac{2}{5} Gl_i^2 z \frac{d^3w}{dx^3}, \]  

\[ m_{12}^{s} = m_{21}^{s} = -Gl_i^2 \frac{d^2w}{dx^2} \]  

(21)

Governing equations can be obtained with the aid of minimum total potential energy principle. According to minimum total potential energy principle

\[ \delta \Pi = \delta U - \delta W = 0 \]  

(22)

Where \( \Pi \) is the total potential energy. Furthermore the first variations of strain energy is represented with \( \delta U \) and the work done by external forces is represented with \( \delta W \). The first variation of the strain energy \( \delta U \) can be expressed as
\[ \delta U = \int_0^L \left( \sigma_{ij} \delta e_{ij} + p \delta \gamma_i + \tau_{ijk} \delta \eta_{ijk} + m'_{ij} \delta \eta_{ijk} + m'_{ijk} \delta \chi_{ijk} \right) \, dA \, dx \]

\[ = \int_0^L \left( (EI + GA(2l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) \frac{d^4 w}{dx^4} - 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^6 w}{dx^6} - 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^4 w}{dx^4} \right) \delta w \, dx \]

\[ + \int_0^L \left( -(EI + GA(2l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) \frac{d^3 w}{dx^3} + 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^5 w}{dx^5} \delta \left( \frac{dw}{dx} \right) + 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^3 w}{dx^3} \delta \left( \frac{d^2 w}{dx^2} \right) \right) \, dx \]

(23)

The first variation of the work done by external forces can be stated as

\[ \delta W = \int_0^L -\left( k_w \delta w + (P - k_p) \frac{d^2 w}{dx^2} \delta w \right) \, dx + \left[ V + (P - k_p) \frac{d^2 w}{dx^2} \right] \delta w - M \delta \left( \frac{d^2 w}{dx^2} \right) - M^{nc} \delta \left( \frac{d^2 w}{dx^2} \right) \right] \, dx \]

(24)

Where the axial compressive force is represented as \( P \). Likewise the shear force, classical and non-classical bending moments are represented with \( V, M \) and \( M^{nc} \), respectively. Winkler modulus and Pasternak modulus of the double parameter elastic foundation are stated as \( k_w \) and \( k_p \), respectively.

By substituting Eqs. (24) and (26) into Eq. (23) (by setting the \( \delta w = 0 \)), the equilibrium equations for a Euler-Bernoulli beam can be obtained as follows
\[ \delta w: (EI + GA(2l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) \frac{d^4 w}{dx^4} - 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^3 w}{dx^3} + (P - k_p) \frac{d^2 w}{dx^2} = 0 \quad (25) \]

To solve Eq. (25), boundary conditions need to be implemented in. Simply supported boundary conditions placed at \( x = 0 \) and \( x = L \) can be expressed as follow

\begin{align*}
(EI + GA(l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) \frac{d^4 w}{dx^4} - 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^3 w}{dx^3} + V + (p - k_p) \frac{dw}{dx} & = 0 \\
\text{or} \quad \delta w & = 0 \\
-(EI + GA(l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) \frac{d^4 w}{dx^4} + 2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^3 w}{dx^3} = M \quad \text{or} \quad \delta \left( \frac{d^2 w}{dx^2} \right) = 0 \\
-2GI(l_0^2 + \frac{2}{5} l_1^2) \frac{d^3 w}{dx^3} = M'' \quad \text{or} \quad \delta \left( \frac{d^2 w}{dx^2} \right) = 0 \\
\end{align*} \quad (26–28)

By using Eqs. (26–28), the boundary conditions (classical and possible non-classical) can be stated as follow

\[ w = 0, M = 0, w'' = 0 \quad (29) \]

Where

\[ w'' = \frac{d^2 w}{dx^2} \quad (30) \]
In case of simply supported boundary conditions:

\[ B_w^{(4)} - D_w^{(6)} + Nw'' = 0 \]  \hspace{1cm} (31)

The solution of Eq. (31) can be expressed as

\[ w(x) = C_1 + C_2 x + C_3 \sin Kx + C_4 \cos Kx + C_5 \sinh Mx + C_6 \cosh Mx \]  \hspace{1cm} (32)

Where

\[ K = \left( \frac{-B + \sqrt{B^2 + 4DN}}{2D} \right)^{1/2}, \quad M = \left( \frac{B + \sqrt{B^2 + 4DN}}{2D} \right)^{1/2} \]  \hspace{1cm} (33)

\( C_i, (i = 1, 2, \ldots, 6) \) are integral constants. These constants can be calculated by using boundary conditions. By substituting simply supported beam’s boundary conditions stated in Eq. (29, 30) into Eq. (32) we obtain

\[ C_i = 0 \] excluding \( C_3 \sin KL = 0 \)  \hspace{1cm} (34)

The non-trivial solution of Eq. (34) can be stated as
\[
\sin KL = 0 \quad (35a)
\]

\[
K = \frac{n\pi}{L} \quad (n = 1, 2, \ldots), \quad N_{cr} = \frac{\pi^2}{L^2} \left( B + \frac{\pi^2 D}{L^2} \right) \quad (35b)
\]

To solve Eq. (36), Navier’s solution procedure can be applied as follows

\[
w(x) = \sum_{n=1}^{\infty} W_n \sin \left( \frac{n\pi x}{L} \right) \quad (36)
\]

By using Navier’s solution, the critical buckling loads for simply supported nanowire on double parameter elastic foundation can be stated as

For MSGT

\[
P_{(n)} = \frac{n^2 \pi^2}{L^2} \left[ (EI + GA(2l_0^2 + \frac{8}{15} l_1^2 + l_2^2)) + \frac{n^2 \pi^2}{L^2} (2GI(l_0^2 + \frac{2}{5} l_1^2)) \right] + \frac{L^2}{n^2 \pi^2} + k_p \quad (37a)
\]

For MCST

\[
P_{(n)} = \frac{n^2 \pi^2}{L^2} \left[ (EI + GA(l_2^2)) \right] + \frac{L^2}{n^2 \pi^2} + k_p \quad (37b)
\]

3.2. Nonlocal elasticity theory (NET)
According to A.C. Eringen [88] the constitutive equation of nonlocal elasticity theory (NET) can be stated as follows

\[ [1 - (e_o a)^2 \nabla^2] \sigma_{ij} = C_{ijkl} \]  

(38)

Where \( \sigma_{ij} \) is the nonlocal tensile tensor, \( C_{ijkl}(x') \) is the local or classical tensile tensor at any \( x' \) point, \( a \) is a constant related to the characteristics of each material, and \( e_o \) is the nonlocal parameter which is chosen in a range for each material.

The displacement of a thin beam (Euler-Bernoulli) can be stated as

\[ u_i(x, z) = -z \frac{dw(x)}{dx} \]

\[ u_2(x, z) = 0 \]

(39)

\[ u_3(x, z) = w(x) \]

Where \( u_1, u_2, u_3 \) are the \( x, y, z \) components of displacement vector, and \( w \) is symbolizing transverse displacement of the beam. According to thin beam theory the relation between stress and displacement can be expressed as
\[ \varepsilon_{11} = \frac{du}{dx} = -z \frac{d^2w}{dx^2}, \quad \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 \] (40)

Where \( \varepsilon_{11} \) symbolizes the axial stress. Also, the stress-strain equation according to thin beam theory can be expressed as

\[ \sigma_{11} = -Ez \frac{d^2w}{dx^2}, \quad \sigma_{22} = \sigma_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \] (41)

According to Eq. (38), the nonlocal stress-strain equation can be expressed as

\[ \sigma_{11} - \mu \frac{d^2\varepsilon_{11}}{dx^2} = E\varepsilon_{11}, \quad \mu = (e_0 a)^2, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0 \] (42)

\[ \tau_{12} = \tau_{21} = 0, \quad \tau_{13} = \tau_{31} = 0, \quad \tau_{23} = \tau_{32} = 0 \]

To obtain governing equations, minimum total energy principle is used. According to minimum total energy principle

\[ \delta \Pi = \delta U - \delta W = 0 \] (43)
Where \( \prod \) symbolize the total potential energy, \( \delta U \) and \( \delta W \) are the first variation of stress and total energy from external loads respectively. According to thin beam theory \( \delta U \) and \( \delta W \) can be expressed as

\[
\delta U = \int_{a}^{L} \left[ \sigma_{11} \delta \varepsilon_{11} \right] \, dx = \int_{a}^{L} \left[ - \frac{d}{dx} \left( - \frac{d^2 \delta w}{dx^2} \right) \right] \, dx
\]  
(44)

\[
\delta W = \int_{a}^{L} \left[ P \frac{dw}{dx} \frac{dw}{dx} + q w(x) \right] \, dx
\]  
(45)

By implementing Eqs. (44, 45) into Eq. (43) we obtain

\[
\int_{a}^{L} \left( - M \frac{d^2 \delta w}{dx^2} \right) \, dx - \int_{a}^{L} \left( P \frac{dw}{dx} \frac{dw}{dx} + q \delta w(x) \right) \, dx = 0
\]  
(46)

Where \( P \) is the axial load.

By partially integrating Eq. (46) we obtain the buckling equation and boundary conditions as follow

\[
\delta w : \frac{d^2 M}{dx^2} = \delta w \left( P \frac{dw}{dx} \right) - q = d^2 M
\]  
(47)
\[ \frac{dM}{dx} - P \frac{dw}{dx} = 0 \quad , \quad M = 0 \]  

(48)

The nonlocal moment can be written as follows by using Eq. (42)

\[ M - \mu \frac{d^2 M}{dx^2} = -EI \frac{d^2 w}{dx^2} \]  

(49)

Using Eq. (47) in Eq. (49) we obtain the nonlocal moment as follow

\[ M = \mu \left( \frac{d}{dx} \left( P \frac{dw}{dx} \right) - q \right) - EI \frac{d^2 w}{dx^2} \]  

(50)

Using the fourth order derivative of nonlocal moment into Eq. (47) we obtain

\[ \delta w : \frac{d^2}{dx^2} \left( -EI \frac{d^2 w}{dx^2} \right) + \mu \frac{d^2}{dx^2} \left( \frac{d}{dx} \left( P \frac{dw}{dx} \right) - q \right) + \]  

\[ q - \frac{d}{dx} \left( P \frac{dw}{dx} \right) = 0 \]  

(51)

Nonlocal boundary conditions are as follow
\[
\frac{d}{dx}\left(\mu\left(\frac{d}{dx}\left(P\frac{dw}{dx}\right)-q\right)-EI\frac{d^2w}{dx^2}\right)-P\frac{dw}{dx}=0
\]

(52)

\[
\mu\left(\frac{d}{dx}\left(P\frac{dw}{dx}\right)-q\right)-EI\frac{d^2w}{dx^2}=0
\]

The relation between load and elastic foundation can be stated as

\[
p(x) = k_ww - k_p\frac{d^2w}{dx^2}
\]

(53)

Implementing Eq. (53) into Eq. (51)

\[
\left(-EI + P\mu - k_p\mu\right)\frac{d^4w}{dx^4} + \left(k_w\mu - P + k_p\right)\frac{d^2w}{dx^2} - k_w'w = 0
\]

(54)

In case of simply supported nanobeams, the fundamental boundary conditions can be stated as

\[
\delta[w]_0^L = 0, \quad \delta\left[\frac{dw}{dx}\right]_0^L = 0
\]

(55)

And natural boundary conditions
\[
\left[ (-EI + P \mu - k_p \mu) \frac{d^2 w}{dx^2} + \mu k_w w \right]_0^L \quad \text{and}
\]
\[
\left[ (-EI + P \mu - k_p \mu) \frac{d^3 w}{dx^3} + (k_w \mu - P) \frac{dw}{dx} \right]_0^L
\]

The nonlocal buckling equation is given in Eq. (54). To simplify the equation, the expressions stated below can be used

\[ A = -EI + P \mu - k_p \mu \]
\[ B = k_w \mu - P + k_p \]
\[ C = k_w \]

The simplified version of Eq. (54) yields to after implementing Eq. (57)

\[ Aw'' + Bw'' - Cw = 0 \]  

(58)

To solve Eq. (58) we can assume \( w = e^{rx} \). Then Eq. (58) can be stated as

\[ Ar^4 e^{rx} + Br^2 e^{rx} - Ce^{rx} = 0 \]  

(59)

Roots of Eq. (59) are as follow
\[ r_{1,2} = \pm i \sqrt{\frac{B - \sqrt{B^2 + 4AC}}{2A}}, \quad r_{3,4} = \pm \sqrt{\frac{B^2 + 4AC}{2A} - \frac{B}{2A}} \] (60)

\[ \psi = \sqrt{\frac{B - \sqrt{B^2 + 4AC}}{2A}} \quad \zeta = \sqrt{\frac{B^2 + 4AC}{2A} - \frac{B}{2A}} \] (61)

By implementing roots into Eq. (58) and solve we obtain

\[ w = C_1 \sin \psi x + C_2 \cos \psi x + C_3 \cosh \zeta x + C_4 \sinh \zeta x \] (62)

As stated before, \( C_1, C_2, C_3, \) and \( C_4 \) are constants which can be found by using boundary conditions.

The first derivative of Eq. (62) can be stated as

\[ w' = \psi C_1 \cos \psi x - \psi C_2 \sin \psi x + \zeta C_3 \sinh \zeta x + \zeta C_4 \cosh \zeta x \] (63)

The second derivative of Eq. (62) can be stated as
\[
M = -\psi^2 C_1 \sin \psi x - \psi^2 C_2 \cos \psi x + \zeta^2 C_3 \cosh \zeta x + \\
\zeta^2 C_4 \sinh \zeta x - \\
e_0 a^2 k_w (C_2 \cos \psi x + C_3 \cosh \zeta x + C_1 \sin \psi x + C_4 \sin \zeta x) \\
\frac{EI - Pe_0 a^2 + e_0 a^2 k_p}{EI - Pe_0 a^2 + e_0 a^2 k_p} \tag{64}
\]

Similarly, the third derivative of Eq. (62) can be stated as

\[
V = -(- k_w e_0 a^2 + P)(\psi^2 C_2 \cos \psi x - \zeta^2 C_3 \cosh \zeta x + \\
\psi^2 C_1 \sin \psi x - \zeta^2 C_4 \sinh \zeta x) - \psi C_1 \cos \psi x + \\
\zeta C_4 \cosh \zeta x + \psi C_2 \sin \psi x + \zeta C_3 \sinh \zeta x) / \\
(EI - Pe_0 a^2 + e_0 a^2 k_p) \tag{65}
\]

Boundary conditions of a simply supported beam can be stated as follows

\[
w(0) = M(0) = w(l) = M(l) = 0 \tag{66}
\]

By substituting Eq. (66) into Eqs. (64, 65) we obtain following equations

\[
w(0) = C_2 + C_3 = 0 \tag{67}
\]
\[ M(0) = -\psi^2 - \frac{e_0 a^2 k_w}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} C_2 + \]
\[ \zeta^2 - \frac{e_0 a^2 k_w}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} C_3 = 0 \] (68)

\[ w(l) = C_1 \sin \psi l + C_2 \cos \psi l + C_3 \cosh \zeta l + C_4 \sinh \zeta l = 0 \] (69)

\[ M(l) = -C_1 \left( \psi^2 \sin \psi l + \frac{e_0 a^2 k_w \sin \psi l}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} \right) - \]
\[ C_2 \left( \psi^2 \cos \psi l + \frac{e_0 a^2 k_w \cos \psi l}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} \right) + \]
\[ C_3 \left( \zeta^2 \cosh \zeta l - \frac{e_0 a^2 k_w \cosh \zeta l}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} \right) + \]
\[ C_4 \left( \zeta^2 \sinh \zeta l - \frac{e_0 a^2 k_w \sinh \zeta l}{EI - P^* e_0 a^2 + e_0 a^2 * k_p} \right) = 0 \] (70)

As it is stated before \( C_1, C_2, C_3, \) and \( C_4 \) are constants and can be found by using Eqs. (67-70). To solve these four equations with four unknown constants, we can write Eqs. (67-70) in matrix form as follows:

\[
\begin{bmatrix}
0 & 1 & A & 0 \\
0 & -\psi^2 + \frac{e_0 a^2 C}{A} & \zeta^2 + \frac{e_0 a^2 C}{A} & 0 \\
\sin \psi l & \cos \psi l & \cosh \zeta l & \sinh \zeta l \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} = 0
\]
\( \hat{\lambda}_1 = \left( \frac{\psi^2 \sin \psi}{A} \right), \quad \hat{\lambda}_2 = \left( \frac{\psi^2 \cos \psi}{A} \right) \) 
\( \hat{\lambda}_3 = \left( \frac{\zeta^2 \cosh \zeta}{A} \right), \quad \hat{\lambda}_4 = \left( \frac{\zeta^2 \sinh \zeta}{A} \right) \) 

By taking the determinant of the matrix given in Eq. (71) we obtain

\[
\sin \psi \sinh \zeta \left( \psi^2 + \zeta^2 \right)^2 = 0 \quad (72)
\]

There are three possibilities to equalize Eq. (72). These three possibilities can be stated as follow

\[
\left( \psi^2 + \zeta^2 \right)^2 = 0 \quad (73)
\]
\[
\sin \psi = 0 \quad (74)
\]
\[
\sinh \zeta = 0 \quad (75)
\]

The non-trivial solution can be found as follow

\[
\sin \psi = 0, \quad \psi l = n\pi, \quad n=0,1,2,\ldots \quad (76)
\]
\[
\psi^2 l^2 = n^2 \pi^2 \quad (77)
\]
\[
B - \sqrt{B^2 + 4AC} \quad \frac{\pi^2}{2A}
\]

By substituting the values of \(A, B,\) and \(C\) given in Eq. (57) to Eq. (78) we can obtain the final form of nonlocal buckling equation.

\[
P(n) = \frac{(EI + k_p \mu) \left( \frac{n \pi}{L} \right)^4 + (k_w \mu + k_p) \left( \frac{n \pi}{L} \right)^2 + k_w}{\mu \left( \frac{n \pi}{L} \right)^4 + \left( \frac{n \pi}{L} \right)^2}
\]

3.3. Surface elasticity theory (SET)

Gurtin and Murdoch have proposed the surface constitutive as follow [108, 109]

\[
\tau_{11} = \tau_0 + (2\mu_0 + \lambda_0)u_{11}
\]

\[
\tau_{n1} = \tau_0 u_{n,1}
\]

Where Lamé constants are symbolized with \(\mu_0\) and \(\lambda_0\), residual surface stress with \(\tau_0\). The displacement of a Timoshenko beam can be stated as

\[
u_1 = z\phi(x,t)
\]

\[
u_2 = w(x,t)
\]
Where \( u_1, u_2 \) are the components of displacement vector respectively, and \( w \) is symbolizing transverse displacement of the beam. The relation between strain and displacement can be expressed as

\[
\varepsilon_i = \frac{du_i}{dx} = -z \frac{d^2w}{dx^2}
\]

\( \varepsilon_{22} = 0 \)  

\[
\varepsilon_{12} = \frac{1}{2} \left( \frac{du_x}{dx} + \frac{du_y}{dx} \right) = \frac{1}{2} \left( \frac{dw_{(x,t)}}{dx} + \phi_{(x,t)} \right)
\]

To obtain surface stress field, Eq. (81) need to be substituted in Eq. (80)

\[
\tau_{11} = \tau_0 - z(2\mu_0 + \lambda_0) \frac{d^2w(x)}{dx^2}
\]

\[
\tau_{n1} = \tau_0 \frac{dw(x)}{dx} \quad n_2
\]

By using Eq. (83) we can obtain both top and bottom surface layer’s vertical stresses in case of \( n_2 = 1 \)

\[
\tau_{21} = \tau_0 \frac{dw(x)}{dx} \quad \text{top layer}
\]
\[ \tau_{21} = -\tau_0 \frac{dw(x)}{dx} \] bottom layer

By substituting Eq. (84) and Eq. (81) we obtain vertical stress as follow

\[ \sigma_{22} = \frac{2z}{H} \tau_0 \frac{d^2w(x)}{dx^2} - \rho_0 \frac{d^2w}{dt^2} \quad (85) \]

The non-zero bulk stresses can be expressed by using Eq. (85) as follow:

\[ \sigma_{11} = E(z) \frac{d\phi}{dx} + \frac{2vz}{H} \tau_0 \frac{d^2w}{dx^2} - \rho_0 \frac{d^2w}{dt^2} \quad (86) \]

\[ \sigma_{12} = GK \left( \frac{dw(x)}{dx} + \phi \right) \quad (87) \]

\[ \sigma_{22} = \frac{2z}{H} \tau_0 \frac{d^2w(x)}{dx^2} - \rho_0 \frac{d^2w}{dt^2} \quad (88) \]

In Eq. (87), \( K \) represents the shear correction coefficient, which is neglected for Euler-Bernoulli beams.

The stress field of the beam can be found by using Eqs. (86-88) and Eq. (83). Consequently, the governing equation including surface effect for a Timoshenko beam can be stated as follows:

\[ GKA \frac{d^2w(x)}{dx^2} + \frac{d\phi}{dx} + \tau_0 s^* \frac{d^2w}{dx^2} - q(x) = (\rho A + \rho_0 s^*) \frac{d^2w}{dt^2} \quad (89) \]

\[ (EI + (2\mu_0 + \lambda_0)I') \frac{d^2\phi}{dx^2} + \frac{2v\tau_0}{H} \frac{d^3w}{dx^3} - GKA \frac{d^2w(x)}{dx} + \phi = (\rho A + \rho_0 s^*) \frac{d^2w}{dt^2} \quad (90) \]

Where \( I' \) represents the perimeter moment of inertia and \( s^* \) can be calculated by the following equation.
\[ s^* = \int_{s} n^2 ds \quad (91) \]

To calculate these values for ZnO nanowire with circular cross-section:

\[ H = 2 \tau_0 D \quad (92) \]
\[ s^* = \frac{\pi D}{2} \quad (93) \]
\[ I^* = \frac{\pi D^3}{8} \quad (94) \]

To obtain the governing equation for a Euler-Bernoulli beam, the rotational inertia need to be ignored in Eq. (90) as follows

\[ GKA \left( \frac{dw(x)}{dx} + \phi \right) = \left( EI + (2 \mu_0 + \lambda_0)I^* \right) \frac{d^2 \phi}{dx^2} \]
\[ + \frac{2vI\tau_0}{H} \frac{d^3 w}{dx^3} - \frac{2vI\rho_0}{H} \frac{d^3 w}{dx dt} \quad (95) \]

By taking the first derivative of Eq. (95) and using with Eq. (89)

\[ \left( EI + (2 \mu_0 + \lambda_0)I^* - \frac{2vI\tau_0}{H} \right) \frac{d^4 w}{dx^4} - \tau_0 s^* \frac{d^2 w}{dx^2} + q(x) \]
\[ = -(\rho A + \rho_0 s^*) \frac{d^2 w}{dx^2} - \frac{2vI\rho_0}{H} \frac{d^4 w}{dx^3 dt^2} \quad (96) \]

By simplifying Eq. (96), we obtain
\[
\left( EI + (2\mu_0 + \lambda_0)I^* - \frac{2vI_0}{H} \right) \frac{d^4 w}{dx^4} + P - \tau_0 s^* \frac{d^2 w}{dx^2} + q(x) = 0
\]  
(97)

Where

\[
q(x) = H \frac{d^2 w}{dx^2} - k_w w + k_p \frac{d^2 w}{dx^2}
\]  
(98)

The general solution of Eq. (97) can be obtained by using the following equation

\[
w(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 x + C_4 + w_q(x)
\]  
(99)

Where

\[
\beta = \sqrt{\frac{P - \tau_0 s^*}{EI + (2\mu_0 + \lambda_0)I^* - \frac{2vI_0}{H}}}
\]  
(100)

And \( C_1, C_2, C_3, C_4 \) need to be calculated by using boundary conditions. By substituting the boundary conditions given in Eq. (66) we obtain

\[
C_1 + C_4 = 0
\]  
\[
-\beta^2 C_1 = 0
\]  
(101)

\[
C_1 \cos \beta L + C_2 \sin \beta L + C_3 L + C_4 = 0
\]

\[
-\beta^2 \cos \beta L - C_2 \beta^2 \sin \beta L = 0
\]

By solving above equations, we can obtain the buckling formulation of Euler-Bernoulli beam including surface effect on double parameter elastic foundation as follow
\[ P(n) = \frac{\overline{EI} \left( \frac{n\pi}{L} \right)^4 + (H + k_p) \left( \frac{n\pi}{L} \right)^2 k_w}{\left( \frac{n\pi}{L} \right)^2} \]  

(102)

Where \( \overline{EI} \) is the flexural rigidity and can be calculated as follow

\[ \overline{EI} = EI + EI^* \]  

(103)

### 3.4 Finite element model

The stiffness matrix obtained from bending effect can be stated as

\[ K = \int_{t_1}^{t_2} \int_0^L EI \phi^\tau \phi \, u \, dx \, dt \]

\[ K_{wy} = \int_{t_1}^{t_2} \int_0^L k_w \phi^\tau \phi \, dx \, dt \]

\[ K_{wyo} = \int_{t_1}^{t_2} \int_0^L (e_o a)^2 k_w \phi^\tau \phi \, dx \, dt \]

\[ K_{py} = \int_{t_1}^{t_2} \int_0^L k_p \phi^\tau \phi \, dx \, dt \]

\[ K_{pyo} = \int_{t_1}^{t_2} \int_0^L (e_o a)^2 k_p \phi^\tau \phi \, dx \, dt \]  

(104)

\[ K_{gy} = \int_{t_1}^{t_2} \int_0^L P \phi^\tau \phi \, dx \, dt \]

\[ K_{gyo} = \int_{t_1}^{t_2} \int_0^L (e_o a)^2 P \phi^\tau \phi \, dx \, dt \]
\[ F_y = \int_{t_i}^{t_f} \int_0^L q \delta \hat{y} \, dx \, dt \]

\[ F_{yo} = \int_{t_i}^{t_f} \int_0^L (e_o \alpha)^2 q \delta \hat{y} \, dx \, dt \]

By applying nondimensional shape functions

\[ K = K_e = \int_0^1 \left[ \frac{EI}{L} \frac{\partial^2 \phi^T}{\partial \xi^2} \frac{\partial^2 \phi}{\partial \xi^2} L \delta \hat{\xi} \right] = \int_0^1 \frac{EI}{L} \left[ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \right] \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \end{bmatrix} d\hat{\xi} \]

\[ = \int_0^1 \frac{EI}{L} \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \end{bmatrix} \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \end{bmatrix} d\hat{\xi} \]

\[ K_e = \frac{EI}{L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \]

\[ (105) \]

Similarly the effect of Winkler foundation can be stated as \[110\]

\[ K_w = K_{wy} + K_{wyo} = \int_0^1 [k_w \phi^T \phi L \delta \hat{\xi}] + \int_0^1 (e_o \alpha)^2 \frac{1}{L^2} k_w \frac{\partial \phi^T}{\partial \xi} \frac{\partial \phi}{\partial \xi} L \delta \hat{\xi} \]
\[ = \int_{0}^{L} k_p L \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]

\[ \frac{1}{L} \int_{0}^{L} (e_i a)^2 k_p \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]

\[ + \frac{1}{L} \int_{0}^{L} \frac{(e_i a)^2 k_p}{L} \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]

\[ K_w = \frac{k_w}{420} \begin{bmatrix} 156L & 22L^2 & 54L & -13L^2 \\ 22L^2 & 4L^3 & 13L^2 & -3L^3 \\ 54L & 13L^2 & 156L & -22L^2 \\ -13L^2 & -3L^3 & -22L^2 & 4L^3 \end{bmatrix} \]

\[ + \frac{(e_i a)^2 k_w}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \]

The matrix form of Pasternak foundation can be stated as

\[ K_k = K_{py} + K_{pyo} = \int_{0}^{L} \frac{1}{L^2} k_p L \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]

\[ + \frac{1}{L^2} \int_{0}^{L} (e_i a)^2 k_p L \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]

\[ \frac{1}{L} \int_{0}^{L} \frac{(e_i a)^2 k_p}{L^3} \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \\ \phi_1' \phi_2' \phi_3' \phi_4' \\ \phi_1'' \phi_2'' \phi_3'' \phi_4'' \end{bmatrix} \partial \xi \]
\[ K_k = \frac{k_p}{30L} \begin{bmatrix}
36 & 3L & -36 & 3L \\
3L & 4L^2 & -3L & -L^2 \\
-36 & -3L & 36 & -3L \\
3L & -L^2 & -3L & 4L^2
\end{bmatrix} \]

\[ + \left( e_0a \right)^2 \frac{k_p}{L^2} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix} \]

The matrix caused by axial load

\[ K_g = K_{g_0} + K_{g_0y} = \int_0^L \frac{1}{L} P \frac{\partial^2 \phi^T}{\partial \xi} \frac{\partial^2 \phi}{\partial \xi} \, \, \, \, \partial^2 \xi + \int_0^L \left( e_0a \right)^2 P \frac{\partial^2 \phi^T}{\partial \xi^2} \frac{\partial^2 \phi}{\partial \xi^2} \, \, \, \, \partial^2 \xi \]

\[ = \frac{1}{L} \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\
\phi_{11}' & \phi_{12}' & \phi_{13}' & \phi_{14}' \\
\phi_{11}'' & \phi_{12}'' & \phi_{13}'' & \phi_{14}'' \\
\phi_{11}''' & \phi_{12}''' & \phi_{13}''' & \phi_{14}'''
\end{bmatrix} \partial^2 \xi \]

\[ + \frac{1}{L^3} \left( e_0a \right)^2 \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\
\phi_{11}' & \phi_{12}' & \phi_{13}' & \phi_{14}' \\
\phi_{11}'' & \phi_{12}'' & \phi_{13}'' & \phi_{14}'' \\
\phi_{11}''' & \phi_{12}''' & \phi_{13}''' & \phi_{14}'''
\end{bmatrix} \partial^2 \xi \]

\[ (108) \]

\[ K_g = \frac{P}{30L} \begin{bmatrix}
36 & 3L & -36 & 3L \\
3L & 4L^2 & -3L & -L^2 \\
-36 & -3L & 36 & -3L \\
3L & -L^2 & -3L & 4L^2
\end{bmatrix} \]

\[ + \left( e_0a \right)^2 \frac{P}{L^2} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix} \]
In Fig. (7) the solving method in finite element is plotted. Basicly, finite element analysis slice the object to numerous piece and connect the intersections.

To obtain results, the following eigenvalue problem need to be solved

\[ [K - \lambda K_e] = 0 \]  \hspace{1cm} (109)

The stiffness matrix is

\[ [K] = [K_c] + [K_n] + [K_s] \]  \hspace{1cm} (110)

If the axial load is any compression load

\[ K_g = [K_g] \]  \hspace{1cm} (111)

4. Numerical Results and Discussions

The stability analysis of silicon carbide and boron nitride nanotubes and nanowires resting on elastic substrate is investigated in present study. To model the nanostructures, Euler-Bernoulli beam theory is employed. As the analysis are made in nano-scale, to take small-scale effect into consideration, different small-scale effective theories are used. Nonlocal elasticity theory, surface elasticity theory and their combination, modified strain gradient theory and modified couple stress theories are used and compared to see their influence into buckling results. Also, finite element analysis was applied to nanostructures both by using continuum model and using computer softwares. In figures, nonlocal elasticity theory is represented as NET, surface elasticity theory as SET, the combination of nonlocal elasticity theory and surface elasticity theory as NSET, modified strain gradient theory as MSGT, modified couple stress theory as MCST. To obtain dimensionless analysis results,
Winkler and Pasternak foundation parameters are used dimensionless as \( K_w = \frac{k_w L^4}{EI} \) and
\[ K_p = \frac{k_p L^2}{EI}. \]

In Fig. (8) the mode shape of nanostructures is plotted by using ANSYS computer software. Size effective theories parameters were chosen as \( E_s=35.3 \text{, } \mu=4 \text{ nm, } l_0= l_1= l_2=0.5 \) [45]. The effect of Winkler parameter was investigated for all size effective theories in Fig. (9). The Dimensionless buckling loads were calculated with the change in Winkler foundation parameter. In the calculation for Fig. (9) the effect of Pasternak foundation was neglected by choosing the Pasternak foundation parameter value zero. As it can be clearly seen from Fig. (9), the effect of Winkler foundation gets dramatically lower on higher modes. It can also be observed that nonlocal elasticity theory lower the buckling load by fragile the nanostructure while modified couple stress theory, modified strain gradient theory, and surface elasticity theory give higher results by strengthening the nanostructure. Between size effective theories, modified strain gradient theory always give highest results while nonlocal elasticity theory gives the lowest for all modes.

The Dimensionless buckling loads were calculated with the change in Pasternak foundation parameter in Fig. (10). The effect of Winkler foundation was neglected by choosing the Winkler foundation parameter value zero. Similarly to previous results, the effect of foundation get lower but not as lower as Winkler foundation.

In Table 1, critical buckling loads of nanostructures are calculated and compared. To validate results, comparison with Naidu and Rao [111] is given and results seems to be in good harmony. In Table 1, N represent the element number used for finite element analysis. Finite element method results are both given for various nonlocal elasticity parameters and
classic analysis. Similar to Fig. (9,10) lowest results are obtained for the highest nonlocal parameter ($e_0a=10$ nm). Choosing higher value of size effective parameter for MCST and MSGT ends with higher buckling load value.

5. Concluding Remarks

The effect of double parameter elastic foundation on buckling of silicon carbide and boron nitride nanotubes and nanowires is investigated in current study. To model the nanostructures, Euler-Bernoulli beam model and computer softwares are used. As the beams are in nano size to take the small-scale into consideration, three different small-scale effect theories is used and compared with finite element results. Small-scale effective theories used are nonlocal elasticity theory, surface elasticity theory and their combination, modified strain gradient theory and modified couple stress theories. The substrate is modeled by using two parameter (Winkler and Pasternak) elastic foundation model. Buckling loads are calculated for maximum fifth mode number. To see the validation of calculations, comparative results with literature is given in the table. Comparative results are in good harmony. To conclude, from small-scale theories used in the study, highest buckling loads are obtained in case of modified strain gradient theories used and lowest in case of nonlocal elasticity theory. Furthermore, the effect of foundation on buckling getting lower with the increase in mode number.

Acknowledgements

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References


[21] Peng, T., Guo, W., Zhang, Q. et al. “Uniform coaxial CNT@Li2MnSiO4@C as advanced cathode material for lithium-ion battery“, *Electrochim Acta*, 291, pp. 1-8 (2018).


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a) BNNT b) SiCNT
Fig. 10. Buckling analysis results with the change in Pasternak foundation for first five modes respectively

a) BNNT b) SiCNT
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**Brief technical biography of authors**
Hayri Metin Numanoğlu

Hayri Metin Numanoğlu currently working on his M.Sc. thesis on nanoscale mechanics in Akdeniz University Civil Eng. Dept. His research interest on vibration analysis of nano and micro scaled continuous components by analytical and finite element method.

Kadir Mercan

Kadir Mercan was graduated from Süleyman Demirel University civil engineering program in 2013 with first degree. He is currently working as research assistant in Mehmet Akif Ersoy University, Architecture and Engineering Faculty, Civil Engineering Department, Mechanical Division. He is also currently studying PhD degree under the advise of Prof. Ömer Civalek in Akdeniz University after having master degree there.

Ömer Civalek

Professor Civalek has been a Professor at the Faculty of Engineering, University of Akdeniz. He holds two Ph.D. degrees in Structural and Mechanical Engineering, one from Dokuz Eylül University in Structural Engineering and the other from the University of Fırat in Applied Mechanics. He has authored 220 refereed journal papers (about 125 in SCI Journals) with 6000 citations, over 30 papers presented at various conferences, and 40 papers in various national journals. His research emphasis has been on Solid Mechanics, Vibration, buckling analyses of plates and shells, Computational Mechanics, Modeling of Nanostructures and composites mechanics.