Control of Natural Frequency of Beams using Different Pre-tensioning Cable Patterns

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Abstract

Sometimes natural frequency of beams is not within the allowable range, regardless of the appropriate shear and bending design. Moreover, every so often structural designers are faced with cases in which it is not possible to increase the beam height. Steel cable is utilized in this research to control the natural frequency of beam due to its important advantages such as low weight, small cross-sectional area and high tensile strength. For the first time, theoretical relations are developed to calculate the increase in pre-tensioning force of steel cables under external loading based on the method of least work. Moreover, the natural frequency of steel beams with different support conditions without cable and with different patterns of cable is calculated based on Rayleigh's method. To verify the theoretical relations, the steel beam is modeled in the ABAQUS software. The obtained results show the theoretical relations can appropriately predict the natural frequency of beams with different support conditions and cable patterns. In addition, simply supported as well as fixed supported beams are pre-stressed with v-shaped and modified v-shaped patterns of the cable. According to the obtained results, the modified v-shaped pattern of the cable is more efficient than the v-shaped pattern one.

Keywords: natural frequency; steel beam; cable; pre-tensioning; least work; Rayleigh's method

1. Introduction

Reckoned as important components of a structure, cables are materials which can tolerate tensile force, and generally increase the stiffness and bearing capacity of a structure [1]. Nowadays, cables are increasingly used in structures. Hou and Tagawa applied cable-cylinder bracing in the seismic retrofitting of steel flexural frames. From their view point, through this retrofitting method, the lateral strength of the storey augments without decreasing the ductility of flexural frame [2]. Fanaie et al. presented theoretical relations for the cable-cylinder bracing system using a rigid cylinder like steel cylinder. They verified the results by finite element ABAQUS software [3]. They also studied seismic behavior of steel flexural frames strengthened with cable-cylinder bracing and obtained reasonable results [4]. Giaccu investigated the non-linear dynamic behavior of pre-tensioned-cable cross-braced structures in the presence of slackening in the braces. They

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concluded there is a direct correlation between equivalent frequency and slackening in the braces [5].

Pre-tensioning of steel beams through high strength cables is one of the most efficient methods so as to decrease the required steel and increase their bearing capacity. The pre-tensioning technique was primarily used in reinforced concrete structures; however, for the first time, it was utilized by Dischinger and Magnel in steel beams. Pre-tensioned steel structures are constructed all over the world, especially in America, Russia, and Germany. This fact shows the structural and economic merits of pre-stressed steel beams compared to non-prestressed ones. The pre-tensioning technique is appropriate to construct new structures as well as strengthening the existing ones [6].

Some researchers have studied pre-stressed beams using tendons. Le et al. experimentally evaluated the application of either unbonded CFRP tendons or steel tendons on precast T-section segmental concrete beams and tested under cyclic loads. They concluded CFRP tendons can be well in replacement of steel tendons and the beams achieve both good strength and ductility capacity [7]. Pisani analysed the simply supported concrete beam externally prestressed under sustained loads and introduced two numerical methods able to describe the time evolution of both the stress distribution and the displacements of a simply supported concrete beam externally prestressed and presented the example to verify the precision of the methods [8]. Lou et al. numerically studied the flexural response of continuous externally fiber reinforced polymer (FRP) prestressed concrete beams having various linearly transformed cable profiles. They observed the cable shift by linear transformation does not affect the basic performance at all stages of loading up to failure and the secondary moments vary linearly with the cable shift [9]. Nie et al. presented theoretical relations to calculate the deflection as well as yield and ultimate moments of simply supported pre-stressed steel-concrete composite beam considering the slip effect. They verified the suggested formulas with the experimental results [10]. Zhou et al. presented the experimental study as well as numerical model of prestressed composite beams subjected to fire and positive moment. They observed the fire resistance of composite beams prestressed with external tendons was highly influenced by the stress in the cable strands [11]. Troitsky evaluated the behavior of pre-stressed steel beam using cables, and observed the increase in the stiffness and decrease in the deformation of the beam [6]. Belletti and Gasperi studied the behavior of pre-stressed simply supported steel I-shaped beams by tendons with focusing on two parameters, namely, the number of deviators and the value of pre-stressing force [12]. Park et al. analytically and experimentally evaluated the flexural behavior of steel I-beam pre-stressed with externally unbonded tendons. They observed a considerable increase in the yielding and ultimate bearing capacity of steel I-beam [13]. Kambal and Jia derived a finite-element formulation to investigate the effectiveness of applying the prestressing technique with respect to the flexural behavior of a simply supported steel box girder and they verified it by experimental results [14]. Zhang examined the analytical solutions of the symmetric and antisymetric elastic lateral-torsional buckling (LTB) of prestressed steel I-beams, with rectilinear tendons, under equal end moments and verified the correctness of the analytical solutions by those simulated by using ANSYS [15].

A number of researchers have investigated the dynamic behavior of pre-tensioned beams. Noble et al. studied the results of dynamic impact testing on externally axially loaded steel rectangular hollow sections (RHSs) and compared the response to that of externally post-tensioned steel RHSs. As well as, they tested the validity of the "compression-softening" effect for post-tensioned sections. They concluded the "compression-softening" theory is not valid for pre- or post-tensioned sections [16]. Cao et al. investigated the vibration performance of the arch prestressed concrete truss (APT) girder subjected to the on-site heel-drop and jumping impact tests and performed theoretical analyses. They concluded the theoretical fundamental natural frequency is in general agreement with the experimental result [17]. Miyamoto et al. have studied the dynamic behavior of the pre-tensioned simply supported composite beam with external tendon. They derived the natural frequency equation of pre-tensioned beam based on the flexural vibration equation and verified the predicted equation by comparing it with the results of the dynamic experiment [18]. Park et al. analytically and experimentally studied the strengthening effect of bridges using external pre-tensioned tendons and concluded that strengthening reduces the mid span deflection by 10-24% [19].

In controlling the natural frequency of the beam, structural designers have been faced with cases in which it is not possible to increase the beam height due to architectural limitations, or cases in which beam frequency limitation was not considered during design, and problem of vibration is observed after implementation. In this study, the natural frequency of steel beams has been evaluated without cable and with various support conditions and different cable patterns.

The increase in pre-tensioning force of steel cable subjected to external loading is determined using method of least work. Then, Rayleigh's method is applied to develop the natural frequency relations of steel beam equipped with cable. In order to validate the obtained natural frequency relations, the results of theoretical relations are compared with those of finite element model of the beams.

2. Pre-tensioning symmetric I-shaped steel beam with steel cable

Symmetric I-shaped steel beam has been considered with different support conditions, namely, simply supported and fixed supported beam. As shown in Figure 1, pre-stressed cables with different patterns have been used in both sides of beam web, and subjected to external loading. In the frequency analysis, beam movement is of back and forth type; therefore, As observed in the figure, regarding the v-shaped pattern, the cable is fixed at both ends to one of the flange (top or bottom) of the beam in both sides of the web. Then, in the middle of the beam, it passes through the reel on the another flange causing the pattern with two inclined cables. Moreover, in the modified v-shaped pattern, the cable is fixed at both ends to the one of flange (top or bottom) of the beam in both sides of the web, and then it passes through two reels on the another flange producing a pattern with one horizontal cable in the middle and two inclined ones in each side.

The following assumptions are taken into account to analyze pre-stressed symmetric I-shaped steel beams with steel cable:

1- The materials of steel beam and cable are linearly elastic;

2-The deformations are small;

3- Shear deformation is neglected;

4- The friction loss in the region of the cable deformation and the relaxation of steel cable are ignored;

5- Steel beam section is rolled; therefore, it is compact;

3. Natural Frequency

The natural frequency of system with distributed mass and rigidity which is usually considered as single degree of freedom system and known as generalized single degree of freedom system, can be calculated using the Rayleigh’s method. This method is based on the principle of conservation of energy. The principle of conservation of energy states that the total energy in a freely vibrating undamped system is constant (i.e., it does not vary with time).

4. Calculating the Natural Circular Frequency of Beam by Rayleigh’s Method

The simple harmonic motion of a beam under free vibration can be defined as follows:

$$u(x,t) = z_0\psi(x)\sin\omega_n t' \tag{1}$$

where $\psi(x)$ is an assumed shape function defining the form of deflections and satisfying the displacement boundary conditions. The shape function can be determined from deflections thanks to a selected set of static forces. One common selection for these forces is the weight of structure applied in an appropriate direction. Also, z_0 is the amplitude of generalized coordinate z(t). ω_n is the natural circular frequency of beam. The velocity of beam is equal to:

$$\dot{u}(x,t) = \omega_n z_0 \psi(x)\cos\omega_n t' \tag{2}$$

The maximum potential energy of the system over a vibration cycle is equal to its strain energy associated with the maximum displacement $u_o(x)$:

$$E_{So} = \int_0^L \frac{1}{2} EI(x)\left[u_o''(x)\right]^2 dx \tag{3}$$

The maximum kinematic energy of the system over a vibration cycle is associated with the maximum velocity $\dot{u}_0(x)$:

$$E_{Ko} = \int_0^L \frac{1}{2} m(x)\left[\dot{u}_o(x)\right]^2 dx \tag{4}$$

Using Equations (1) and (2), the maximum displacement $u_0(x)$ and maximum velocity $\dot{u}_0(x)$ are defined as follows:

$$u_o(x) = z_o \psi(x) \tag{5}$$

$$\dot{u}_o(x) = \omega_n z_o \psi(x) = \omega_n u_o(x) \tag{6}$$

The natural circular frequency of beam is obtained by replacing Equations (5) and (6) in Equations (3) and (4), and using the principle of conservation of energy, equating maximum strain energy E_{So} to maximum kinematic energy E_{Ko} gives:

$$\omega_n^2 = \frac{\int_0^L EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx} \tag{7}$$

In which $m(x)$ is the mass per unit length of the beam, $EI(x)$ is flexural rigidity and L is the beam length.

Equation (7) is the Rayleigh's quotient for a system with distributed mass and rigidity.

Rayleigh's method can be used to calculate the natural frequency of beam with different support conditions without cable as well as different patterns of cable.

4.1. Calculating the Natural Frequency of a Simply Supported Beam without Cable

To determine the shape function of the simply supported beam, as shown in Figure 2(a), the boundary conditions are as follows:

$$u_o(0) = u_o(l_b) = 0 \quad , \quad u_o'(\frac{l_b}{2}) = 0$$

With this limitation, two different shape functions can be assumed.

4.1.1. Calculating natural frequency of simply supported beam based on the assumed shape function

The hypothetical shape function satisfying the displacement boundary conditions is considered as follows:

$$\psi(x) = \sin\frac{\pi x}{l_b}$$

(8)

The natural circular frequency of the simply supported beam is obtained using the Rayleigh's quotient formula (Equation (7)) by replacing the assumed shape function according to Equation (8) as follows:

$$\omega_n^{\,2} = \frac{\int_0^{l_b} EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^{l_b} m(x)\left[\psi(x)\right]^2 dx} = \frac{(EI)_b \int_0^{l_b} (-\frac{\pi^2}{l_b^{\,2}} \sin\frac{\pi x}{l_b})^2 dx}{\frac{q_D}{g}\int_0^{l_b}\left(sin\frac{\pi x}{l_b}\right)^2 dx} = \frac{\pi^4 (EI)_b\, g}{l_b^{\,4} q_D} \tag{9}$$

$$\rightarrow \omega_n = \frac{\pi^2}{l_b^{\,2}}\sqrt{\frac{(EI)_b\, g}{q_D}}$$

In which m(x) is the mass per unit length equal to $\frac{q_D}{g}$ (q_D is uniform dead load per unit length and g is the gravity acceleration), l_b is the beam length and $(EI)_b$ is the flexural rigidity of the beam.

The natural frequency of the simply supported beam is obtained using the natural circular frequency formula (Equation (9)) as follows:

$$f = \frac{\omega_n}{2\pi} = \frac{\pi}{2l_b^{\,2}}\sqrt{\frac{(EI)_b\, g}{q_D}} \tag{10}$$

4.1.2. Calculating the natural frequency of simply supported beam based on the shape function resulting from the deflection curve

A possible approach is to select the shape function based on the deflection curve due to static force. The bending moment equation of the simply supported beam due to vertical structural weight (as in Figure 2(b)) in order to determine the deflection curve, is as follows:

$$M(x) = \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{11}$$

The internal bending moment is equal to:

$$M(x) = -(EI)_b u_o''(x) \tag{12}$$

By replacing Equation (11) in Equation (12) and imposing boundary conditions, the deflection curve is obtained as follows:

$$u_o(x) = \frac{5 q_D l_b^{\,4}}{384 (EI)_b}\left[\frac{16}{5}(\frac{x}{l_b}) - \frac{32}{5}(\frac{x}{l_b})^3 + \frac{16}{5}(\frac{x}{l_b})^4\right] \tag{13}$$

If the deflection in the mid-span of the simply supported beam (as Figure 2(a)) is assumed as the amplitude of generalized coordinate $z_o = u_o(\frac{l_b}{2}) = \frac{5q_D l_b^4}{384(EI)_b}$, the shape function is obtained as follows:

$$\psi(x) = \frac{16}{5}(\frac{x}{l_b}) - \frac{32}{5}(\frac{x}{l_b})^3 + \frac{16}{5}(\frac{x}{l_b})^4 \tag{14}$$

The natural circular frequency of the simply supported beam using the Rayleigh quotient formula (Equation (7)) by replacing the obtained shape function from the deflection curve according to Equation (14) is obtained as follows:

$$\omega_n^2 = \frac{\int_0^{l_b} EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^{l_b} m(x)\left[\psi(x)\right]^2 dx} = \frac{(EI)_b \int_0^{l_b}(-\frac{192x}{5l_b^3} + \frac{192x^2}{5l_b^4})^2 dx}{\frac{q_D}{g}\int_0^{l_b}\left(\frac{16}{5}(\frac{x}{l_b}) - \frac{32}{5}(\frac{x}{l_b})^3 + \frac{16}{5}(\frac{x}{l_b})^4\right)^2 dx} = \frac{3024(EI)_b g}{31 l_b^4 q_D} \tag{15}$$

$$\rightarrow \omega_n = \frac{12}{l_b^2}\sqrt{\frac{21(EI)_b g}{31 q_D}}$$

Then the natural frequency of the simply supported beam is obtained as follows:

$$f = \frac{\omega_n}{2\pi} = \frac{6}{\pi l_b^2}\sqrt{\frac{21(EI)_b g}{31 q_D}} \tag{16}$$

4.2. Calculating the natural frequency of the fixed supported beam without cable

To determine the shape function of the fixed supported beam, as shown in Figure 3(a), the boundary conditions are as follows:

$$u_o(0) = u_o(l_b) = 0 \quad , \quad u_o'(0) = u_o'(\frac{l_b}{2}) = u_o'(l_b) = 0$$

Considering this limitation two different shape functions can be assumed.

4.2.1. Determining natural frequency of fixed supported beam based on assumed shape function

The assumed shape function satisfying the displacement boundary conditions is considered as follows:

$$\psi(x) = 1 - \cos\frac{2\pi x}{l_b} \tag{17}$$

The natural circular frequency of the fixed supported beam using the Rayleigh's quotient formula (Equation (7)) by replacing the assumed shape function according to Equation (17), is obtained as follows:

$$\omega_n{}^2 = \frac{\int_0^{l_b} EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^{l_b} m(x)\left[\psi(x)\right]^2 dx} = \frac{(EI)_b \int_0^{l_b} (\frac{4\pi^2}{l_b{}^2}\cos\frac{2\pi x}{l_b})^2 dx}{\frac{q_D}{g}\int_0^{l_b}\left(1-cos\frac{2\pi x}{l_b}\right)^2 dx} = \frac{16\pi^4 (EI)_b g}{3 l_b{}^4 q_D} \quad (18)$$

$$\rightarrow \omega_n = \frac{4\pi^2}{l_b{}^2}\sqrt{\frac{(EI)_b g}{3q_D}}$$

Then the natural frequency of the fixed supported beam is obtained as follows:

$$f = \frac{\omega_n}{2\pi} = \frac{2\pi}{l_b{}^2}\sqrt{\frac{(EI)_b g}{3q_D}} \quad (19)$$

4.2.2. Calculating natural frequency of the fixed supported beam based on the shape function obtained from the elastic curve

The bending moment equation of the fixed supported beam due to vertical structural weight (as in Figure 3(b)) in order to determine the deflection curve, is as follows:

$$M(x) = -\frac{q_D l_b{}^2}{12} + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \quad (20)$$

The internal bending moment is equal to:

$$M(x) = -(EI)_b u_o''(x) \quad (21)$$

By replacing Equation (20) in Equation (21) and imposing boundary conditions, the deflection curve is obtained as follows:

$$u_o(x) = \frac{q_D l_b{}^4}{384(EI)_b}\left[16(\frac{x}{l_b})^2 - 32(\frac{x}{l_b})^3 + 16(\frac{x}{l_b})^4\right] \quad (22)$$

If the deflection in the mid-span of the fixed supported beam (as in Figure 3(a)) is assumed as a amplitude of generalized coordinate $z_o = u_o(\frac{l_b}{2}) = \frac{q_D l_b{}^4}{384(EI)_b}$, the shape function is obtained as follows:

$$\psi(x) = 16(\frac{x}{l_b})^2 - 32(\frac{x}{l_b})^3 + 16(\frac{x}{l_b})^4 \quad (23)$$

The natural circular frequency of the fixed supported beam using the Rayleigh's quotient formula (Equation (7)) by replacing the shape function obtained from the deflection curve according to Equation (23), is obtained as follows:

$$\omega_n^2=\frac{\int_0^{l_b} EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^{l_b} m(x)\left[\psi(x)\right]^2 dx}=\frac{(EI)_b\int_0^{l_b}\left(\frac{32}{l_b^{\,2}}-\frac{192x}{l_b^{\,3}}+\frac{192x^2}{l_b^{\,4}}\right)^2 dx}{\frac{q_D}{g}\int_0^{l_b}\left(16\left(\frac{x}{l_b}\right)^2-32\left(\frac{x}{l_b}\right)^3+16\left(\frac{x}{l_b}\right)^4\right)^2 dx}=\frac{504(EI)_b\, g}{l_b^{\,4} q_D} \qquad (24)$$

$$\rightarrow \omega_n=\frac{6}{l_b^{\,2}}\sqrt{\frac{14(EI)_b\, g}{q_D}}$$

Then the natural frequency of the fixed supported beam is obtained as follows:

$$f=\frac{\omega_n}{2\pi}=\frac{3}{\pi l_b^{\,2}}\sqrt{\frac{14(EI)_b\, g}{q_D}} \qquad (25)$$

4.3. Calculating the natural frequency of simply supported beam with v-shaped pattern cable

One possible way to select the shape function in order to calculate natural frequency of the beam with cable using the Rayleigh's method is based on the deflection curve due to static force. One common selection for these forces is the weight of structure applied in vertical direction.

The cable length increased by Δl, and its pre-tensioning force, F_{pt}, increased by ΔF, in beam with cable under uniform dead load. As the structure is statically indeterminate, the static equilibrium equations are not enough to calculate ΔF. The increase in the force in the cable can be calculated using the method of least work.

Regarding the simply supported beam with v-shaped pattern cable, as shown in Figure 4, the increase in pre-tensioning force of the steel cable is equal to ΔF. Therefore, axial force of the beam is equal to $\Delta F\cos\theta$. Also, with regard to the symmetry of structure and loading (as in Figure 4), the maximum strain energy of the simply supported beam with v-shaped pattern of cable including strain energy resulting from bending and axial force of the beam and strain energy from axial force of the cable can be obtained for half of the beam and by duplicating it, the maximum strain energy for the whole beam is obtained as follows:

$$E_{So}=2\times\frac{1}{2(EI)_b}\int_0^{\frac{l_b}{2}} M(x)^2 dx+2\times\frac{\Delta F^2 l_c}{2(AE)_c}+\frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} \qquad (26)$$

In which l_b and l_c are the lengths of beam and inclined cable, A_b and A_c are cross section area of beam and cable on both sides of the web, E_b and E_c are elasticity modulus of beam and cable, respectively, and I_b is the moment of inertia of beam, θ is the angle of inclined cable with the horizontal axis, and $M(x)$ is the bending moment of the simply supported beam with v-shaped pattern of cable for the $0 \le x \le \frac{l_b}{2}$ range.

Due to the symmetry of structure and loading, the bending moment diagram for the right half of the beam is exactly similar to the left half of the beam; therefore, the bending moment of the simply supported beam with v-shaped pattern of cable under uniform distributed dead load for the half of the beam is obtained as follows:

For the $0 \le x \le \frac{l_b}{2}$ region:

$$M(x) = \Delta F \cos\theta y_0 - \Delta F \sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{27}$$

In which q_D is the uniform distributed dead load per unit length, y_0 is the distance of neutral axis to the connection point of steel cable to the beam flange (half of the height of beam web).

By replacing Equation (27) in Equation (26), the maximum strain energy equation is obtained as follows:

$$\begin{aligned} E_{So} &= 2\times\frac{1}{2(EI)_b}\int_0^{\frac{l_b}{2}}\left(\Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2}\right)^2 dx + 2\times\frac{\Delta F^2 l_c}{2(AE)_c} \\ &+ \frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} = \frac{1}{(EI)_b}\left\{\begin{aligned}&\frac{q_D{}^2 l_b{}^5}{240} + \frac{\Delta F^2 l_b{}^3 \sin^2\theta}{24} + \frac{\Delta F^2 l_b y_0{}^2\cos^2\theta}{2} \\ &-\frac{\Delta F^2 l_b{}^2 y_0 \sin\theta\cos\theta}{4} - \frac{5 q_D \Delta F l_b{}^4 \sin\theta}{192} + \frac{q_D \Delta F l_b{}^3 y_0\cos\theta}{12}\end{aligned}\right\} \\ &+ \frac{\Delta F^2 l_c}{(AE)_c} + \frac{\Delta F^2 l_b \cos^2\theta}{2(AE)_b} \end{aligned} \tag{28}$$

Calculating the increase in pre-tensioning force of the cable (ΔF) through the method of least work, the relation of whole strain energy is differentiated with respect to ΔF and the obtained result is equated to zero:

$$\frac{\partial E_{So}}{\partial(\Delta F)} = 0 \tag{29}$$

The relation for calculating the increase of pre-tensioning force of the cable (ΔF) is obtained as follows:

$$\Delta F=\frac{5q_Dl_b^4\sin\theta-16q_Dl_b^3y_0\cos\theta}{16\left(l_b^3\sin^2\theta+12l_by_0^2\cos^2\theta-6l_b^2y_0\sin\theta\cos\theta+\frac{24(EI)_b\,l_c}{(AE)_c}+\frac{12I_bl_b\cos^2\theta}{A_b}\right)}$$
$$=q_D\mu$$
(30)

In Equation (30), μ is as follows:

$$\mu=\frac{5l_b^4\sin\theta-16l_b^3y_0\cos\theta}{16\left(l_b^3\sin^2\theta+12l_by_0^2\cos^2\theta-6l_b^2y_0\sin\theta\cos\theta+\frac{24(EI)_b\,l_c}{(AE)_c}+\frac{12I_bl_b\cos^2\theta}{A_b}\right)} \tag{31}$$

By replacing Equation (30) in Equation (28), the maximum strain energy of the beam E_{So} is obtained as follows:

$$E_{So}=\frac{q_D^{\ 2}}{960(EI)_b}\left\{\begin{array}{l}4l_b^5+40\mu^2l_b^3\sin^2\theta+480\mu^2l_by_0^2\cos^2\theta-240\mu^2l_b^2y_0\sin\theta\cos\theta\\ -25\mu l_b^4\sin\theta+80\mu l_b^3y_0\cos\theta+\frac{960\mu^2(EI)_b\,l_c}{(AE)_c}+\frac{480\mu^2I_bl_b\cos^2\theta}{A_b}\end{array}\right\} \tag{32}$$
$$=\frac{q_D^{\ 2}\beta}{960(EI)_b}$$

In Equation (32), β is as follows:

$$\beta=4l_b^5+40\mu^2l_b^3\sin^2\theta+480\mu^2l_by_0^2\cos^2\theta-240\mu^2l_b^2y_0\sin\theta\cos\theta-25\mu l_b^4\sin\theta$$
$$+80\mu l_b^3y_0\cos\theta+\frac{960\mu^2(EI)_b\,l_c}{(AE)_c}+\frac{480\mu^2I_bl_b\cos^2\theta}{A_b} \tag{33}$$

With regard to the symmetry of structure and loading, the maximum kinematic energy of the simply supported beam with v-shaped pattern of cable can be obtained using Equation (4) and replacing the maximum velocity according to Equation (6) for half of the beam and by duplicating it, the maximum kinematic energy is obtained for the whole beam as follows:

$$E_{Ko}=2\times\int_0^{\frac{l_b}{2}}\frac{1}{2}m(x)\left(\dot{u}_o(x)\right)^2dx=\int_0^{\frac{l_b}{2}}m(x)\left(\omega_nu_o(x)\right)^2dx \tag{34}$$

In which $m(x)$ is mass per unit length of the beam equal to $\frac{q_D}{g}$ (mass of cable is neglected) and $u_o(x)$ is the deflection curve of the simply supported beam with v-shaped pattern cable for $0\le x\le\frac{l_b}{2}$.

To determine the deflection curve, the internal bending moment is equal to:

$$M(x) = -(EI)_b u_o''(x) \tag{35}$$

The deflection curve in Equation (35) should satisfy the displacement boundary conditions. For the simply supported beam with v-shaped pattern of cable, the boundary conditions for half of the beam are:

$$u_o(0) = 0 \quad , \quad u_o'(\frac{l_b}{2}) = 0$$

By replacing Equation (27) in Equation (35) and imposing the above boundary conditions, the deflection curve for half of the beam is obtained as follows:

$$u_o(x) = \frac{1}{(EI)_b}\begin{bmatrix} \frac{q_D l_b{}^3 x}{24} - \frac{\Delta F l_b{}^2 \sin\theta x}{8} + \frac{\Delta F l_b y_0 \cos\theta x}{2} - \frac{\Delta F y_0 \cos\theta x^2}{2} \\ + \frac{\Delta F \sin\theta x^3}{6} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24} \end{bmatrix} \tag{36}$$

By replacing Equation (36) in Equation (34), the maximum kinematic energy equation of the beam is obtained as follows:

$$\begin{aligned} E_{Ko} &= \omega_n{}^2 \frac{q_D}{g} \int_0^{\frac{l_b}{2}} \left(\frac{1}{(EI)_b} \begin{bmatrix} \frac{q_D l_b{}^3 x}{24} - \frac{\Delta F l_b{}^2 \sin\theta x}{8} + \frac{\Delta F l_b y_0 \cos\theta x}{2} - \frac{\Delta F y_0 \cos\theta x^2}{2} \\ + \frac{\Delta F \sin\theta x^3}{6} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24} \end{bmatrix} \right)^2 dx \\ &= \omega_n{}^2 \frac{q_D}{(EI)_b{}^2 g} \begin{Bmatrix} \frac{31 q_D{}^2 l_b{}^9}{725760} + \frac{17 \Delta F^2 l_b{}^7 \sin^2\theta}{40320} + \frac{\Delta F^2 l_b{}^5 y_0{}^2 \cos^2\theta}{240} - \frac{61 \Delta F^2 l_b{}^6 y_0 \sin\theta\cos\theta}{23040} \\ - \frac{277 q_D \Delta F l_b{}^8 \sin\theta}{1032192} + \frac{17 q_D \Delta F l_b{}^7 y_0 \cos\theta}{20160} \end{Bmatrix} \end{aligned} \tag{37}$$

By replacing Equation (30) in Equation (37), the maximum kinematic energy equation of the beam E_{Ko} is obtained as follows:

$$\begin{aligned} E_{Ko} &= \omega_n{}^2 \frac{q_D{}^3}{46448640 (EI)_b{}^2 g} \begin{Bmatrix} 1984 l_b{}^9 + 19584 \mu^2 l_b{}^7 \sin^2\theta + 193536 \mu^2 l_b{}^5 y_0{}^2 \cos^2\theta \\ -122976 \mu^2 l_b{}^6 y_0 \sin\theta\cos\theta - 12465 \mu l_b{}^8 \sin\theta \\ +39168 \mu l_b{}^7 y_0 \cos\theta \end{Bmatrix} \\ &= \omega_n{}^2 \frac{q_D{}^3 \gamma}{46448640 (EI)_b{}^2 g} \end{aligned} \tag{38}$$

In Equation (38), γ is as follows:

$$\begin{aligned} \gamma = {} & 1984 l_b{}^9 + 19584 \mu^2 l_b{}^7 \sin^2\theta + 193536 \mu^2 l_b{}^5 y_0{}^2 \cos^2\theta - 122976 \mu^2 l_b{}^6 y_0 \sin\theta\cos\theta \\ & -12465 \mu l_b{}^8 \sin\theta + 39168 \mu l_b{}^7 y_0 \cos\theta \end{aligned} \tag{39}$$

The natural circular frequency of the simply supported beam with v-shaped pattern of cable using the Rayleigh's method and the principle of conservation of energy, is obtained as follows:

$$\omega_n^{\,2} = \frac{48384(EI)_b g\beta}{q_D\gamma} \rightarrow \omega_n = 48\sqrt{\frac{21(EI)_b g\beta}{q_D\gamma}} \tag{40}$$

Then the natural frequency of the simply supported beam with v-shaped pattern of the cable is obtained as follows:

$$f_n = \frac{\omega_n}{2\pi} = \frac{24}{\pi}\sqrt{\frac{21(EI)_b g\beta}{q_D\gamma}} \tag{41}$$

4.4. Calculating the natural frequency of the simply supported beam with modified v-shaped pattern of cable

Concerning the simply supported beam with the modified v-shaped pattern of cable, as shown in Figure 5, the force of horizontal cable should be equal to the horizontal component of the force of inclined cable to keep the bending moment continuous in the slope change region of the cable. Therefore, if the increase in pre-tensioning force of steel cable is assumed as ΔF in the slope parts, then it will be equal to $\Delta F\cos\theta$ in the horizontal part; hence, the axial force of the beam is equal to $\Delta F\cos\theta$.

Also regarding symmetry of the structure and loading (as in Figure 5), the maximum strain energy of the simply supported beam with modified v-shaped pattern of cable for the whole beam is obtained as follows:

$$E_{So} = 2\times\frac{1}{2(EI)_b}\left\{\int_0^a M_1(x)^2dx + \int_a^{\frac{l_b}{2}} M_2(x)^2dx\right\} + 2\times\frac{\Delta F^2 l_c}{2(AE)_c} + \frac{(\Delta F\cos\theta)^2(l_b-2a)}{2(AE)_c} + \frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} \tag{42}$$

In which a is the distance of support to the point of change in cable slope (horizontal projection of inclined cable) and $M_1(x)$ and $M_2(x)$ arethe bending moments of the simply supported beam along with modified v-shaped pattern of cable for $0 \le x \le a$ and $a \le x \le \frac{l_b}{2}$, respectively.

Due to symmetry of the structure and loading, the bending moment of the simply supported beam with modified v-shaped pattern of cable subjected to uniform distributed dead load for half of the beam is obtained as follows:

For $0 \le x \le a$;

$$M_1(x) = \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{43}$$

For $a \le x \le \frac{l_b}{2}$;

$$M_2(x) = -\Delta F \cos\theta y_0 + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{44}$$

By replacing Equations (43) and (44) in Equation (42), the maximum strain energy is obtained as follows:

$$\begin{aligned}
E_{So} &= 2\times\frac{1}{2(EI)_b}\left\{\begin{array}{l}\int_0^a (\Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2})^2 dx \\ +\int_a^{\frac{l_b}{2}} (-\Delta F\cos\theta y_0 + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2})^2 dx\end{array}\right\} + 2\times\frac{\Delta F^2 l_c}{2(AE)_c} \\
&+ \frac{(\Delta F\cos\theta)^2(l_b - 2a)}{2(AE)_c} + \frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} \\
&= \frac{1}{(EI)_b}\left\{\begin{array}{l}\frac{q_D{}^2 l_b{}^5}{240} + \frac{\Delta F^2 a^3 \sin^2\theta}{3} + \frac{\Delta F^2 l_b y_0{}^2 \cos^2\theta}{2} - \Delta F^2 y_0 a^2 \sin\theta\cos\theta + \frac{q_D \Delta F a^4 \sin\theta}{4} \\ -\frac{q_D \Delta F l_b a^3 \sin\theta}{3} - \frac{2q_D \Delta F y_0 a^3 \cos\theta}{3} + q_D \Delta F l_b y_0 a^2 \cos\theta - \frac{q_D \Delta F l_b{}^3 y_0 \cos\theta}{12}\end{array}\right\} \\
&+ \frac{\Delta F^2 l_c}{(AE)_c} + \frac{\Delta F^2 (l_b - 2a)\cos^2\theta}{2(AE)_c} + \frac{\Delta F^2 l_b \cos^2\theta}{2(AE)_b}
\end{aligned} \tag{45}$$

The increase in pre-tensioning force of the cable (ΔF) calculated through the method of least work is defined as follows:

$$\begin{aligned}
\Delta F &= \frac{-3q_D a^4 \sin\theta + 4q_D l_b a^3 \sin\theta + 8q_D y_0 a^3 \cos\theta - 12 q_D l_b y_0 a^2 \cos\theta + q_D l_b{}^3 y_0 \cos\theta}{4\left(\begin{array}{l}2a^3\sin^2\theta + 3l_b y_0{}^2 \cos^2\theta - 6y_0 a^2 \sin\theta\cos\theta \\ + \frac{3(EI)_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] + \frac{3I_b l_b \cos^2\theta}{A_b}\end{array}\right)} \\
&= q_D \mu
\end{aligned} \tag{46}$$

In Equation (46), μ is as follows:

$$\mu = \frac{-3a^4 \sin\theta + 4l_b a^3 \sin\theta + 8y_0 a^3 \cos\theta - 12 l_b y_0 a^2 \cos\theta + l_b{}^3 y_0 \cos\theta}{4\left(\begin{array}{l}2a^3\sin^2\theta + 3l_b y_0{}^2 \cos^2\theta - 6y_0 a^2 \sin\theta\cos\theta \\ + \frac{3(EI)_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] + \frac{3I_b l_b \cos^2\theta}{A_b}\end{array}\right)} \tag{47}$$

By replacing Equation (46) in Equation (45), the maximum strain energy of the beam E_{So} is obtained as follows:

$$E_{So}=\frac{{q_D}^2}{240(EI)_b}\left\{\begin{array}{l}{l_b}^5+80\mu^2a^3\sin^2\theta+120\mu^2l_b{y_0}^2\cos^2\theta-240\mu^2y_0a^2\sin\theta\cos\theta\\+60\mu a^4\sin\theta-80\mu l_ba^3\sin\theta-160\mu y_0a^3\cos\theta+\mu l_by_0a^2\cos\theta\\-20\mu{l_b}^3y_0\cos\theta+\frac{120\mu^2(EI)_b}{(AE)_c}\left[2l_c+(l_b-2a)\cos^2\theta\right]\\+\frac{120\mu^2I_bl_b\cos^2\theta}{A_b}\end{array}\right\}$$

$$=\frac{{q_D}^2\beta}{240(EI)_b} \tag{48}$$

In Equation (48) β is as follows:

$$\begin{aligned}\beta&={l_b}^5+80\mu^2a^3\sin^2\theta+120\mu^2l_b{y_0}^2\cos^2\theta-240\mu^2y_0a^2\sin\theta\cos\theta+60\mu a^4\sin\theta\\&-80\mu l_ba^3\sin\theta-160\mu y_0a^3\cos\theta+\mu l_by_0a^2\cos\theta-20\mu{l_b}^3y_0\cos\theta\\&+\frac{120\mu^2(EI)_b}{(AE)_c}\left[2l_c+(l_b-2a)\cos^2\theta\right]+\frac{120\mu^2I_bl_b\cos^2\theta}{A_b}\end{aligned} \tag{49}$$

Regarding the symmetry of structure and loading, the maximum kinematic energy of the simply supported beam with modified v-shaped pattern of cable for the whole beam is obtained as follows:

$$\begin{aligned}E_{Ko}&=2\times\left\{\int_0^a\frac{1}{2}m(x)\left(\dot{u}_{o1}(x)\right)^2dx+\int_a^{\frac{l_b}{2}}\frac{1}{2}m(x)\left(\dot{u}_{o2}(x)\right)^2dx\right\}\\&=\int_0^a m(x)\left(\omega_nu_{o1}(x)\right)^2dx+\int_a^{\frac{l_b}{2}}m(x)\left(\omega_nu_{o2}(x)\right)^2dx\end{aligned} \tag{50}$$

In which $u_{o1}(x)$ and $u_{o2}(x)$ are the deflection curves of the simply supported beam with modified v-shaped pattern of cable for the $0\le x\le a$ and $a\le x\le\frac{l_b}{2}$,respectively.

To determine the deflection curve, the internal bending moment is equal to:

$$M_1(x)=-(EI)_b{u_{o1}}''(x) \tag{51}$$

$$M_2(x)=-(EI)_b{u_{o2}}''(x) \tag{52}$$

The deflection curve in Equations (51) and (52) should satisfy the displacement boundary conditions. For the simply supported beam with the modified v-shaped pattern of cable, the boundary conditions for half of the beam are as follows:

$$u_{o1}(0)=0\quad,\quad {u_{o2}}'(\frac{l_b}{2})=0\quad,\quad u_{o1}(a)=u_{o2}(a)\quad,\quad {u_{o1}}'(a)={u_{o2}}'(a)$$

By replacing Equations (43) and (44) in Equations (51) and (52), and imposing the above boundary conditions, the deflection curve for half of the beam is obtained as follows:

For $0 \le x \le a$;

$$u_{o1}(x) = \frac{1}{(EI)_b}\begin{bmatrix} \frac{q_D {l_b}^3 x}{24} - \frac{\Delta F a^2 \sin\theta x}{2} + 2\Delta F y_0 a \cos\theta x - \frac{\Delta F l_b y_0 \cos\theta x}{2} \\ -\frac{\Delta F y_0 \cos\theta x^2}{2} + \frac{\Delta F \sin\theta x^3}{6} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24} \end{bmatrix} \tag{53}$$

For $a \le x \le \frac{l_b}{2}$;

$$u_{o2}(x) = \frac{1}{(EI)_b}\begin{bmatrix} -\frac{\Delta F a^3 \sin\theta}{3} + \Delta F y_0 a^2 \cos\theta + \frac{q_D {l_b}^3 x}{24} - \frac{\Delta F l_b y_0 \cos\theta x}{2} \\ +\frac{\Delta F y_0 \cos\theta x^2}{2} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24} \end{bmatrix} \tag{54}$$

By replacing Equations (53) and (54) in Equation (50), the maximum kinematic energy of the beam is obtained as follows:

$$E_{Ko}=\omega_n^{2}\frac{q_D}{g}\left\{\begin{aligned}&\int_0^a\left(\frac{1}{(EI)_b}\left[\begin{aligned}&\frac{q_D l_b^{3}x}{24}-\frac{\Delta F a^2\sin\theta x}{2}+2\Delta F y_0 a\cos\theta x-\frac{\Delta F l_b y_0\cos\theta x}{2}\\&-\frac{\Delta F y_0\cos\theta x^2}{2}+\frac{\Delta F\sin\theta x^3}{6}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{aligned}\right]\right)^2 dx\\&+\int_a^{\frac{l_b}{2}}\left(\frac{1}{(EI)_b}\left[\begin{aligned}&-\frac{\Delta F a^3\sin\theta}{3}+\Delta F y_0 a^2\cos\theta+\frac{q_D l_b^{3}x}{24}-\frac{\Delta F l_b y_0\cos\theta x}{2}\\&+\frac{\Delta F y_0\cos\theta x^2}{2}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{aligned}\right]\right)^2 dx\end{aligned}\right\}$$

$$=\omega_n^{2}\frac{q_D}{(EI)_b^{2}g}\left\{\begin{aligned}&\frac{31q_D^{2}l_b^{9}}{725760}-\frac{2\Delta F^2a^7\sin^2\theta}{35}+\frac{\Delta F^2 l_b a^6\sin^2\theta}{18}-\frac{\Delta F^2y_0^{2}a^5\cos^2\theta}{2}\\&+\frac{7\Delta F^2 l_b y_0^{2}a^4\cos^2\theta}{12}-\frac{\Delta F^2 l_b^{3}y_0^{2}a^2\cos^2\theta}{12}+\frac{\Delta F^2 l_b^{5}y_0^{2}\cos^2\theta}{240}\\&+\frac{41\Delta F^2 y_0 a^6\sin\theta\cos\theta}{120}-\frac{11\Delta F^2 l_b y_0 a^5\sin\theta\cos\theta}{30}\\&+\frac{\Delta F^2 l_b^{3}y_0 a^3\sin\theta\cos\theta}{36}+\frac{q_D\Delta F a^8\sin\theta}{2880}-\frac{q_D\Delta F l_b a^7\sin\theta}{840}\\&+\frac{q_D\Delta F l_b^{3}a^5\sin\theta}{360}-\frac{q_D\Delta F l_b^{5}a^3\sin\theta}{360}-\frac{q_D\Delta F y_0 a^7\cos\theta}{1260}\\&+\frac{q_D\Delta F l_b y_0 a^6\cos\theta}{360}-\frac{q_D\Delta F l_b^{3}y_0 a^4\cos\theta}{144}+\frac{q_D\Delta F l_b^{5}y_0 a^2\cos\theta}{120}\\&-\frac{17q_D\Delta F l_b^{7}y_0\cos\theta}{20160}\end{aligned}\right\} \quad (55)$$

By replacing Equation (46) in Equation (55), the maximum kinematic energy of the beam E_{Ko} is obtained as follows:

$$E_{Ko}=\omega_n^{2}\frac{q_D^{3}}{725760(EI)_b^{2}g}\left\{\begin{aligned}&31l_b^{9}-41472\mu^2a^7\sin^2\theta+40320\mu^2l_b a^6\sin^2\theta\\&-362880\mu^2y_0^{2}a^5\cos^2\theta+423360\mu^2l_b y_0^{2}a^4\cos^2\theta\\&-60480\mu^2l_b^{3}y_0^{2}a^2\cos^2\theta+3024\mu^2l_b^{5}y_0^{2}\cos^2\theta\\&+247968\mu^2y_0a^6\sin\theta\cos\theta-266112\mu^2l_b y_0a^5\sin\theta\cos\theta\\&+20160\mu^2l_b^{3}y_0a^3\sin\theta\cos\theta+252\mu a^8\sin\theta-864\mu l_b a^7\sin\theta\\&+2016\mu l_b^{3}a^5\sin\theta-2016\mu l_b^{5}a^3\sin\theta-576\mu y_0a^7\cos\theta\\&+2016\mu l_b y_0a^6\cos\theta-5040\mu l_b^{3}y_0a^4\cos\theta\\&+6048\mu l_b^{5}y_0a^2\cos\theta-612\mu l_b^{7}y_0\cos\theta\end{aligned}\right\} \quad (56)$$

$$=\omega_n^{2}\frac{q_D^{3}\gamma}{725760(EI)_b^{2}g}$$

In Equation (56) γ is as follows;

$$\begin{aligned}
\gamma &= 31l_b^{\,9} - 41472\mu^2 a^7 \sin^2\theta + 40320\mu^2 l_b a^6 \sin^2\theta - 362880\mu^2 y_0^{\,2} a^5 \cos^2\theta \\
&+ 423360\mu^2 l_b y_0^{\,2} a^4 \cos^2\theta - 60480\mu^2 l_b^{\,3} y_0^{\,2} a^2 \cos^2\theta + 3024\mu^2 l_b^{\,5} y_0^{\,2} \cos^2\theta \\
&+ 247968\mu^2 y_0 a^6 \sin\theta\cos\theta - 266112\mu^2 l_b y_0 a^5 \sin\theta\cos\theta \\
&+ 20160\mu^2 l_b^{\,3} y_0 a^3 \sin\theta\cos\theta + 252\mu a^8 \sin\theta - 864\mu l_b a^7 \sin\theta \\
&+ 2016\mu l_b^{\,3} a^5 \sin\theta - 2016\mu l_b^{\,5} a^3 \sin\theta - 576\mu y_0 a^7 \cos\theta + 2016\mu l_b y_0 a^6 \cos\theta \\
&- 5040\mu l_b^{\,3} y_0 a^4 \cos\theta + 6048\mu l_b^{\,5} y_0 a^2 \cos\theta - 612\mu l_b^{\,7} y_0 \cos\theta
\end{aligned} \tag{57}$$

The natural circular frequency of the simply supported beam with modified v-shaped pattern of cable using the Rayleigh's method and the principle of conservation of energy, is obtained as follows:

$$\omega_n^{\,2} = \frac{3024(EI)_b g\beta}{q_D\gamma} \rightarrow \omega_n = 12\sqrt{\frac{21(EI)_b g\beta}{q_D\gamma}} \tag{58}$$

The natural frequency of the simply supported beam with modified v-shaped pattern of cable is obtained as follows:

$$f_n = \frac{\omega_n}{2\pi} = \frac{6}{\pi}\sqrt{\frac{21(EI)_b g\beta}{q_D\gamma}} \tag{59}$$

4.5. Calculating the natural frequency of the fixed supported beam with v-shaped pattern of cable

In the fixed supported beam along with the v-shaped pattern of cable, as shown in Figure 6, the increase in pre-tensioning force of steel cable is equal to ΔF. Therefore, axial force of the beam is equal to $\Delta F\cos\theta$.

Also, regarding the symmetry of structure and loading (as in Figure 6), the maximum strain energy of the fixed supported beam with v-shaped pattern of cable for the whole beam is obtained as follows:

$$E_{So} = 2\times\frac{1}{2(EI)_b}\int_0^{\frac{l_b}{2}} M(x)^2 dx + 2\times\frac{\Delta F^2 l_c}{2(AE)_c} + \frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} \tag{60}$$

In which $M(x)$ is the bending moment of the fixed supported beam with v-shaped pattern of cable for $0 \le x \le \frac{l_b}{2}$.

It should be mentioned that the fixed supported beam along with v-shaped pattern of the cable has two degrees of indeterminacy (the increase in pre-tensioning force of the cable, ΔF and the moment at fixed end, *M*). The increase in the pre-tensioning force of the steel cable and the fixed end moment can be calculated using the method of least work. Thus, the bending moment of fixed supported beam with v-shaped pattern of cable subjected to uniform distributed dead load for the half of beam is obtained as follows:

For $0 \le x \le \frac{l_b}{2}$:

$$M\left(x\right) = -M + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{61}$$

In which *M* is the moment at fixed end.

By replacing Equation (61) in Equation (60), the maximum strain energy formula is obtained as follows:

$$\begin{aligned} E_{So} &= 2\times\frac{1}{2\left(EI\right)_b}\int_0^{\frac{l_b}{2}}\left(-M + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2}\right)^2 dx + 2\times\frac{\Delta F^2 l_c}{2\left(AE\right)_c} \\ &+ \frac{(\Delta F\cos\theta)^2 l_b}{2\left(AE\right)_b} = \frac{1}{\left(EI\right)_b}\left\{\begin{aligned} &\frac{q_D{}^2 l_b{}^5}{240} + \frac{\Delta F^2 l_b{}^3\sin^2\theta}{24} + \frac{\Delta F^2 l_b y_0{}^2\cos^2\theta}{2} \\ &-\frac{\Delta F^2 l_b{}^2 y_0\sin\theta\cos\theta}{4} - \frac{5q_D\Delta F l_b{}^4\sin\theta}{192} + \frac{q_D\Delta F l_b{}^3 y_0\cos\theta}{12} \\ &+\frac{M\Delta F l_b{}^2\sin\theta}{4} - M\Delta F l_b y_0\cos\theta - \frac{q_D M l_b{}^3}{12} + \frac{M^2 l_b}{2} \end{aligned}\right\} \\ &+ \frac{\Delta F^2 l_c}{\left(AE\right)_c} + \frac{\Delta F^2 l_b\cos^2\theta}{2\left(AE\right)_b} \end{aligned} \tag{62}$$

Calculating the moment at fixed end (*M*) through the method of least work, the relation of whole strain energy is differentiated with respect to *M* and the obtained result is equated to zero:

$$\frac{\partial E_{So}}{\partial M} = 0 \tag{63}$$

The calculated bending moment at the fixed end (*M*) is determined as follows:

$$M = \frac{q_D l_b{}^2}{12} - \frac{\Delta F l_b\sin\theta}{4} + \Delta F y_0\cos\theta \tag{64}$$

To calculate the increase in pre-tensioning force of cable (ΔF) through the method of least work, the relation of whole strain energy is differentiated with respect to ΔF and the obtained result is equated to zero:

$$\frac{\partial E_{So}}{\partial(\Delta F)} = 0 \tag{65}$$

The calculated increase in pre-tensioning force of cable (ΔF) is obtained using Equation (64) as follows:

$$\Delta F=\frac{q_D l_b^{\,4}\sin\theta}{4\left(l_b^{\,3}\sin^2\theta+\dfrac{96(EI)_b\,l_c}{(AE)_c}+\dfrac{48I_b l_b\cos^2\theta}{A_b}\right)}=q_D\mu \tag{66}$$

.In Equation (66), μ is as follows:

$$\mu=\frac{l_b^{\,4}\sin\theta}{4\left(l_b^{\,3}\sin^2\theta+\dfrac{96(EI)_b\,l_c}{(AE)_c}+\dfrac{48I_b l_b\cos^2\theta}{A_b}\right)} \tag{67}$$

If $\sin\theta=\dfrac{2y_0}{l_c}$ and $\cos\theta=\dfrac{l_b}{2l_c}$ are replaced in Equation (64) (as in Figure 6), it is observed that the fixed end moment is the fixed end moment of the beam without cable as follows:

$$M=\frac{q_D l_b^{\,2}}{12} \tag{68}$$

Replacing Equations (66) and (68) into Equation (62), the maximum strain energy of the beam E_{So} is obtained as follows:

$$E_{So}=\frac{q_D^{\,2}}{2880(EI)_b}\left\{\begin{aligned}&2l_b^{\,5}+120\mu^2 l_b^{\,3}\sin^2\theta+1440\mu^2 l_b y_0^{\,2}\cos^2\theta-720\mu^2 l_b^{\,2}y_0\sin\theta\cos\theta\\&-15\mu l_b^{\,4}\sin\theta+\frac{2880\mu^2(EI)_b\,l_c}{(AE)_c}+\frac{1440\mu^2 I_b l_b\cos^2\theta}{A_b}\end{aligned}\right\}$$
$$=\frac{q_D^{\,2}\beta}{2880(EI)_b} \tag{69}$$

In Equation (69), β is as follows:

$$\beta=2l_b^{\,5}+120\mu^2 l_b^{\,3}\sin^2\theta+1440\mu^2 l_b y_0^{\,2}\cos^2\theta-720\mu^2 l_b^{\,2}y_0\sin\theta\cos\theta$$
$$-15\mu l_b^{\,4}\sin\theta+\frac{2880\mu^2(EI)_b\,l_c}{(AE)_c}+\frac{1440\mu^2 I_b l_b\cos^2\theta}{A_b} \tag{70}$$

Regarding the symmetry of structure and loading, the maximum kinematic energy of the fixed support beam with v-shaped pattern of cable for the whole beam is obtained as follows:

$$E_{Ko}=2\times\int_0^{\frac{l_b}{2}}\frac{1}{2}m(x)\big(\dot{u}_o(x)\big)^2dx=\int_0^{\frac{l_b}{2}}m(x)\big(\omega_n u_o(x)\big)^2dx \tag{71}$$

In which $u_o(x)$ is the deflection curve of the fixed supported beam with v-shaped pattern of cable for the $0\le x\le\dfrac{l_b}{2}$ range.

The bending moment of the fixed supported beam with v-shaped pattern of the cable subjected to uniform distributed dead load, in order to determine the deflection curve, for the half of beam and by replacing Equation (68) is as follows:

$$M\left(x\right) = -M + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2}$$
$$= -\frac{q_D l_b^{\,2}}{12} + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{72}$$

To determine the deflection curve, the internal bending moment is equal to:

$$M(x) = -(EI)_b u_o''(x) \tag{73}$$

The deflection curve in Equation (73) should satisfy the displacement boundary conditions. For the fixed supported beam with v-shaped pattern of cable, boundary conditions for the half of the beam are:

$$u_o(0) = 0 \quad , \quad u_o'(0) = u_o'(\frac{l_b}{2}) = 0$$

By replacing Equation (72) in Equation (73) and imposing the above boundary conditions, the deflection curve for the half of the beam is obtained as follows:

$$u_o(x) = \frac{1}{(EI)_b}\left[\frac{q_D l_b^{\,2} x^2}{24} - \frac{\Delta F y_0\cos\theta x^2}{2} + \frac{\Delta F\sin\theta x^3}{6} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24}\right] \tag{74}$$

By replacing Equation (74) in Equation (71), the maximum kinematic energy of the beam is obtained as follows:

$$E_{Ko} = \omega_n^{\,2}\frac{q_D}{g}\int_0^{\frac{l_b}{2}}\left(\frac{1}{(EI)_b}\left[\frac{q_D l_b^{\,2} x^2}{24} - \frac{\Delta F y_0\cos\theta x^2}{2} + \frac{\Delta F\sin\theta x^3}{6} - \frac{q_D l_b x^3}{12} + \frac{q_D x^4}{24}\right]\right)^2 dx$$
$$= \omega_n^{\,2}\frac{q_D}{(EI)_b^{\,2} g}\left\{\begin{aligned}&\frac{q_D^{\,2} l_b^{\,9}}{725760} + \frac{\Delta F^2 l_b^{\,7}\sin^2\theta}{32256} + \frac{\Delta F^2 l_b^{\,5} y_0^{\,2}\cos^2\theta}{640} - \frac{\Delta F^2 l_b^{\,6} y_0\sin\theta\cos\theta}{2304}\\&+\frac{37 q_D\Delta F l_b^{\,8}\sin\theta}{3096576} - \frac{29 q_D\Delta F l_b^{\,7} y_0\cos\theta}{322560}\end{aligned}\right\} \tag{75}$$

By replacing Equation (66) in Equation (75), the maximum kinematic energy E_{Ko} formula is obtained as follows:

$$E_{Ko} = \omega_n^{\,2}\frac{q_D^{\,3}}{46448640(EI)_b^{\,2} g}\left\{\begin{aligned}&64 l_b^{\,9} + 1440\mu^2 l_b^{\,7}\sin^2\theta + 72576\mu^2 l_b^{\,5} y_0^{\,2}\cos^2\theta\\&-20160\mu^2 l_b^{\,6} y_0\sin\theta\cos\theta + 555\mu l_b^{\,8}\sin\theta\\&-4176\mu l_b^{\,7} y_0\cos\theta\end{aligned}\right\}$$
$$= \omega_n^{\,2}\frac{q_D^{\,3}\gamma}{46448640(EI)_b^{\,2} g} \tag{76}$$

In Equation (76), γ is as follows,

$$\gamma = 64 l_b^{9} + 1440\mu^2 l_b^{7}\sin^2\theta + 72576\mu^2 l_b^{5} y_0^{2}\cos^2\theta - 20160\mu^2 l_b^{6} y_0 \sin\theta\cos\theta \\ + 555\mu l_b^{8}\sin\theta - 4176\mu l_b^{7} y_0\cos\theta \tag{77}$$

The natural circular frequency of the fixed supported beam with v-shaped pattern of the cable using the Rayleigh's method and the principle of conservation of energy, is obtained as follows:

$$\omega_n^{2} = \frac{16128(EI)_b g\beta}{q_D\gamma} \rightarrow \omega_n = 48\sqrt{\frac{7(EI)_b g\beta}{q_D\gamma}} \tag{78}$$

The natural frequency of the fixed supported beam with v-shaped pattern of the cable is obtained as follows:

$$f_n = \frac{\omega_n}{2\pi} = \frac{24}{\pi}\sqrt{\frac{7(EI)_b g\beta}{q_D\gamma}} \tag{79}$$

4.6. Calculating the natural frequency of fixed supported beam with modified v-shaped pattern of cable

In the fixed supported beam with the modified v-shaped pattern of the cable, as shown in Figure 7, assuming the increase in pre-tensioning force of steel cable to be equal to ΔF in the inclined parts, the increase in pre-tensioning force of the cable in the horizontal part is $\Delta F\cos\theta$. Therefore, axial force of the beam is equal to $\Delta F\cos\theta$. Regarding the symmetry of structure and loading (as in Figure 7), the maximum strain energy of the fixed supported beam with modified v-shaped pattern of cable for the whole beam is obtained as follows:

$$E_{So} = 2\times\frac{1}{2(EI)_b}\left\{\int_0^a M_1(x)^2dx + \int_a^{\frac{l_b}{2}} M_2(x)^2dx\right\} + 2\times\frac{\Delta F^2 l_c}{2(AE)_c} + \frac{(\Delta F\cos\theta)^2(l_b-2a)}{2(AE)_c} \\ + \frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b} \tag{80}$$

In which $M_1(x)$ and $M_2(x)$ are the bending moments of the fixed supported beam with modified v-shaped pattern of the cable for the $0\le x\le a$ and $a\le x\le\frac{l_b}{2}$ ranges.

It should be mentioned that the fixed supported beam with the modified v-shaped pattern of the cable has two degrees of indeterminacy, the increase in pre-tensioning force of the cable (ΔF), and the moment at fixed end (*M*).The increase in the pre-tensioning force of the steel cable and fixed end moment can be calculated using the method of least work. Thus, the bending moment of the fixed supported beam with the modified v-shaped pattern of cable is obtained subjected to uniform distributed dead load for the half of beam as follows:

For the $0\le x\le a$ range;

$$M_1(x) = -M + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{81}$$

For the $a \le x \le \frac{l_b}{2}$ range;

$$M_2(x) = -M - \Delta F\cos\theta y_0 + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{82}$$

By replacing Equations (81) and (82) in Equation (80), the maximum strain energy is obtained as follows:

$$\begin{aligned}
E_{So} &= 2\times\frac{1}{2(EI)_b}\left\{\begin{array}{l}\int_0^a\left(-M+\Delta F\cos\theta y_0-\Delta F\sin\theta x+\frac{q_D l_b x}{2}-\frac{q_D x^2}{2}\right)^2 dx\\ +\int_a^{\frac{l_b}{2}}\left(-M-\Delta F\cos\theta y_0+\frac{q_D l_b x}{2}-\frac{q_D x^2}{2}\right)^2 dx\end{array}\right\}\\
&+2\times\frac{\Delta F^2 l_c}{2(AE)_c}+\frac{(\Delta F\cos\theta)^2(l_b-2a)}{2(AE)_c}+\frac{(\Delta F\cos\theta)^2 l_b}{2(AE)_b}\\
&=\frac{1}{(EI)_b}\left\{\begin{array}{l}\frac{q_D{}^2 l_b{}^5}{240}+\frac{\Delta F^2 a^3\sin^2\theta}{3}+\frac{\Delta F^2 l_b y_0{}^2\cos^2\theta}{2}-\Delta F^2 y_0 a^2\sin\theta\cos\theta+\frac{q_D\Delta F a^4\sin\theta}{4}\\ -\frac{q_D\Delta F l_b a^3\sin\theta}{3}-\frac{2q_D\Delta F y_0 a^3\cos\theta}{3}+q_D\Delta F l_b y_0 a^2\cos\theta-\frac{q_D\Delta F l_b{}^3 y_0\cos\theta}{12}\\ +M\Delta F a^2\sin\theta-4M\Delta F y_0 a\cos\theta+M\Delta F l_b y_0\cos\theta-\frac{q_D M l_b{}^3}{12}+\frac{M^2 l_b}{2}\end{array}\right\}\\
&+\frac{\Delta F^2 l_c}{(AE)_c}+\frac{\Delta F^2(l_b-2a)\cos^2\theta}{2(AE)_c}+\frac{\Delta F^2 l_b\cos^2\theta}{2(AE)_b}
\end{aligned} \tag{83}$$

To calculate the moment at fixed end (*M*) through the method of least work, the relation of whole strain energy is differentiated with respect to *M* and the obtained result is equated to zero:

$$\frac{\partial E_{So}}{\partial M} = 0 \tag{84}$$

The calculated bending moment at the fixed end (*M*) is obtained as follows:

$$M = \frac{q_D l_b{}^2}{12} - \frac{\Delta F a^2\sin\theta}{l_b} + \frac{4\Delta F y_0 a\cos\theta}{l_b} - \Delta F y_0\cos\theta \tag{85}$$

To calculate the increase in pre-tensioning force of cable (ΔF) through the method of least work, the relation of whole strain energy is differentiated with respect to ΔF and the obtained result is equated to zero:

$$\frac{\partial E_{So}}{\partial(\Delta F)} = 0 \tag{86}$$

The calculated increase in pre-tensioning force of the cable (ΔF) is obtained using the Equation (85) as follows:

$$\Delta F = \frac{\begin{gathered}-3q_D l_b a^4 \sin\theta + 4q_D {l_b}^2 a^3 \sin\theta - q_D {l_b}^3 a^2 \sin\theta \\ +8q_D l_b y_0 a^3 \cos\theta - 12 q_D {l_b}^2 y_0 a^2 \cos\theta + 4q_D {l_b}^3 y_0 a \cos\theta\end{gathered}}{4\left(\begin{gathered}-3a^4 \sin^2\theta + 2l_b a^3 \sin^2\theta - 48{y_0}^2 a^2 \cos^2\theta + 24 l_b {y_0}^2 a\cos^2\theta + 24 y_0 a^3 \sin\theta\cos\theta \\ -12 l_b y_0 a^2 \sin\theta\cos\theta + \frac{3l_b (EI)_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] + \frac{3I_b {l_b}^2 \cos^2\theta}{A_b}\end{gathered}\right)} \tag{87}$$

$$= q_D \mu$$

In Equation (87), μ is as follows,

$$\mu = \frac{-3l_b a^4 \sin\theta + 4{l_b}^2 a^3 \sin\theta - {l_b}^3 a^2 \sin\theta + 8 l_b y_0 a^3 \cos\theta - 12 {l_b}^2 y_0 a^2 \cos\theta + 4{l_b}^3 y_0 a\cos\theta}{4\left(\begin{gathered}-3a^4 \sin^2\theta + 2l_b a^3 \sin^2\theta - 48{y_0}^2 a^2 \cos^2\theta + 24 l_b {y_0}^2 a\cos^2\theta + 24 y_0 a^3 \sin\theta\cos\theta \\ -12 l_b y_0 a^2 \sin\theta\cos\theta + \frac{3l_b (EI)_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] + \frac{3I_b {l_b}^2 \cos^2\theta}{A_b}\end{gathered}\right)} \tag{88}$$

By replacing Equations (85) and (87) in Equation (83), the maximum strain energy of the beam E_{So} is obtained as follows:

$$E_{So} = \frac{{q_D}^2}{1440(EI)_b\, l_b}\left\{\begin{aligned}&{l_b}^6 - 720\mu^2 a^4 \sin^2\theta + 480\mu^2 l_b a^3 \sin^2\theta - 11520\mu^2 {y_0}^2 a^2 \cos^2\theta \\ &+5760\mu^2 l_b {y_0}^2 a\cos^2\theta + 5760\mu^2 y_0 a^3 \sin\theta\cos\theta \\ &-2880\mu^2 l_b y_0 a^2 \sin\theta\cos\theta + 360\mu l_b a^4 \sin\theta - 480\mu {l_b}^2 a^3 \sin\theta \\ &+120\mu {l_b}^3 a^2 \sin\theta - 960\mu l_b y_0 a^3 \cos\theta + 1440\mu {l_b}^2 y_0 a^2 \cos\theta \\ &-480\mu {l_b}^3 y_0 a\cos\theta + \frac{720\mu^2 (EI)_b\, l_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] \\ &+\frac{720\mu^2 I_b {l_b}^2 \cos^2\theta}{A_b}\end{aligned}\right\} \tag{89}$$

$$= \frac{{q_D}^2 \beta}{1440(EI)_b\, l_b}$$

In Equation (89), β is as follows,

$$\beta = l_b^{\,6} - 720\mu^2 a^4 \sin^2\theta + 480\mu^2 l_b a^3 \sin^2\theta - 11520\mu^2 y_0^{\,2} a^2 \cos^2\theta + 5760\mu^2 l_b y_0^{\,2} a \cos^2\theta$$
$$+5760\mu^2 y_0 a^3 \sin\theta\cos\theta - 2880\mu^2 l_b y_0 a^2 \sin\theta\cos\theta + 360\mu l_b a^4 \sin\theta - 480\mu l_b^{\,2} a^3 \sin\theta$$
$$+120\mu l_b^{\,3} a^2 \sin\theta - 960\mu l_b y_0 a^3 \cos\theta + 1440\mu l_b^{\,2} y_0 a^2 \cos\theta - 480\mu l_b^{\,3} y_0 a \cos\theta \tag{90}$$
$$+\frac{720\mu^2 (EI)_b l_b}{(AE)_c}\left[2l_c + (l_b - 2a)\cos^2\theta\right] + \frac{720\mu^2 I_b l_b^{\,2}\cos^2\theta}{A_b}$$

Regarding symmetry of the structure and loading, the maximum kinematic energy of the fixed supported beam with modified v-shaped pattern of cable for the whole beam is obtained as follows:

$$E_{Ko} = 2\times\left\{\int_0^a \frac{1}{2} m(x)\left(\dot{u}_{o1}(x)\right)^2 dx + \int_a^{\frac{l_b}{2}} \frac{1}{2} m(x)\left(\dot{u}_{o2}(x)\right)^2 dx\right\}$$
$$= \int_0^a m(x)\left(\omega_n u_{o1}(x)\right)^2 dx + \int_a^{\frac{l_b}{2}} m(x)\left(\omega_n u_{o2}(x)\right)^2 dx \tag{91}$$

In which $u_{o1}(x)$ and $u_{o2}(x)$ are the deflection curves of the fixed supported beam with modified v-shaped pattern of the cable for the $0 \le x \le a$ and $a \le x \le \frac{l_b}{2}$ ranges, respectively.

By replacing Equation (85) in Equations (81) and (82), the bending moment of the fixed supported beam with modified v-shaped pattern of cable under uniform distributed dead load in order to determine the deflection curve for the half of beam is obtained as follows:

For the $0 \le x \le a$ range;

$$M_1(x) = -M + \Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2}$$
$$= -\frac{q_D l_b^{\,2}}{12} + \frac{\Delta F a^2 \sin\theta}{l_b} - \frac{4\Delta F y_0 a\cos\theta}{l_b} + 2\Delta F\cos\theta y_0 - \Delta F\sin\theta x + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{92}$$

For the $a \le x \le \frac{l_b}{2}$ range;

$$M_2(x) = -M - \Delta F\cos\theta y_0 + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2}$$
$$= -\frac{q_D l_b^{\,2}}{12} + \frac{\Delta F a^2 \sin\theta}{l_b} - \frac{4\Delta F y_0 a\cos\theta}{l_b} + \frac{q_D l_b x}{2} - \frac{q_D x^2}{2} \tag{93}$$

Determining the deflection curve, the internal bending is equal to:

$$M_1(x) = -(EI)_b u_{o1}''(x) \tag{94}$$

$$M_2(x) = -(EI)_b u_{o2}''(x) \tag{95}$$

The deflection curve in Equations (94) and (95) should satisfy the displacement boundary conditions. For the fixed supported beam with the modified v-shaped pattern of cable, the boundary conditions for the middle beam are:

$$u_{o1}(0)=0 \quad , \quad u_{o1}{}'(0)=u_{o2}{}'(\frac{l_b}{2})=0 \quad , \quad u_{o1}(a)=u_{o2}(a) \quad , \quad u_{o1}{}'(a)=u_{o2}{}'(a)$$

By replacing Equations (92) and (93) in Equations (94) and (95) and imposing the above boundary conditions, the deflection curve for half of beam is obtained as follows:

For the $0 \le x \le a$ range;

$$u_{o1}(x)=\frac{1}{(EI)_b}\left[\begin{array}{l}\frac{q_D l_b{}^2 x^2}{24}-\frac{\Delta F a^2 \sin\theta x^2}{2l_b}+\frac{2\Delta F y_0 a\cos\theta x^2}{l_b}-\Delta F y_0 \cos\theta x^2 \\ +\frac{\Delta F \sin\theta x^3}{6}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{array}\right] \tag{96}$$

For the $a \le x \le \frac{l_b}{2}$ range;

$$u_{o2}(x)=\frac{1}{(EI)_b}\left[\begin{array}{l}-\frac{\Delta F a^3 \sin\theta}{3}+\Delta F y_0 a^2 \cos\theta+\frac{\Delta F a^2 \sin\theta x}{2}-2\Delta F y_0 a\cos\theta x \\ -\frac{\Delta F a^2 \sin\theta x^2}{2l_b}+\frac{2\Delta F y_0 a\cos\theta x^2}{l_b}+\frac{q_D l_b{}^2 x^2}{24}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{array}\right] \tag{97}$$

By replacing Equations (96) and (97) in Equation (91), the maximum kinematic energy formula of the beam is obtained as follows:

$$E_{Ko}=\omega_n^{2}\frac{q_D}{g}\left\{\begin{array}{l}\displaystyle\int_0^a\left(\frac{1}{(EI)_b}\left[\begin{array}{l}\displaystyle\frac{q_D l_b^{2}x^2}{24}-\frac{\Delta Fa^2\sin\theta x^2}{2l_b}+\frac{2\Delta Fy_0 a\cos\theta x^2}{l_b}-\Delta Fy_0\cos\theta x^2\\ \displaystyle+\frac{\Delta F\sin\theta x^3}{6}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{array}\right]\right)^2 dx\\ \displaystyle+\int_a^{\frac{l_b}{2}}\left(\frac{1}{(EI)_b}\left[\begin{array}{l}\displaystyle-\frac{\Delta Fa^3\sin\theta}{3}+\Delta Fy_0a^2\cos\theta+\frac{\Delta Fa^2\sin\theta x}{2}-2\Delta Fy_0a\cos\theta x\\ \displaystyle-\frac{\Delta Fa^2\sin\theta x^2}{2l_b}+\frac{2\Delta Fy_0a\cos\theta x^2}{l_b}+\frac{q_D l_b^{2}x^2}{24}-\frac{q_D l_b x^3}{12}+\frac{q_D x^4}{24}\end{array}\right]\right)^2 dx\end{array}\right\}$$

$$=\omega_n^{2}\frac{q_D}{(EI)_b^{2}g}\left\{\begin{array}{l}\displaystyle\frac{q_D^{2}l_b^{9}}{725760}-\frac{\Delta F^2a^8\sin^2\theta}{72l_b}-\frac{\Delta F^2a^7\sin^2\theta}{42}+\frac{\Delta F^2l_ba^6\sin^2\theta}{18}-\frac{\Delta F^2l_b^{2}a^5\sin^2\theta}{36}\\ \displaystyle+\frac{\Delta F^2l_b^{3}a^4\sin^2\theta}{240}-\frac{2\Delta F^2y_0^{2}a^6\cos^2\theta}{15l_b}-\frac{2\Delta F^2y_0^{2}a^5\cos^2\theta}{15}+\frac{\Delta F^2l_by_0^{2}a^4\cos^2\theta}{2}\\ \displaystyle-\frac{\Delta F^2l_b^{2}y_0^{2}a^3\cos^2\theta}{3}+\frac{\Delta F^2l_b^{3}y_0^{2}a^2\cos^2\theta}{15}+\frac{4\Delta F^2y_0a^7\sin\theta\cos\theta}{45l_b}\\ \displaystyle+\frac{\Delta F^2y_0a^6\sin\theta\cos\theta}{9}-\frac{\Delta F^2l_by_0a^5\sin\theta\cos\theta}{3}+\frac{7\Delta F^2l_b^{2}y_0a^4\sin\theta\cos\theta}{36}\\ \displaystyle-\frac{\Delta F^2l_b^{3}y_0a^3\sin\theta\cos\theta}{30}+\frac{q_D\Delta Fa^8\sin\theta}{2880}-\frac{q_D\Delta Fl_ba^7\sin\theta}{840}+\frac{q_D\Delta Fl_b^{2}a^6\sin\theta}{864}\\ \displaystyle-\frac{q_D\Delta Fl_b^{5}a^3\sin\theta}{2160}+\frac{q_D\Delta Fl_b^{6}a^2\sin\theta}{6720}-\frac{q_D\Delta Fy_0a^7\cos\theta}{1260}+\frac{q_D\Delta Fl_by_0a^6\cos\theta}{360}\\ \displaystyle-\frac{q_D\Delta Fl_b^{2}y_0a^5\cos\theta}{360}+\frac{q_D\Delta Fl_b^{5}y_0a^2\cos\theta}{720}-\frac{q_D\Delta Fl_b^{6}y_0a\cos\theta}{1680}\end{array}\right\} \qquad (98)$$

By replacing Equation (87) in Equation (98), the maximum kinematic energy of the beam E_{Ko} is obtained as follows:

$$E_{Ko} = \omega_n^{\ 2} \frac{q_D^{\ 3}}{725760(EI)_b^{\ 2} l_b g} \left\{ \begin{array}{l} l_b^{10} - 10080\mu^2 a^8 \sin^2\theta - 17280\mu^2 l_b a^7 \sin^2\theta + 40320\mu^2 l_b^{\ 2} a^6 \sin^2\theta \\ -20160\mu^2 l_b^{\ 3} a^5 \sin^2\theta + 3024\mu^2 l_b^{\ 4} a^4 \sin^2\theta - 96768\mu^2 y_0^{\ 2} a^6 \cos^2\theta \\ -96768\mu^2 l_b y_0^{\ 2} a^5 \cos^2\theta + 362880\mu^2 l_b^{\ 2} y_0^{\ 2} a^4 \cos^2\theta \\ -241920\mu^2 l_b^{\ 3} y_0^{\ 2} a^3 \cos^2\theta + 48384\mu^2 l_b^{\ 4} y_0^{\ 2} a^2 \cos^2\theta \\ +64512\mu^2 y_0 a^7 \sin\theta\cos\theta + 80640\mu^2 l_b y_0 a^6 \sin\theta\cos\theta \\ -241920\mu^2 l_b^{\ 2} y_0 a^5 \sin\theta\cos\theta + 141120\mu^2 l_b^{\ 3} y_0 a^4 \sin\theta\cos\theta \\ -24192\mu^2 l_b^{\ 4} y_0 a^3 \sin\theta\cos\theta + 252\mu l_b a^8 \sin\theta - 864\mu l_b^{\ 2} a^7 \sin\theta \\ +840\mu l_b^{\ 3} a^6 \sin\theta - 336\mu l_b^{\ 6} a^3 \sin\theta + 108\mu l_b^{\ 7} a^2 \sin\theta \\ -576\mu l_b y_0 a^7 \cos\theta + 2016\mu l_b^{\ 2} y_0 a^6 \cos\theta - 2016\mu l_b^{\ 3} y_0 a^5 \cos\theta \\ +1008\mu l_b^{\ 6} y_0 a^2 \cos\theta - 432\mu l_b^{\ 7} y_0 a \cos\theta \end{array} \right\} \tag{99}$$

$$= \omega_n^{\ 2} \frac{q_D^{\ 3}\gamma}{725760(EI)_b^{\ 2} l_b g}$$

In Equation (99), γ is as follows;

$$\begin{aligned} \gamma = {} & l_b^{10} - 10080\mu^2 a^8 \sin^2\theta - 17280\mu^2 l_b a^7 \sin^2\theta + 40320\mu^2 l_b^{\ 2} a^6 \sin^2\theta \\ & -20160\mu^2 l_b^{\ 3} a^5 \sin^2\theta + 3024\mu^2 l_b^{\ 4} a^4 \sin^2\theta - 96768\mu^2 y_0^{\ 2} a^6 \cos^2\theta \\ & -96768\mu^2 l_b y_0^{\ 2} a^5 \cos^2\theta + 362880\mu^2 l_b^{\ 2} y_0^{\ 2} a^4 \cos^2\theta - 241920\mu^2 l_b^{\ 3} y_0^{\ 2} a^3 \cos^2\theta \\ & +48384\mu^2 l_b^{\ 4} y_0^{\ 2} a^2 \cos^2\theta + 64512\mu^2 y_0 a^7 \sin\theta\cos\theta + 80640\mu^2 l_b y_0 a^6 \sin\theta\cos\theta \\ & -241920\mu^2 l_b^{\ 2} y_0 a^5 \sin\theta\cos\theta + 141120\mu^2 l_b^{\ 3} y_0 a^4 \sin\theta\cos\theta - 24192\mu^2 l_b^{\ 4} y_0 a^3 \sin\theta\cos\theta \\ & +252\mu l_b a^8 \sin\theta - 864\mu l_b^{\ 2} a^7 \sin\theta + 840\mu l_b^{\ 3} a^6 \sin\theta - 336\mu l_b^{\ 6} a^3 \sin\theta \\ & +108\mu l_b^{\ 7} a^2 \sin\theta - 576\mu l_b y_0 a^7 \cos\theta + 2016\mu l_b^{\ 2} y_0 a^6 \cos\theta - 2016\mu l_b^{\ 3} y_0 a^5 \cos\theta \\ & +1008\mu l_b^{\ 6} y_0 a^2 \cos\theta - 432\mu l_b^{\ 7} y_0 a \cos\theta \end{aligned} \tag{100}$$

The natural circular frequency of the fixed supported beam with modified v-shaped pattern of the cable using the Rayleigh's method and the principle of conservation of energy, is obtained as follows:

$$\omega_n^{\ 2} = \frac{504(EI)_b g\beta}{q_D\gamma} \rightarrow \omega_n = 6\sqrt{\frac{14(EI)_b g\beta}{q_D\gamma}} \tag{101}$$

The natural frequency of fixed supported beam with modified v-shaped pattern of the cable is obtained as follows:

$$f_n = \frac{\omega_n}{2\pi} = \frac{3}{\pi}\sqrt{\frac{14(EI)_b g\beta}{q_D\gamma}} \tag{102}$$

5. Finite element modeling of steel beams pre-stressed with steel cable

Simply supported and fixed supported beams have been designed based on Load and Resistance Factor Design (LRFD) method using AISC360-10 code [20].Then the natural frequency of the simply supported beam is obtained based on the assumed shape function and shape function resulting from the elastic deflection curve corresponding to Equations (10) and (16) and natural frequency of the fixed supported beam is obtained based on the assumed shape function and shape function resulting from the elastic deflection curve corresponding to Equations (19) and (25), respectively. The beams are designed such that their natural frequency is smaller than the minimum permissible frequency of 5 Hz. Table 1 shows the beam properties with different support conditions and their natural frequencies based on different assumed shape functions. It should be noted that the length of loading span is 1.5 m for the beams with different support conditions; dead and live loads are 450 and 200 kg/m^2 respectively.

The beams with different support conditions without cables and with different patterns of cables were modeled in ABAQUS finite element software. Figure 8 shows the finite element model of the beam with different patterns of cables. The beams and cables have been modeled in 3-dimentional coordinate with shell and truss elements (as wire), respectively. The weld's connector is used to connect the cable to one of flange of the beam at two ends to constrain their corresponding degree of freedom. Moreover, coupling constraint is used to connect the cable to the another flange of the beam so as to model the deviator's performance. Uniform distributed load is applied as a surface traction type on the top flange. Predefined field tool is used to create the initial pre-tensioning stress in the cable as well. In this research, mesh size was used as 5 percent of beam length. Figure 9 shows the position of cables in beams with different support conditions.

For better presenting the behavior of beam with different support conditions and different patterns of cable, first it has been modeled in the software without cable, and then with different patterns of cable; and the obtained results have been compared with each other.

The steel material of beams considered in this research is ST-37; yield stress is 240MPa; modulus of elasticity of steel is 200 GPa; and Poisson's ratio is 0.3. The material of steel cable is in accordance with the ASTM A416M standard [21]. 7-wire strand (grade 270 (1860)) is considered for steel cable with low relaxation, minimum ultimate strength (f_{pu}) of 270 ksi (1860 MPa), minimum yield strength at 1% extension of 52.74 kip (234.6 KN), elasticity module of 28.5×10^{6}psi (196501.8 MPa) and Poisson's ratio of 0.3.

6. Verification of theoretical relations of natural frequency with results of ABAQUS models

Frequency analysis of ABAQUS software has been employed so as to analyze the beams with different support conditions (Table 1), without cable and with different patterns of cable. The 7-wire strand steel cable with low relaxation is considered for beams with different support conditions as two cables on each side of the beam web with a cross section area of 140 mm^2 in accordance with ASTM A416M standards. As a result, the entire steel cable cross-section is equal to 560 mm^2. Pre-tensioning of the steel cable is considered as 600 MPa. Controlling the accuracy of theoretical relations, natural frequency obtained from modeling is compared to those of the theoretical relations for the beams with different support conditions and different patterns of cable. The results of natural frequency obtained from modeling are compared: 1) with those of Equations 10, 16, 41 and 59 for simply supported beams without cable and with different patterns of cable; 2) with those of Equations19, 25, 79 and 102 for fixed supported beams without cable and with different patterns of cable, Table 2.

As it is observed from Table 2, the theoretical relations well predict the natural frequency of the beam Also, it is observed that the natural frequency of the beam has increased when the pre-tensioned steel cable is used compared to that of the beam without cable; therefore, using the cable increases the natural frequency of the beam with different support conditions. Also, the natural frequency of the simply supported and fixed supported beam with the modified v-shaped pattern of cable is greater than the v-shaped pattern of cable. As a result, the modified v-shaped pattern of cable is proposed as a more appropriate pattern compared to the v-shaped pattern of cable due to more suitable results observed.

7. The effects of horizontal cable length on natural frequency of simply supported and fixed supported beams along with the modified v-shaped pattern of cable

Equations (59) and (102) are considered to calculate the natural frequency of simply supported and fixed supported beams along with the modified v-shaped pattern of cable according to Table 1 for simply supported and fixed supported beams, 560 mm^2 for total cross-section of steel cable, and for different values of the horizontal cable length ($l_b - 2a$ of Figures 5 and 7). Figures 10 and 11 depict the curves of natural frequency for simply supported and fixed supported beams along with the modified v-shaped pattern of cable for various lengths of the horizontal cable.

According to Figures 10 and 11, if the horizontal cable length is zero in the simply supported as well as fixed supported beams along with the modified v-shaped pattern of cable, their natural frequencies are 4.84 Hz and 4.86 Hz, respectively. These values are the results of natural frequency of simply supported and fixed supported beams along with the v-shaped pattern of cable (Table 2). Natural frequency increases with an increase in the horizontal cable length. Finally, for horizontal cable lengths of 3.9 m and 3.4 m, natural frequencies are maximum; 5.193 Hz and 4.896 Hz, respectively, in the simply supported and fixed supported beam along with the modified v-shaped pattern of cable. Since then, natural frequency of the beam reduces with an increase in the horizontal cable length. The mentioned values of natural frequency are 4.81 Hz and 4.78 Hz, respectively, when the lengths of horizontal cable and beam are alike. These values are the results of natural frequency of simply supported and fixed supported beams without cable (Table 2). The reason is that for keeping the bending moment in the slope change region of the cable continuous, the force of horizontal cable should be equal to the horizontal component of the inclined cable force. Therefore, if the inclined cable becomes vertical in its special status (in the case the horizontal cable length is equal to that of the beam length), the horizontal component of the vertical cable force becomes equal to zero; and consequently, the force of the horizontal cable becomes zero. As the length of the vertical cable, which is equal to the distance between two flanges of the beam remains constant, no force is created in the length of the cable. Therefore, the cable has no effect on the beam behavior, and the beam natural frequency is exactly similar to that of the beam without cable.

8. Sensitivity analysis on the cross-section of steel cable

In sensitivity analysis on the cross-section of the steel cable, different amounts of the 7-wire strand steel cable cross-section with low relaxation have been considered for beams with different support conditions as an equal number of cables on both sides of the beam web with an area of 140 mm^2 in accordance with ASTM A416 standard and stable pre-tensioned stress of 600 MPa. Tables 3 and 4 present the natural frequency in the beams with different support conditions and different patterns of cable modeled in ABAQUS software for different cross-sections of the steel cable.

According to Tables 3 and 4, natural frequency is increased in the beams with different support conditions and different patterns of cable with the increase in steel cable cross-section area due to the increase in stiffness in the beam along with cable.

9. Sensitivity analysis on the pre-tensioning stress of the steel cable

In sensitivity analysis on the pre-tensioning stress of the steel cable, 7-wire strand steel cable with low relaxation for beams with different support conditions is used in the form of four cables on each side of the beam web with an area of 140 mm^2 in accordance to the ASTM A16M standard. As a result, the overall steel cable cross-section is equal to 1120 mm^2. Table 5 presents the values of natural frequency of the beams with different support conditions and various patterns of cable modeled in ABAQUS software, for different values of pre-tensioning of the steel cable.

As it is observed in Table 5, the natural frequency of beams with different support conditions and different patterns of cable remain stable with an increase in the pre-tensioning stress of steel cable.

10. Conclusion

Cables, due to their low weights, small cross sections and high tensile strengths, are reckoned as proper alternatives for pre-tensioning the steel beams subjected to external loads. In this research, cables are employed to pre-stress the beams with different support conditions in which the natural frequency is not within the allowable range, despite appropriate design under bending and shear. Theoretical equations have been derived to calculate the increase in pre-tensioning force of the cable as well as the natural frequency of simply supported and fixed supported beams with and without cable. The results obtained from the finite element model and theoretical equations, are briefly summarized as follows:

1. The moment at fixed end in fixed supported beam with the v-shaped pattern of cable is equal to the moment at fixed end in the beam without cable ($\frac{ql_b^2}{12}$) but in the fixed supported beam along with the modified v-shaped pattern of cable, the moment at fixed end depends on external loading and total force of the cable as well.
2. Comparing the results obtained from theoretical equations and those of finite element model demonstrates that the theoretical equations developed in this article can properly predict the natural frequency of simply supported and fixed supported beams without cable and along with different patterns of cable;
3. Adding cable to the beam results in increasing the natural frequency of the beam with different support conditions and different patterns of cable;
4. The natural frequency is more in the simply supported and fixed supported beams along with the modified v-shaped pattern of cable, compared to that of v-shaped pattern. Therefore, modified v-shaped pattern of cable is recommended as more appropriate one;
5. The effects of horizontal cable length on the natural frequency of simply supported and fixed supported beams along with the modified v-shaped pattern of cable have been studied. According to the obtained results, if the length of horizontal cable is equal to zero, the natural frequency of beam along with the v-shaped pattern of cable is obtained. If the length of horizontal cable increases, natural frequency increases. By increasing in the length of horizontal cable, natural frequency decreases as well. When the lengths of horizontal cable and beam are equal, the natural frequency results of simply supported and fixed supported beams without cable are obtained;
6. In beams with different support conditions and different patterns of cable, the natural frequency is increased by increasing the cross section of steel cable, considering equal pre-tensioning. Moreover, proper values of steel cable cross sections are obtained, as per which the natural frequency criterion of beam with different support conditions and different patterns of cable is satisfied.
7. By increasing in pre-tensioning in the steel cables of equal cross-sections, the natural frequency is constant in the beams with different support conditions and different patterns of cable.

References

[1] Razavi, M. and Sheidaii, M.R. "Seismic performance of cable zipper-braced frames", Journal of constructional steel research, 74, pp. 49-57 (2012).

[2] Hou, X. and Tagawa, H. "Displacement-restraint bracing for seismic retrofit of steel moment frames", Journal of constructional steel research, 65, pp. 1096-104 (2009).

[3] Fanaie, N., Aghajani, S. and Afsar Dizaj, E. "Theoretical assessment of the behavior of cable bracing system with central steel cylinder", Advances in structural engineering, 19(3), pp. 463-72 (2016).

[4] Fanaie, N., Aghajani, S. and Afsar Dizaj, E. "Strengthening of moment-resisting frame using cable–cylinder bracing", Advances in structural engineering, 19(11), pp. 1-19 (2016).

[5] Giaccu, G.F. "An equivalent frequency approach for determining non-linear effects on pre-tensioned-cable cross-braced structures", Journal of sound and vibration, 422, pp. 62-78 (2018).

[6] Troitsky, M.S. "Prestressed steel bridges-Theory and design", Van Nostrand Reinhold, New York (1990).

[7] Le, T.D., Pham, T.M., Hao, H. et al. "Flexural behaviour of precast segmental concrete beams internally prestressed with unbonded CFRP tendons under four-point loading", Engineering structures, 168, pp. 371-83 (2018).

[8] Pisani, M.A. "Behaviour under long-term loading of externally prestressed concrete beams", Engineering structures, 160, pp. 24-33 (2018).

[9] Lou, T., Lopes, S.M.R. and Lopes, A.V. "Effect of linear transformation on nonlinear behavior of continuous prestressed beams with external FRP cables", Engineering structures, 147, pp. 410-24 (2017).

[10] Nie, J.G., Cai, C.S., Zhou, T.R. et al. "Experimental and analytical study of prestressed steel–concrete composite beams considering slip effect", Journal of structural engineering, 133(4), pp. 530-40 (2007).

[11] Zhou, H., Li, S., Chen, L. et al. "Fire tests on composite steel-concrete beams prestressed with external tendons", Journal of constructional steel research, 143, pp. 62-71 (2018).

[12] Belletti, B. and Gasperi, A. "Behavior of prestressed steel beams", Journal of structural engineering, 136(9), pp. 1131-9 (2010).

[13] Park, S., Kim, T., Kim, K. et al. "Flexural behavior of steel I-beam prestressed with externally unbonded tendons", Journal of constructional steel research, 66, pp.125-32 (2010).

[14] Kambal, M.E.M. and Jia, Y. "Theoretical and experimental study on flexural behavior of prestressed steel plate girders", Journal of constructional steel research, 142, pp. 5-16 (2018).

[15] Zhang, W.F. "Symmetric and antisymmetric lateral-torsional buckling of prestressed steel I-beams", Thin-walled structures, 122, pp. 463-79 (2018).

[16] Noble, D., Nogal, M., O'Connor, A. et al. "Dynamic impact testing on post-tensioned steel rectangular hollow sections; An investigation into the "compression-softening" effect", Journal of sound and vibration, 355, pp. 246-63 (2015).

[17] Cao, L., Liu, J. and Chen, Y.F. "Vibration performance of arch prestressed concrete truss girder under impulse excitation", Engineering structures, 165, pp. 386-95 (2018).

[18] Miyamoto, A., Tei, K., Nakamura, H. et al. "Behavior of prestressed beam strengthened with external tendons", Journal of structural engineering, 126(9), pp. 1033-44 (2000).

[19] Park, Y.H., Park, C. and Park, Y.G. “The behavior of an in-service plate girder bridge strengthened with external prestressing tendons”, Engineering structures, 27, pp. 379-86 (2005).

[20] American Institute of Steel Construction (AISC) ANSI/AISC360–10, Specification for structural steel buildings, Chicago, IL (2010).

[21] American Society for Testing and Materials (ASTM), Standard specification for low-relaxation, seven-wire steel strand for prestressed concrete (ASTM A416M), Philadelphia, Pa.

Biographies

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Fatemeh Partovi received her BS and MS degrees in Civil engineering from the Department of Civil Engineering at K. N. Toosi University of Technology, Tehran, Iran. She graduated in 2017. She achieved 4th and 2nd rank respectively in BS and MS degrees. Her field of research includes pre-tensioning beam, reduced beam section (RBS) and seismic evaluation and retrofitting of wood construction. She has published 2 conference papers in 3rd International Conference on Steel and Structure, Tehran, Iran.

Figure captions:

Figure 1: Pre-stressed symmetric I-shaped steel beams with steel cable under external loading: a) simply supported beam along with v- shaped cable pattern; b) simply supported beam along with modified v- shaped cable pattern; c) fixed supported beam along with v-shaped cable pattern; d) fixed supported beam along with modified v-shaped cable pattern

Figure 2: (a) Deflection curve of simply supported beam; (b) Simply supported beam subjected to vertical structural weight

Figure 3: (a). Deflection curve of fixed supported beam; (b) fixed supported beam subjected to vertical structural weight

Figure 4: Simply supported beam with v-shaped pattern of cable

Figure 5: Simply supported beam along with the modified v-shaped pattern of cable

Figure 6: Fixed supported beam along with the v-shaped pattern of cable

Figure 7: Fixed supported beam along with the modified v-shaped pattern of beam

Figure 8: Finite element model of the beam along with different patterns of cable: a) simply supported beam along with the v-shaped pattern of cable; b) simply supported beam along with the modified v-shaped pattern of cable; c)fixed supported beam along with the v-shaped pattern of cable; d) fixed supported beam along with the modified v-shaped pattern of cable

Figure 9: The locations of cables in the beams: a) simply supported beam along with the v-shaped pattern of cable; b) simply supported beam along with the modified v-shaped pattern of cable; c) fixed supported beam along with the v-shaped pattern of cable; d) fixed supported beam along with the modified v-shaped pattern of cable

Figure 10: Natural frequency of simply supported beam along with the modified v-shaped pattern of cable for different values of horizontal cable length

Figure 11: Natural frequency of fixed supported beam along with the modified v-shaped pattern of cable for different values of horizontal cable length

Table captions:

Table 1: Properties and natural frequency of beams with different support conditions

Table 2: Natural frequency values obtained from modeling and theoretical equations for the beams with different support conditions without cable and with different cable patterns

Table 3: Natural frequency results of simply supported beam along with different patterns of cable in sensitivity analysis on cross-section area of steel cable

Table 4: Natural frequency results of fixed supported beam along with different patterns of cable in sensitivity analysis on cross-section area of steel cable

Table 5: Natural frequency results of beams with different support conditions and different cable patterns in sensitivity analysis on the cable pre-tensioning stress

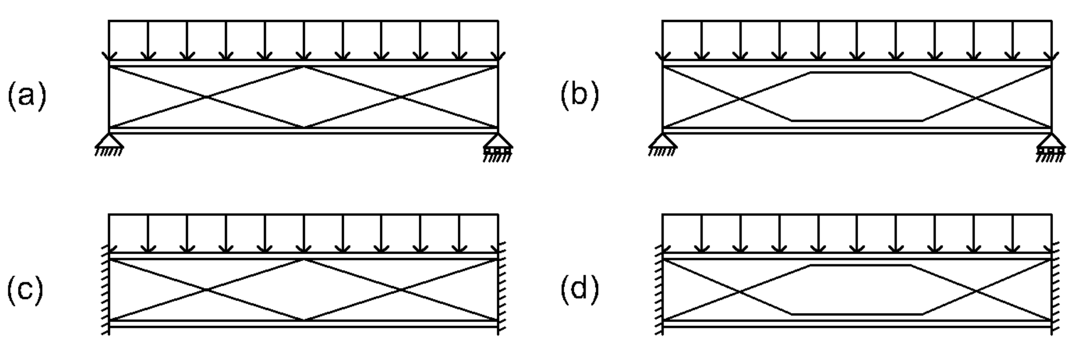
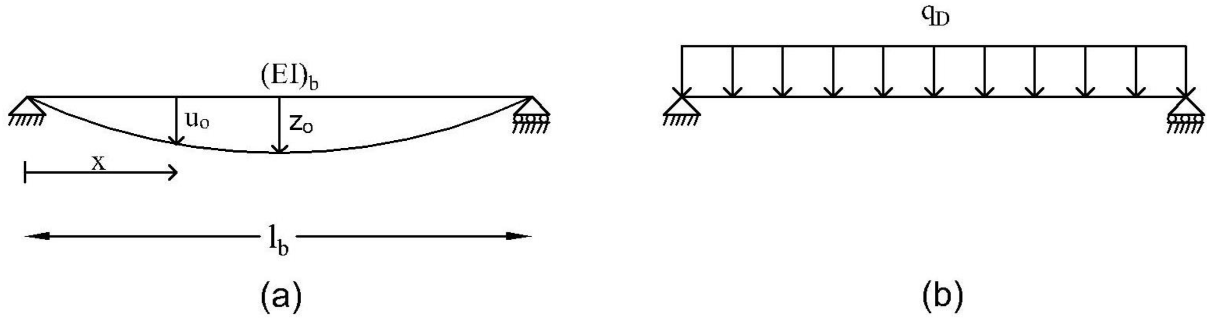


Figure 1



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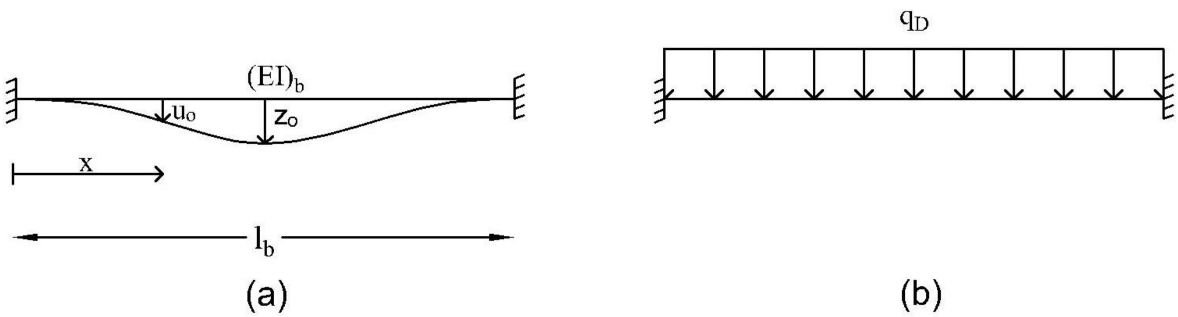

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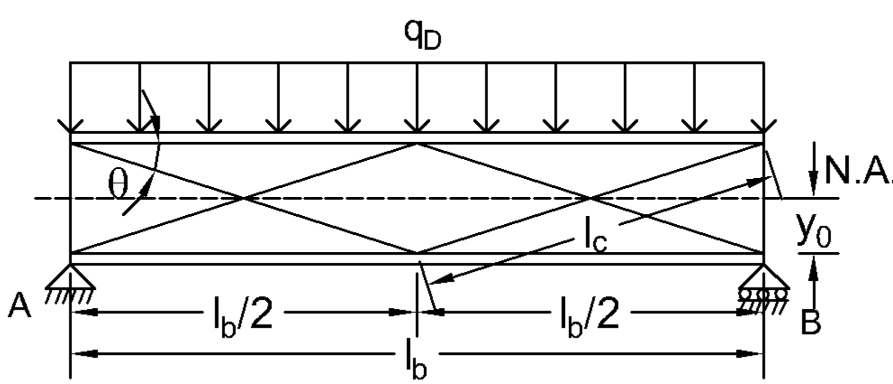

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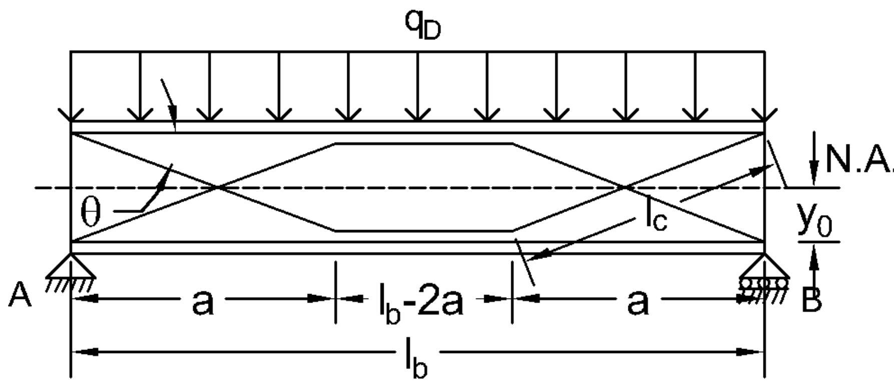
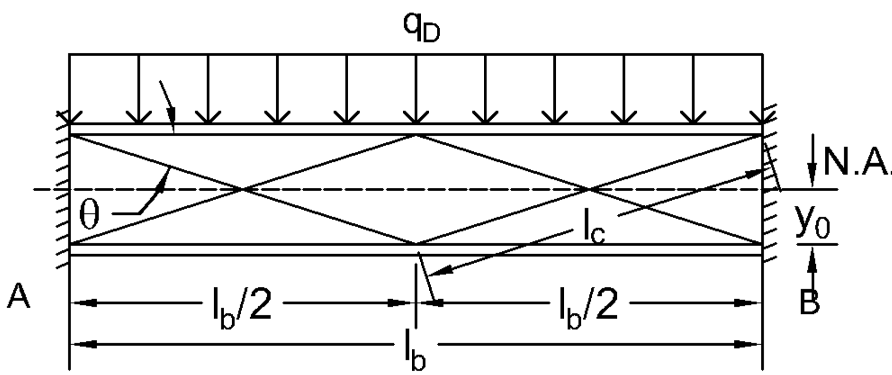


Figure 5



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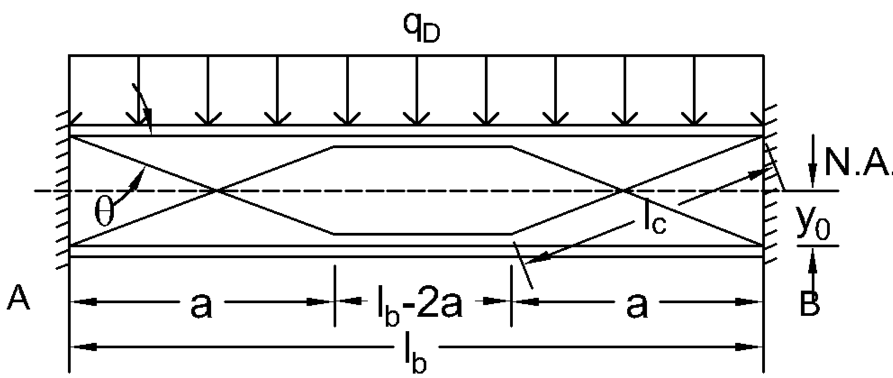
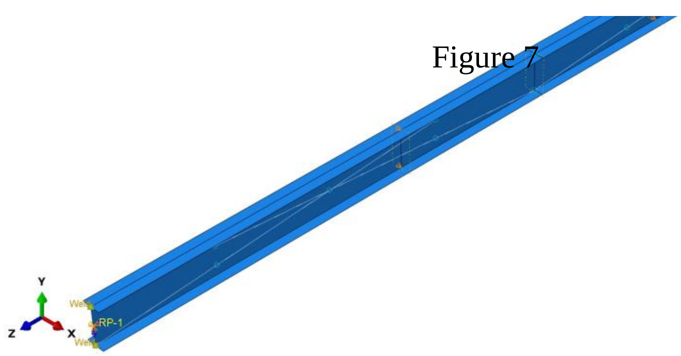
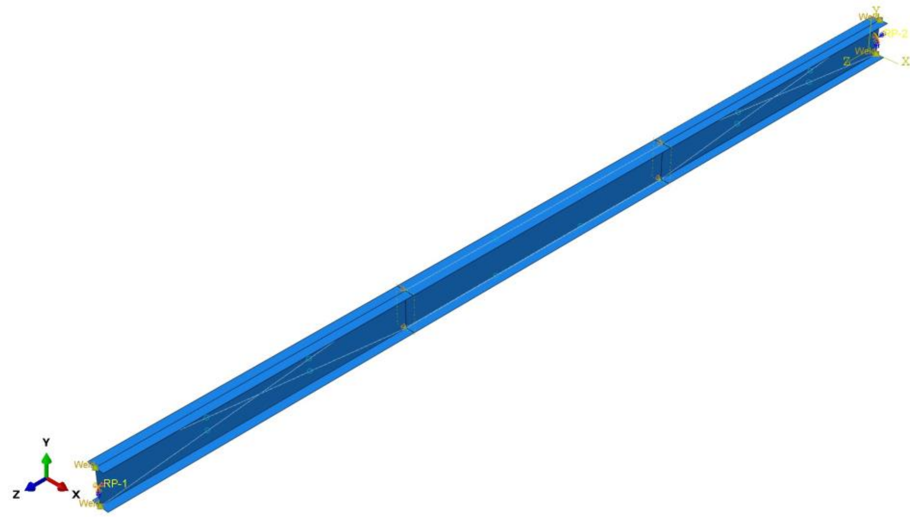


Figure 7

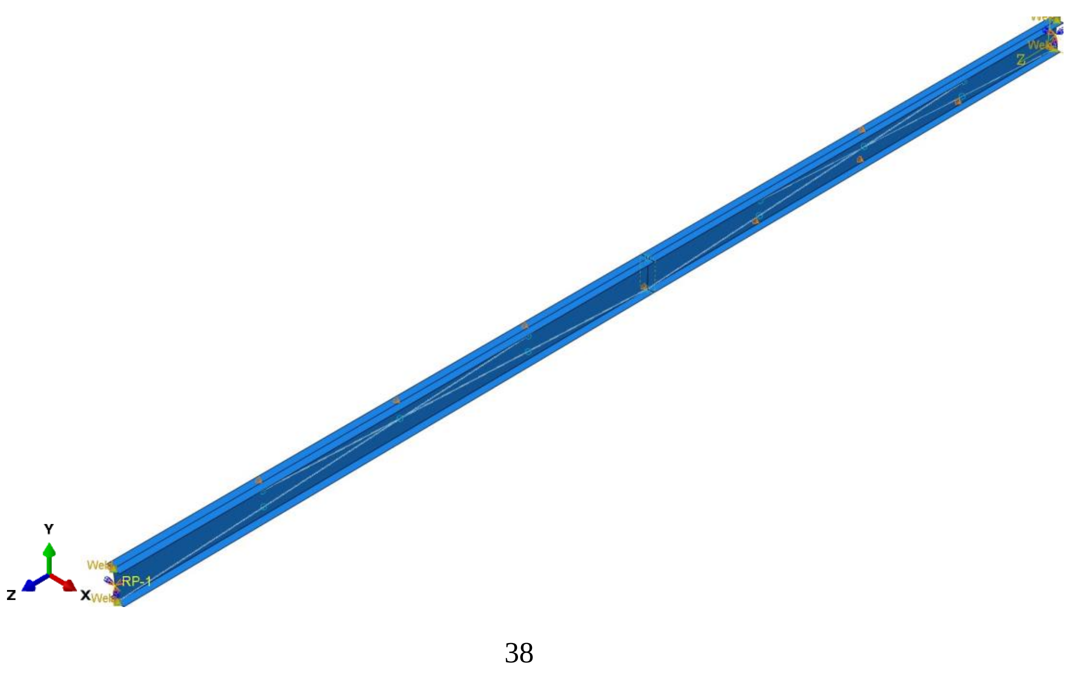
(a)



(b)



(c)



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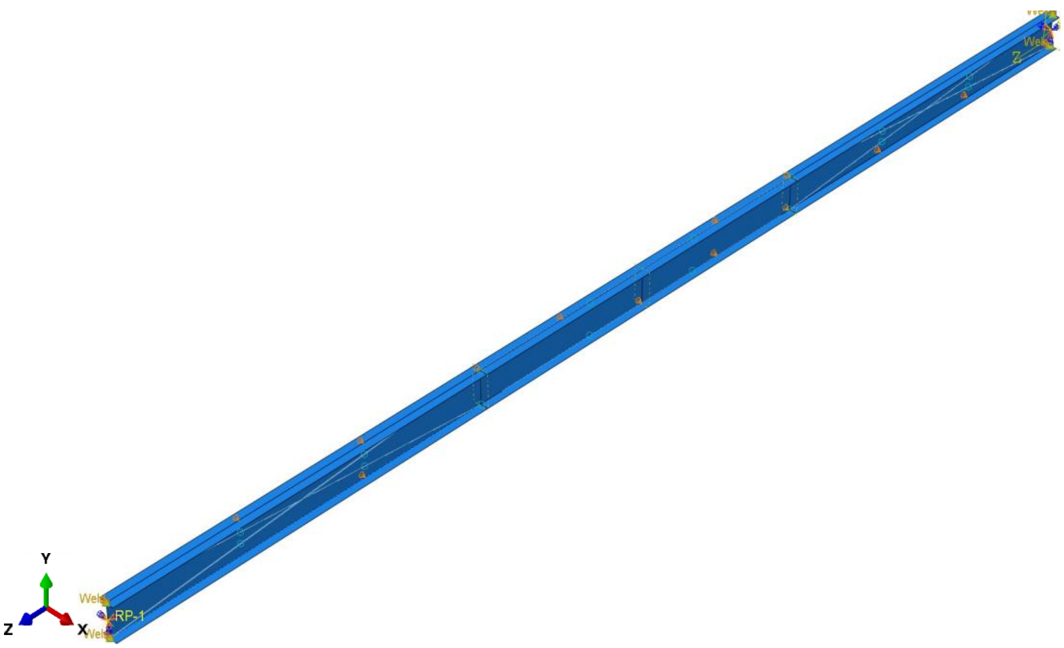

Figure 8

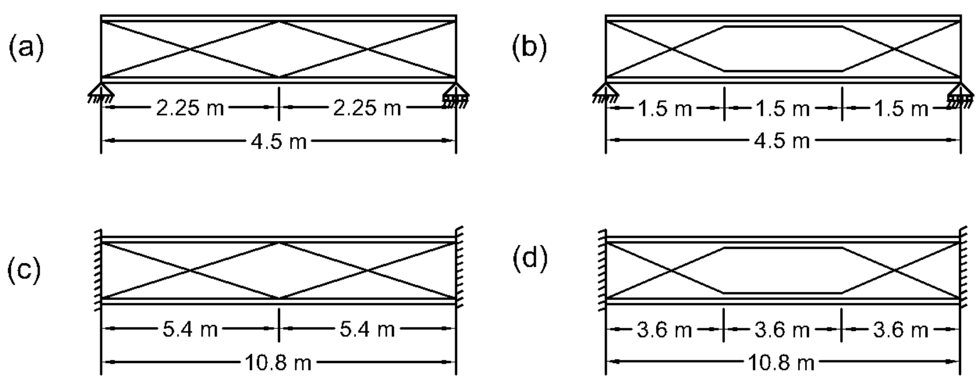

Figure 9

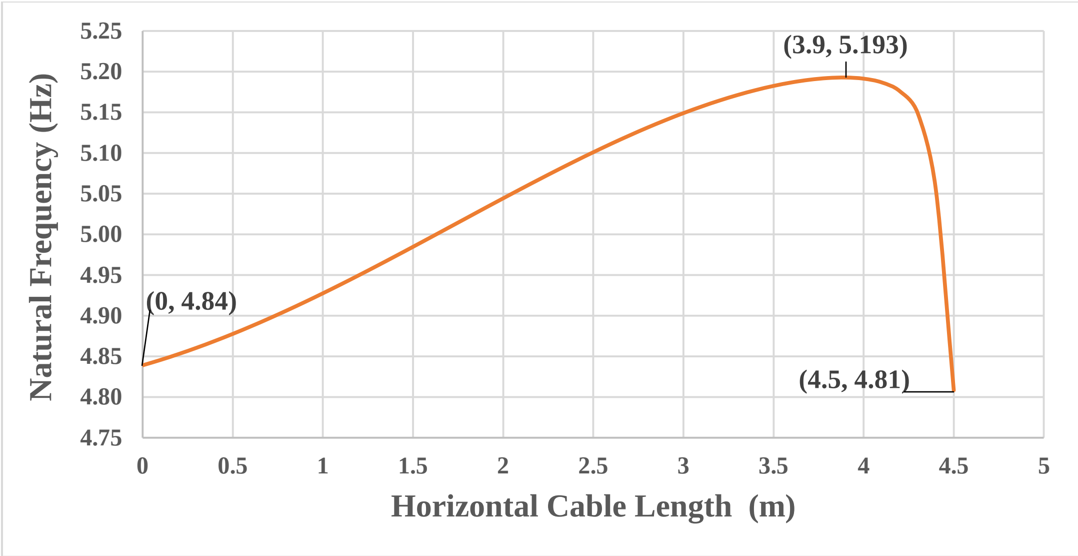

Figure 10

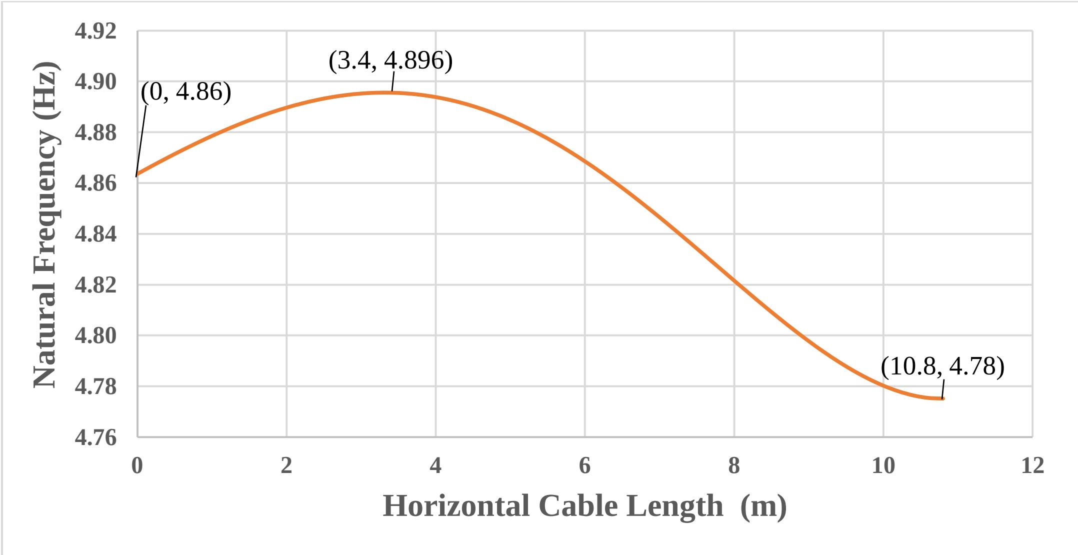

Figure 11

Table 1

Type of beam	Beam span length(m)	Cross-section of beam	Natural frequency based on assumptive shape function (Hz)	Natural frequency based on shape function of the deflection curve (Hz)	Allowable natural frequency (Hz)
Simply supported beam	4.5	IPE180	4.80	4.81	5

Fixed supported beam	10.8	IPE300	4.85	4.78	5

Table 2

Type of beam		Natural frequency of beam obtained from modeling (Hz)	Natural frequency of beam obtained from theoretical equations (Hz)		Allowable natural frequency (Hz)
			based on assumptive shape function	based on shape function of the deflection curve	
Simply supported beam	Without cable	4.74	4.80	4.81	5
	With v-shaped cable pattern	4.83	-	4.84	
	With modified v-shaped cable pattern	5.22	-	4.98	
Fixed supported beam	Without cable	4.61	4.85	4.78	5
	With v-shaped cable pattern	4.83	-	4.86	
	With modified v-shaped cable pattern	4.92	-	4.90	

Table 3

Total cross-section area of steel cable (mm^2)	Natural frequency of simply supported beam along with the v-shaped cable pattern (Hz)	Natural frequency of simply supported beam along with modified v-shaped cable pattern (Hz)	Allowable natural frequency (Hz)
280	4.79	5.00	5
560	4.83	5.22	5
840	4.86	5.40	5
1120	4.89	5.56	5

Table 4

Total cross-section area of steel cable (mm^2)	Natural frequency of fixed supported beam along with the v-shaped cable pattern (Hz)	Natural frequency of fixed supported beam along with modified v-shaped cable pattern (Hz)	Allowable natural frequency(Hz)
280	4.72	4.77	5
560	4.83	4.92	5
840	4.94	5.06	5

1120	5.04	5.21	5

Table 5

Type of beam		cable pre-tensioning stress (MPa)			Allowable natural frequency (Hz)
		400	600	800	
Simply supported beam	Along with the v-shaped pattern	4.89	4.89	4.89	5
	Along with the modified v-shaped cable pattern	5.56	5.56	5.56	
Fixed supported beam	Along with the v-shaped cable pattern	5.04	5.04	5.04	5
	Along with the modified v-shaped cable pattern	5.21	5.21	5.21	

Notations

The following symbols are used in this paper:

$u(x,t)$ = simple harmonic motion of a beam under free vibration;

$\psi(x)$ = shape function;

z_0 = amplitude of generalized coordinate z(t);

ω_n = natural circular frequency;

E_{So} = maximum strain energy;

$u_o(x)$ = maximum displacement;

E_{Ko} = maximum kinematic energy;

$m(x)$ = mass per unit length of the beam;

$EI(x)$ = flexural rigidity;

L = beam length;

q_D = uniform distributed dead load per unit length;

g = gravity acceleration;

f_n = natural frequency;

$M(x)$ = bending moment;

F_{pt} = pre-tensioning force of the steel cable;

ΔF = increase in pre-tensioning force of the steel cable;

Δl = increase in length of steel cable;

l_b = beam length;

l_c = inclined cable length;

A_b = cross section area of beam;

A_c = cross section area of cable on both sides of the web;

E_b = elasticity modulus of beam;

E_c = elasticity modulus of cable;

I_b = moment of inertia of beam;

θ = angle of inclined cable with the horizontal axis;

y_0 = distance of neutral axis to the connection point of steel cable to the beam flange (half of the height of beam web);