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An analytical solution to the bending problem of micro-plate using a new displacement potential function

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Abstract. In this paper, to include small-scale effect, the augmented Love Displacement Potential Functions (DPF) are developed for isotropic micro or nano-scale mediums based on couple stress theory. By substituting the new DPF into the equilibrium equations, the governing equations were simplified to two linear partial differential equations of sixth and fourth orders. Then, the governing differential equations were solved for simply supported rectangular plate using the separation of variable method by satisfying exact boundary conditions without any simplification assumptions. Displacements, bending and torsional moments of rectangular plate were obtained for different length-scale parameters and aspect and Poisson's ratios. The obtained results were compared with other studies, which showed excellent agreement.

1. Introduction

Nowadays, thin beams and plates are widely used in nanoelectromechanical systems (NEMS), microelectromechanical systems (MEMS) and micro/nanotechnologies. Hence, investigating the mechanical behavior of such components has always been an important issue among the researchers. The classical continuum theory of elasticity is a theoretical, idealized, and simplified model with limited ability to predict behaviors of solid body in micro scale. For instance, based on the classical theory, stress concentration factor around a circular hole in an infinite plate is independent of the radius of the hole [1]. Also, in this theory, it is assumed that all material bodies are continuous and therefore, the laws of motion and postulates of constitution are valid for any part of the body regardless of their dimensions [2]. On the contrary, any material body consists of granular or porous materials. Experimental studies such as those of McFarland and Calton [3] have indicated that the classical continuum mechanics is incapable of justifying the size dependency in micro or nano scales due to the lack of material length-scale parameter. In addition, micro structure effects become important when the thickness of the structure is extremely low and comparable to internal length scales of its materials [4]. As a result, a number of theories comprising material length-scale parameters have been proposed. These theories are known to bridge between macroscopic and microscopic physics [2]. Voigt, who was one of the pioneers in formulating the theory of elasticity, introduced couple stresses m_{ij} in addition to the force stresses σ_{ij} [5]. In 1909, Cosserat brothers developed the first mathematical model using couple stress theory. Based on their theory, during loading on a body made of elastic material, the body not only is displaced, but also rotates independently, meaning that three extra freedoms need to be considered for any element [5,6]. Despite its novelty, Cosserat brothers' theory remained

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unknown during their lifetime. Afterwards, in 1968, Eringen introduced the micropolar theory based on the Cosserat theory. Another higher-order continuum theory is the couple stress theory developed by Toupin [7], Mindlin and Tiersten [8], Koiter [9], and Nowacki [10]. The couple stress theory is a special form of the micropolar theory in which rotation vectors for microstructure and macrostructure are considered to be equal [11]. Hence, rotation components are obtained based on displacements [8]. In couple stress theory, in addition to two Lame constants, there are two material length-scale parameters for an isotropic elastic material [12]. Yang et al. [13] utilized an additional equilibrium equation for moments of couples as well as two classical equilibrium equations in order to describe the behavior of the couples. Thus, the couple stress tensor was constrained to be symmetric and the two material length scales were reduced to only one. Yang's studies are known as the modified couple stress theory. The modified couple stress theory has been used in some researches for static and dynamic problems. Park and Gao [14], Ma et al. [15], Kong et al. [12] and Ke and Wang [16] stated that in static analysis, the displacement obtained by the modified couple stress theory was smaller than that by the classical theory. This implies that the size dependency reduces deflection and increases bending rigidity. Also, when size effect is applied to the dynamic behavior of microstructures, natural frequencies of free vibration predicted by the modified couple stress theory are higher than those by the classical model [15,17-23].

Microplates can be considered as important parts of various microsystems. Therefore, the static and dynamic behaviors of the microplate have been studied by many researchers [24-28]. Tsiatas [17] developed a new Kirchhoff plate method for static analysis of isotropic micro-plates based on the modified couple stress theory with only one material-length scale parameter. The proposed model was capable of analyzing plates with complex geometries and boundary conditions.

Even after the introduction of the modified couple stress theory, the couple stress theory was being studied in parallel. Asghari et al. [11] showed that from the two existing material length-scale parameters in the constitutive equations of the couple stress theory, only one of them would appear. Fathalilou et al. [29] showed that bending rigidity of couple stress theory was closer than that of modified couple stress theory to the experimental results.

Due to indeterminacy of the spherical part of the couple-stress tensor and appearance of body couple in the constitutive relations of force-stress tensor, formulation of the couple stress theory is more difficult than classical theory [8,30]. Hadjesfandiari and Dargush [31] developed the consistent couple stress theory for solids. They introduced the skew-symmetric couple stress tensor by utilizing the principle of virtual work and some kinematical considerations. It is necessary to point out that Mindlin and Tiersten [8] derived the general solution to the displacement in isotropic elasticity within the context of the indeterminate couple stress theory. Their formulation included two couple stress parameters, which limited its application. Hadjesfandiari and Dargush [31] showed that these coefficients were not independent. They presented the governing equation with only one length scale parameter.

Utilizing potential functions is a suitable method to simplify and solve elasticity problems. Hence, several potential functions have been developed for elasticity problems within both displacement and stress formulations. Displacement formulation is more common for 3D elasticity problems. There are some displacement potential functions such as Helmholtz decomposition, Galerkin vector, Papkovich-Neuber functions, Love's strain function, and Lame's strain potential function that can be used for isotropic media [1]. It is to be noted that Love [32] originally derived his solutions expressively for the axisymmetric case. Love's displacement potential function was then extended to non-axisymmetric irrotational problems [33]. In honor of Love, a non-axisymmetric DPF, named the non-axisymmetric Love solution, and its completeness were investigated by Tran-Cong [33] for a simply connected and z-convex smooth surface. In non-axisymmetric augmented Love DPF, irrotational limitation is omitted by adding the second function in displacement vectors. Completeness of augmented Love DPF was investigated for elastostatics and elastodynamics problems by Tran-Cong [33], Pak and Eskandari-Ghadi [34], and Eskandari-Ghadi and Pak [35]. One of the useful displacement potential functions is Lekhnitskii-Hu-Nowacki, proposed for transversely isotropic materials. Lekhnitskii presented particular potential functions for the general solution to axisymmetric problems in transversely isotropic materials that were the generalization forms of Love solution [1]. Hu (1953) and Nowacki (1954) developed the Lekhnitskii solution to a general case of elastostatic problems for transversely isotropic media [36]. Wang and Wang [37] proved that Lekhnitskii-Hu-Nowacki's solution was complete for elastostatic problems in transversely isotropic media. Afterwards, Eskandari-Ghadi [36] generalized the Lekhnitskii-Hu-Nowacki solution to dynamic problems. They successfully applied augmented Love, Lekhnitskii-Hu-Nowacki, and Eskandari-Ghadi displacement potential functions to bending, buckling, and vibration of isotropic and transversely isotropic simply supported thick rectangular plates problems [38,39].

In this paper, the augmented Love displacement potential functions are developed to take into account the small effect of micro and nano mediums. By substituting the new DPF in equilibrium equations, the sixth- and fourth-order partial differential equations are obtained as the governing equations. Then, the governing equations are solved for simply supported rectangular plate by separation of the variable method, satisfying exact boundary conditions. The presented method is exact and applicable to all isotropic plates of the problem with any thickness ratio, because there is no simplifying assumption in the procedure, displacement or stress components, thickness ratio, and linear elastic material properties.

2. The governing equations

In continuum mechanics, for an arbitrary part of the material continuum with volume, V, enclosed by boundary surface, S, the linear and angular equilibrium equations for the couple stress theory are written as [1]:

$$\sigma_{ji,j}+F_i=0, \tag{1}$$

$$\mu_{ji,j}+\varepsilon_{ijk}\sigma_{jk}+C_i=0, \tag{2}$$

where σ_{ji} and μ_{ji} are force stress and couple stress tensors, respectively; F_i and C_i are body force and couple per unit volume, respectively; and ε_{ijk} is permutation symbol. Here, the subscript denotes differentiation with respect to spatial variables.

Writing force-stress tensor in terms of displacements and substituting it into the linear equilibrium equation, Eq. (3) is obtained. In this paper, we use the liner equilibrium equation derived by Hadjesfandiari and Dargush [31], which is a different version of the Mindlin and Tiersten [8] equation:

$$\left(\lambda+\mu+\eta\nabla^2\right)\nabla\left(\nabla.\vec{u}\right)+\left(\mu-\eta\nabla^2\right)\nabla^2\vec{u}+F=0, \tag{3}$$

where $\vec{u}$ is the displacement vector of the continuum material, λ and μ are the Lame constants, δ_{ij} is the Kronecker delta, and η is an extra material constant defined in couple stress theory.

In Eq. (3), $\vec{u}$ is of class C^4 on R^3 if u is continuous on R^3 and its position derivative up to the 4th order is at each point in R^3, where R is the set of real numbers. A solution, u, to Eq. (3) of class C^4 on R^3 will be referred to as an elastic response in micro scale of an isotropic material.

Using the energy and the constitutive relations, the force stress tensor and couple stress tensor can be written as [30]:

$$\begin{aligned}\sigma_{ji}=&\lambda e_{kk}\delta_{ij}+2\mu e_{ij}+2\eta\nabla^2\omega_{ji}=\lambda u_{k,k}\delta_{ij}\\&+\mu(u_{i,j}+u_{j,i})-\mu l^2\nabla^2(u_{i,j}-u_{j,i}),\end{aligned} \tag{4}$$

$$\mu_{ji}=4\eta(\omega_{i,j}-\omega_{j,i}), \tag{5}$$

where u_i is displacement vector component of the continuum material, and e_{ij} and ω_{ij} are strain and rotation tensors and can be written as:

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{6}$$

$$\omega_{ij}=\frac{1}{2}(u_{i,j}-u_{j,i}). \tag{7}$$

The operator ∇ is defined as:

$$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}. \tag{8}$$

Also, η can be represented in terms of length scale parameter, l, as [31]:

$$\frac{\eta}{\mu}=l^2. \tag{9}$$

3. Displacement potential functions

In non-axisymmetric Love DPF, only one function is used, i.e., F (see Eqs. (10)-(12)), and for rotation-free augmented Love DPF, function χ is added. The completeness of augmented Love potential functions, as mentioned before, has been investigated for many elastodynamics and elastostatics mechanics problems [33-35], but due to the lack of length scale parameter, the augmented Love displacement potential functions are not suitable for solving of the micromechanics problems [40]. Hence, in this section, by developing the augmented Love potential functions for isotropic materials, we introduce new displacement potential functions that include length scale parameter.

$$u=-(\alpha_1+l^2\nabla^2)\frac{\partial^2 F}{\partial x\partial z}-\frac{\partial\chi}{\partial y}, \tag{10}$$

$$v=-(\alpha_1+l^2\nabla^2)\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial\chi}{\partial x}, \tag{11}$$

$$w=\alpha_2\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)+(1-l^2\nabla^2)\frac{\partial^2 F}{\partial z^2}, \tag{12}$$

where u, v, and w are displacement components in x, y, and z directions, respectively; F and χ are displacements potential functions, respectively; and α_1 and α_2 are defined as:

$$\alpha_1=\frac{\lambda+\mu}{\mu},\qquad \alpha_2=\frac{\lambda+2\mu}{\mu}. \tag{13}$$

Substituting the displacement potential functions into the governing equation (3) with some algebraic manipulations, the governing equations can be written as:

$$-(1-l^2\nabla^2)\frac{\partial}{\partial y}\nabla^2\chi=0, \tag{14}$$

$$(1-l^2\nabla^2)\frac{\partial}{\partial x}\nabla^2\chi=0, \tag{15}$$

$$(1-l^2\nabla^2)\nabla^4 F=0. \tag{16}$$

Based on the first two equations of this system, one may define a vector function, namely G, as $G = ((1-l^2\nabla^2)\nabla^2\chi)e_z$, where e_z is the unite vector in z-direction. Then, the first two equations are translated to curl $G = 0$. The solution to this equation is not unique [41] and, in this problem, it is enough to be zero, meaning that:

$$(1-l^2\nabla^2)\nabla^2\chi = 0, \tag{17}$$

where $F(x,y,z)$ is a class C^6 field and $\chi(x,y,z)$ is a class C^4 field on R^3.

It can be seen that by eliminating the length scale parameter (l), Eqs. (16) and (17) are simplified to the classical form [39]. The proposed displacement potential functions and governing equations are valid for any isotropic medium. In the following, these equations are solved for a simple rectangular plate.

4. Application to plates

The governing differential equations (16) and (17) are solved here for a simply supported rectangular plate with thickness of t, length of a, and width of b (see Figure 1).

As shown in Figure 1, an x, y, z Cartesian coordinates system is used, where the z-axis is along the plate thickness and q is a static lateral load applied to upper surface of plate and written in the form of the Fourier series as:

$$q(x,y) = \sum_m \sum_n q_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \tag{18}$$

where n and m are integers, and q_{mn} is the Fourier loading coefficient written as [42]:

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q_z(x,y).\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy, \tag{19}$$

where $q_z(x,y)$ is static lateral load.

4.1. Boundary conditions

For a simply supported rectangular plate, the boundary conditions on edges can be written as:

$$w = 0, \quad M_x = 0 \quad \text{at} \quad x = 0, a, \tag{20}$$

$$w = 0, \quad M_y = 0 \quad \text{at} \quad y = 0, b, \tag{21}$$

where bending moments M_x and M_y are defined as:

$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x z dz, \qquad M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y z dz. \tag{22}$$

The boundary conditions on the upper ($-t/2$) and lower ($+t/2$) surfaces of the plate can be represented as:

At $z = -\frac{t}{2}$:

$$\sigma_{zz} = -q, \tag{23a}$$

$$\tau_{zx} = 0, \tag{23b}$$

$$\tau_{zy} = 0, \tag{23c}$$

$$\mu_{zx} = 0, \tag{23d}$$

$$\mu_{zy} = 0. \tag{23e}$$

At $z = +\frac{t}{2}$:

$$\sigma_{zz} = 0, \tag{24a}$$

$$\tau_{zx} = 0, \tag{24b}$$

$$\tau_{zy} = 0, \tag{24c}$$

$$\mu_{zx} = 0, \tag{24d}$$

$$\mu_{zy} = 0. \tag{24e}$$

5. Solution procedure

Using the method of separation of variables, Eq. (16) can be solved for the potential function, F, in terms of three scalar functions, f, g, and h, which are functions of x, y, and z, respectively.

$$F(x,y,z) = f(x) \times g(y) \times h(z). \tag{25}$$

Substituting Eq. (25) into Eq. (16), by satisfying the displacement boundary conditions, i.e., Eqs. (20)

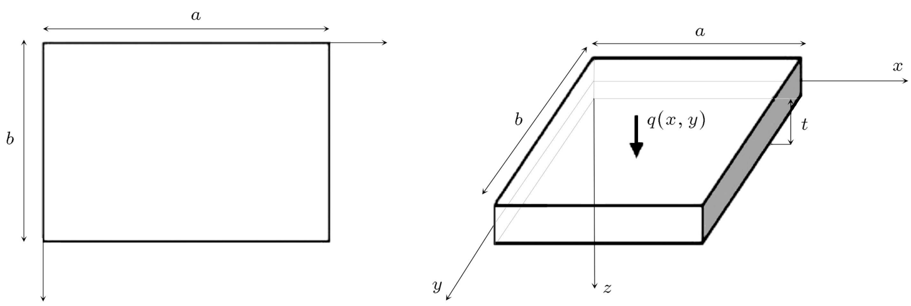

Figure 1. Geometry and loading of a plate in xyz Cartesian coordinates.

and (21), f and g are determined as:

$$f = C_1' \sin \frac{m\pi}{a} x, \tag{26}$$

$$g = C_2' \sin \frac{n\pi}{b} y, \tag{27}$$

where C_1' to C_2' are constant coefficients. Parameters α_m, α_n, and α_{mn} are defined as:

$$\alpha_m = \frac{m\pi}{a}, \qquad \alpha_n = \frac{n\pi}{b}, \tag{28}$$

$$\alpha_{mn} = \sqrt{(\alpha_m)^2 + (\alpha_n)^2}. \tag{29}$$

Substituting f and g into Eq. (16), the governing differential equation is obtained in terms of h:

$$\frac{d^6 h}{dz^6}(-l^2) + \frac{d^4 h}{dz^4}(1 + 3\alpha_{mn}^2 l^2) + \frac{d^2 h}{dz^2}(-2\alpha_{mn}^2 - 3\alpha_{mn}^4 l^2) + h(\alpha_{mn}^4 + \alpha_{mn}^6 l^2) = 0. \tag{30}$$

The solution to Eq. (30) can be written as:

$$h = c_3' \sinh \alpha_{mn} z + c_4' \cosh \alpha_{mn} z + c_5' z \sinh \alpha_{mn} z + c_6' z \cosh \alpha_{mn} z + c_7' z \sinh(\sqrt{\rho_1} \times z) + c_8' z \cosh(\sqrt{\rho_1} \times z), \tag{31}$$

where $\rho_1 = \frac{1}{l^2} + \alpha_{mn}^2$.

Introducing new coefficients C_1 to C_6, the potential function F can be rewritten as:

$$F = \sum_m \sum_n \Bigg[\big(C_1 \sinh \alpha_{mn} z + C_2 \cosh \alpha_{mn} z + C_3 z \sinh \alpha_{mn} z + C_4 z \cosh \alpha_{mn} z + C_5 \sinh(\sqrt{\rho_1} \times z) + C_6 \cosh(\sqrt{\rho_1} \times z) \big) \times \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \Bigg]. \tag{32}$$

Similar to F, the potential function χ can be derived as:

$$\chi(x, y, z) = f_1(x) \times g_1(y) \times h_1(z). \tag{33}$$

Substituting Eq. (33) into Eq. (17) and using the separation of variables technique, by satisfying the moment boundary conditions, i.e., Eqs. (20) and (21), f_1 and g_1 are specified as

$$f_1 = A_1' \cos \frac{m\pi}{a} x, \tag{34}$$

$$g_1 = A_2' \cos \frac{n\pi}{b} y. \tag{35}$$

Substituting f_1, i.e., Eq. (34), and g_1, i.e., Eq. (35), into Eq. (17), the governing differential equation is obtained in terms of h_1 as:

$$\frac{d^4 h_1}{dz^4}(l^2) - \frac{d^2 h_1}{dz^2}(1 + 2\alpha_{mn}^2 l^2) + h_1 \alpha_{mn}^2 (1 + \alpha_{mn}^2 l^2) = 0. \tag{36}$$

Eq. (36) can be solved as:

$$h_1 = A_3' \sinh \alpha_{mn} z + A_4' \cosh \alpha_{mn} z + A_5' \sinh(\sqrt{\rho_1} \times z) + A_6' \cosh(\sqrt{\rho_1} \times z). \tag{37}$$

By introducing new coefficients A_1 to A_4, the potential function χ is expressed as:

$$\chi = \sum_m \sum_n \Bigg[\big(A_1 \sinh \alpha_{mn} z + A_2 \cosh \alpha_{mn} z + A_3 \sinh(\sqrt{\rho_1} \times z) + A_4 \cosh(\sqrt{\rho_1} \times z) \big) \times \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \Bigg]. \tag{38}$$

To find the unknown coefficients C_1-C_6 and A_1-A_4, the remaining boundary conditions must be satisfied. Hence, the displacements components and boundary conditions, Eqs. (23) and (24), are written in terms of f, g, h and f_1, g_1, h_1 as:

$$u = -\alpha_1 f' g h' - l^2 \nabla^2 f' g h' - f_1 g_1' h_1, \tag{39}$$

$$v = -\alpha_1 f g' h' - l^2 \nabla^2 f g' h' - f_1' g_1 h_1, \tag{40}$$

$$w = \alpha_2 f'' g h + \alpha_2 f g'' h + f g h'' - l^2 \nabla^2 f g h'', \tag{41}$$

$$\tau_{zx} = \left(-\lambda + \frac{k}{\alpha_2} - k\right) f' g h'' + \frac{k}{2}\left(-\frac{1}{l^2} + \alpha_{mn}^2\right) f' g h + \frac{k}{\alpha_{mn}^2}\left(-\frac{1}{\alpha_2} + \frac{1}{2}\right) f' g h^{(4)} + \left(-\mu - \frac{k}{2\alpha_2}\right) f_1 g_1' h_1' + \left(\frac{k}{2\alpha_2 \alpha_{mn}^2}\right) f_1 g_1' h_1''', \tag{42}$$

$$\tau_{zy} = \left(-\lambda + \frac{k}{\alpha_2} - k\right) f g' h'' + \frac{k}{2}\left(-\frac{1}{l^2} + \alpha_{mn}^2\right) f g' h + \frac{k}{\alpha_{mn}^2}\left(-\frac{1}{\alpha_2} + \frac{1}{2}\right) f g' h^{(4)} + \left(\mu + \frac{k}{2\alpha_2}\right) f_1' g_1 h_1' + \left(-\frac{k}{2\alpha_2 \alpha_{mn}^2}\right) f_1' g_1 h_1''', \tag{43}$$

$$\mu_{zy} = k\left(h' - \frac{1}{\alpha_{mn}^2} h'''\right) f' g + \frac{k}{\alpha_2}\left(-\frac{1}{\alpha_{mn}^2} h_1'' + h_1\right) f_1 g_1', \tag{44}$$

$$\mu_{zx} = k\left(-h' + \frac{1}{\alpha_{mn}^2}h'''\right)fg' + \frac{k}{\alpha_2}\left(-\frac{1}{\alpha_{mn}^2}h_1'' + h_1\right)f_1'g_1, \tag{45}$$

$$\sigma_{zz} = \left(-\lambda\alpha_{mn}^2 - l^2\lambda\alpha_{mn}^4 - \frac{k}{l^2}\right)fgh' + \left(\mu\alpha_2 + l^2\lambda\alpha_{mn}^2 + \frac{k}{2}\right)fgh''' + \left(-\frac{k}{2\alpha_{mn}^2}\right)fgh^{(5)}, \tag{46}$$

where $k = 2\mu l^2\alpha_2\alpha_m n^2$.

Satisfying boundary conditions (23b), (23c) and (24b), (24c) lead to the zero value of χ. It is noticeable that the length scale parameter remains although potential function χ is eliminated. Thus, the coefficients C_1 to C_6 should be obtained based on boundary conditions, i.e., Eqs. (23) and (24). Eqs. (23b), (23c) and (24b), (24c) can be rewritten as:

$$\left(-\lambda - k + \frac{k}{\alpha_2}\right)h'' + \frac{k}{2}\left(-\frac{1}{l^2} + \alpha_{mn}^2\right)h + \frac{k}{\alpha_{mn}^2}\left(-\frac{1}{\alpha_2} + \frac{1}{2}\right)h^{(4)} = 0 \qquad \text{at} \quad z = \pm\frac{t}{2}. \tag{47}$$

Similarly, μ_{zx} and μ_{zy} can be written as:

At $z = \pm\frac{t}{2}$:

$$\mu_{zx} = k\left(-h' + \frac{1}{\alpha_{mn}^2}h'''\right)fg' = 0, \tag{48}$$

$$\mu_{zy} = k\left(h' - \frac{1}{\alpha_{mn}^2}h'''\right)f'g = 0. \tag{49}$$

It can be seen that Eqs. (48) and (49) are identical and can be rewritten as:

$$\left(-kh' + \frac{k}{\alpha_{mn}^2}h'''\right) = 0 \qquad \text{at} \quad z = \pm\frac{t}{2}. \tag{50}$$

Thus, the previous boundary conditions are reduced to 6 boundary conditions, namely force stresses σ_{zz}, Eqs. (23a) and (24a); shear stresses, Eqs. (23c) and (24c); and couple stresses, Eqs. (23e) and (24e). Therefore, there is a system of 6 independent equations in which 6 constants C_1 to C_6 remain unknown.

Using MATLAB software, the 6 constant coefficients and, consequently, the displacement potential function F can be determined. Having F, the displacements, strains, force stresses, couple stress tensors, and shear forces, bending and twisting moments can be determined.

6. Numerical examples

In this section, it is shown that how the new displacement potential functions can be employed in order to solve the real-case problems. The results obtained by utilizing the new displacement potential functions are compared to the classical data as well as the outcomes of the previous study.

6.1. Verifications

Verification of the proposed displacement potential functions is shown in the following by eliminating length scale parameter (l) in the governing equations (Eqs. (16) and (17)). Therefore, the equations are simplified to the classical form [39], demonstrating the validity of the new displacement potential functions.

To compare the results, non-dimensional displacement of the mid-plane is introduced as $\bar{w} = wD/qa^4$, in which D is rigidity of plate ($D = (Et^3/12(1-v^2))$), and non-dimensional bending moments in x and y directions are introduced as $\bar{M}_x = M_x/qa^2$ and $\bar{M}_y = M_y/qa^2$.

To verify the accuracy of the Presented Work (PW), the central non-dimensional deflection ($\bar{w}_c$) and central non-dimensional bending moment in x and y directions ($\bar{M}_{xc}$ and $\bar{M}_{yc}$) are compared with their exact classical solutions, namely $\bar{w}_c^c$, $\bar{M}_{xc}^c$, and $\bar{M}_{yc}^c$. Considering $t = 0.01$ in contrast to l, which is in micro range, the l/t ratio becomes approximately zero. With the elimination of the l/t ratio, the solutions become similar to those to the classical method for thin plate. Thus, $\bar{w}_c$, $\bar{M}_{xc}$, and $\bar{M}_{yc}$ are adapted as $\bar{w}_c^c$, $\bar{M}_{xc}^c$, and $\bar{M}_{yc}^c$, demonstrating the accuracy of the model developed in this study [43]. According to Table 1, the results of the central deflection in rectangular plate for 4 different aspect ratios are in excellent agreement

Table 1. Comparison of $\bar{w}_c^c$ and $\bar{w}_c$, $\bar{M}_{x_c}^c$ and $\bar{M}_{x_c}$, and $\bar{M}_{yc}^c$ and $\bar{M}_{yc}$ in the rectangular plate for various aspect ratios.

b/a	$\bar{w}_c^c$	$\bar{w}_c$	$\bar{M}_{x_c}^c$	$\bar{M}_{x_c}$	$\bar{M}_{yc}^c$	$\bar{M}_{yc}$
1	0.00406	0.00406	0.047900	0.048236	0.047900	0.048236
2	0.01013	0.01014	0.101700	0.102293	0.046400	0.047449
3	0.01223	0.01227	0.118900	0.119985	0.0406000	0.042726
4	0.01282	0.01292	0.123500	0.125377	0.038400	0.041458

Table 2. Comparison of normalized central deflections of square micro-plate in the PW and Tsiatas results versus $(l/t)^2$ for 3 different Poisson's ratios.

$(l/t)^2$	w_c/w_c^c					
	$v = 0.25$		$v = 0.3$		$v = 0.35$	
	PW	Tsiatas	PW	Tsiatas	PW	Tsiatas
0	1	1	1	1	1	1
0.05	0.648	0. 817	0. 655	0.825	0.684	0.85
0.10	0.516	0.69	0.53	0.7	0.547	0.72
0.20	0.415	0.53	0.43	0.545	0.45	0.565

with those obtained by the analytical solutions of Timoshenko and Krieger [43]. Also, the comparison of $\bar{M}_{xc}$ and $\bar{M}_{yc}$ with $\bar{M}_{xc}^c$ and $\bar{M}_{yc}^c$ is summarized in Table 1.

Also, the presented results in Table 1 show that the central non-dimensional bending moments in rectangular plate for the 4 aspect ratios are consistent with the solutions for the classical method. The slight difference is due to the length scale parameter.

Table 2 shows the normalized central deflection, w_c/w_c^c, for a square plate versus $(l/t)^2$ for 3 different Poisson's ratios ($v = 0.25, 0.3, 0.35$) of PW and Tsiatas result.

According to Table 2, the deflection of the plate decreases by the increment of (l/t). This is similar to the results of Tsiatas, in which the normalized deflection decreased with the increase in (l/t). Also, it is observed that the deflection of the micro-plate slightly decreases with increase in the Poisson's ratio.

6.2. Results and discussion

The results for the analysis of micro plates under uniformly distributed load have been presented in Figures 2-6. Here, the following assumptions have been made: Young's modulus: $E = 2.1 \times 10^{11}$ N/m^2 and Poisson's ratio: $v = 0.3$.

Figure 2 shows non-dimensional displacements of the simply supported square plate ($b/a = 1$) on the line $y = (1/2)b$ for various length-to-thickness, l/t, ratios.

According to Figure 2, displacement of plate decreases with increase in l/t and it is always smaller than the classical deflection. This can be justified with rigidity of microplate defined as $D = (Et^3/12(1-v^2))+ \mu t l^2$. Rigidity of micro plate is the sum of bending rigidity of the plate and the contribution of rotation gradients to the bending rigidity [17]. By considering length scale parameter, the rigidity of couple stress theory increases. Hence, the displacement of plate decreases. It is noticeable that the displacements estimated here are less than those estimated by the classical theory and are adapted to Tsiatas results. In both Tsiatas model and PW, the displacement treatments are equal and for the same value of l/t , the displacement for PW is less than that for the Tsiatas model. It is noticeable that in the couple stress theory, size effect is applied using the l/t ratio.

The effect of the aspect ratio, b/a, on the central displacement of the rectangular microplates for various l/t ratios is illustrated in Figure 3.

The results indicate that by the increasing the aspect ratio (b/a), the central displacement will increase for different l/t ratios. Also, for the ratios of $b/a =$ 3 and 4, the displacements are approximately equal.

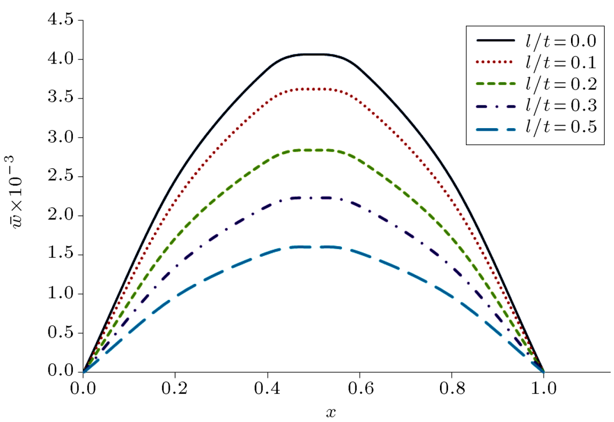

Figure 2. Non-dimensional displacement of the square nano-plate at $y = (1/2)b$ for various values of l/t.

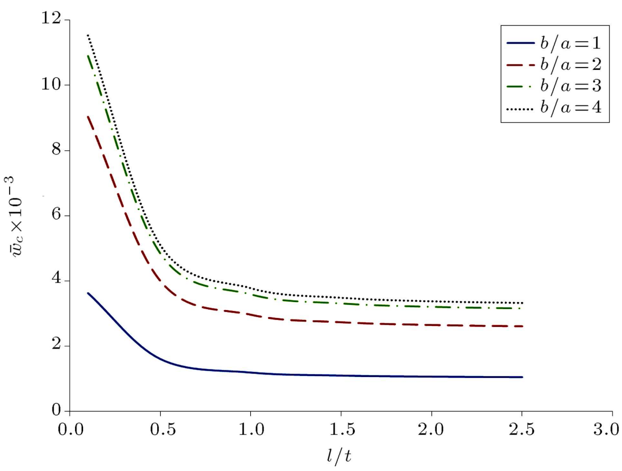

Figure 3. Non-dimensional maximum displacement for various values of l/t and b/a.

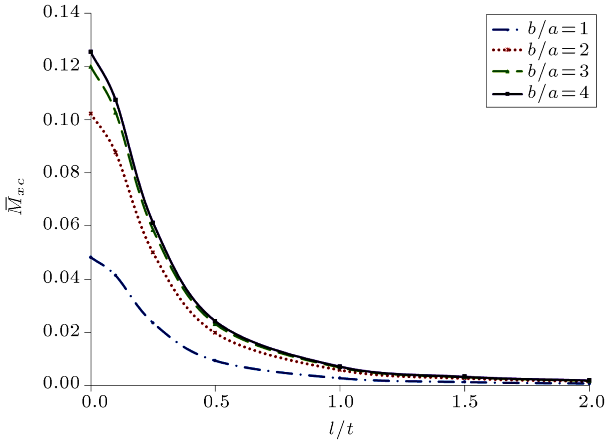

Figure 4. Non-dimensional bending moment ($\bar{M}_{xc}$) for difference values of l/t and b/a.

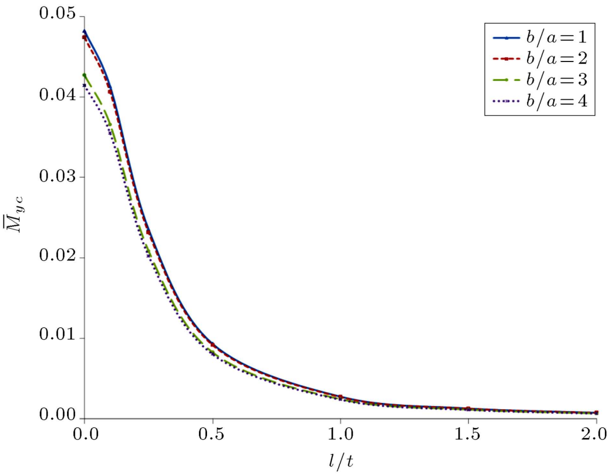

Figure 5. Non-dimensional bending moment ($\bar{M}_{yc}$) for various values of l/t and b/a.

Such behavior can be justified for an infinite length stripe. Figure 3 also shows that by increasing l/t, the central displacement of rectangular plate always decreases nonlinearly. This response is similar to that obtained by Tsiatas [17], and Papargyri-Beskou and Beskos [4].

Maximum bending moment of the rectangular plate in x direction (M_x) depends on the b/a ratio. In this regard, $\bar{M}_{xc}$ for different l/t and b/a ratios under uniform distributed load is determined. Figure 4 shows the variations of $\bar{M}_{xc}$ with respect to different ratios of l/t and different plate dimensions of b/a. It is observed that, unlike for $l/t > 0.5$, for small l/t ratios, with increase in b/a, the maximum bending moments increase. In addition, it is observed that as l/t increases, thickness of the plate becomes smaller than length scale parameter and $\bar{M}_{xc}$ shows convergence behavior for different b/a ratios.

The variation of non-dimensional maximum moment in y direction ($\bar{M}_{yc}$) is depicted in Figure 5. The behavior of $\bar{M}_{yc}$ is similar to that of $\bar{M}_{xc}$ with respect

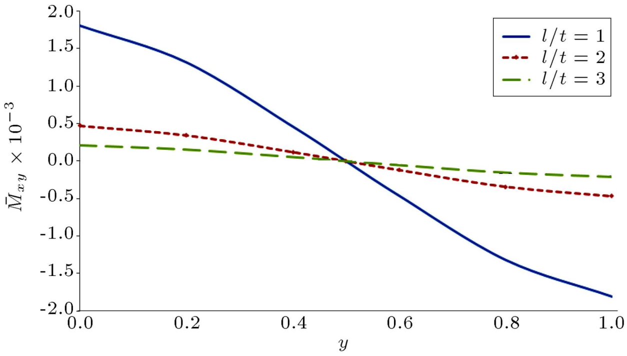

Figure 6. Distribution of non-dimensional torsional moment $\bar{M}_{xy}$ at $x = a$ for various values of l/t.

to the variations of l/t. In contrast, when b/a increases, $\bar{M}_{yc}$ slightly decreases and the convergence between the results improves for all the values of b/a. The twisting moment M_{xy} can be written as:

$$M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} z dz. \tag{51}$$

In order to investigate the influence of twisting moment, distribution of non-dimensional twisting moment, $\bar{M}_{xy} = M_{xy}/qa^2$, in $x = a$ for the square plate is shown in Figure 6. The figure shows that maximum $\bar{M}_{xy}$ occurs at the edges of the plate and increases by decreasing l/t. Also, the differences between the twisting moments are reduced as the l/t ratio increases.

7. Conclusions

In this paper, the augmented Love displacement potential functions were developed for mediums of isotropic micro and nano scales based on couple stress theory. Using the new DPF for an arbitrary isotropic and homogeneous medium in micro or nano scale led to the formation of differential equations in the form of two independent partial differential equations of sixth and fourth orders. These equations were solved for micro rectangular plates using the method of separation of variables. In contrast with the results of the previous researches, the proposed solution did not require simplifying assumptions, especially for shear stress or strain in thickness of plate. The numerical results of this study showed that with the reduction in plate thickness and by considering length parameter, the stiffness of the plate increased and the plate displacement decreased, compared with the classic theory. Moreover, it was observed that in the couple stress theory for rectangular plates, with increase in b/a, the plate maximum displacement, w_c, and the maximum moment, M_{xc}, increased, while the maximum moment M_{yc} of the plate decreased. In addition, w_c, M_{xc}, and M_{yc} decreased by increasing l/t. In the investigation into torsional moment, it was concluded that when the length parameter increased, M_{xy} on a simply supported edge decreased and the convergence of the values of M_{xy} improved.

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