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On accuracy function and distance measures of interval-valued Pythagorean fuzzy sets with application to decision making

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Abstract. In the present paper, a new accuracy function is provided to overcome the limitations of the existing score/accuracy functions for interval-valued Pythagorean Fuzzy Sets (PFSs). The proposed accuracy function is validated and discussed in detail through illustrative examples. Furthermore, a new distance measure for interval-valued Pythagorean fuzzy numbers is proposed and used in terms of the existing weighted averaging operators. Finally, with regard to the proposed accuracy function, distance measure, and weighted averaging operators, a numerical example of Multi-Criteria Decision Making (MCDM) process is presented to validate the methodology.

1. Introduction

Fuzzy sets [1] have been used to express imprecise or vague information in various fields of real-world application. Intuitionistic Fuzzy Set (IFS) proposed by Atanassov [2] has been found a highly adjustable framework to grapple with uncertainty with a certain amount of hesitation arising from imperfect or vague information. The concept of IFS has been widely studied and applied to deal with uncertainties and hesitancy inherent in practical circumstances. The most significant characteristic of an IFS is that it assigns a number from the unit interval [0, 1] to every element in the domain of discourse, a degree of membership, and a degree of non-membership along with the degree of indeterminacy, whose total sum equals unity. In the literature, IFSs and interval-valued IFSs comprehensively span applications to the fields of decision making problems [3], pattern recognition, sales analysis financial services, medical diagnosis, etc.

Pythagorean Fuzzy Set (PFS) [4] is an efficient generalization of IFS characterized by the inequality that the squared sum of membership and non-membership values is less than or equal to one. Yager and Abbasov [5] stated that in some practical Multiple-Criteria Decision Making (MCDM) problems, it is viable that the sum of the degree of the membership and the degree of non-membership in a particular alternative provided by a decision maker may be greater than one, where it would not be feasible to use IFS. PFS has proven more proficient than IFS in representing and handling vagueness, impreciseness, and uncertainties in various decision making processes. Since the span of

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membership degree of PFS is greater than that of IFS, it may be stated that it is more generalized and has wider applicability.

Yager [4] also developed some aggregation operations under the PFS environment. Furthermore, Yager and Abbasov [5] investigated the correspondence between the degrees of Pythagorean membership and complex numbers and found that Pythagorean membership degrees were only subsets of the complex numbers, called Π–i numbers. Utilizing PFSs for solving MCDM problems, Zhang and Xu [6] enhanced the existing technique for order of preference by similarity to ideal solution. PFSs were generalized to interval-valued PFSs by Peng and Yang [7], who provided some interval-valued Pythagorean fuzzy aggregation operators for handling the related problems. Various new operators with different properties and applications to the field of decision making have also been presented by Peng [8]. A ranking method for Pythagorean fuzzy numbers as well as Interval-Valued Pythagorean Fuzzy Numbers (IVPFNs) was proposed by Zhang [9] by taking the idea of closeness index into account.

In the literature, the notion of distance measures plays a key role in the fuzzy set theory and in the application fields such as MCDM problems [10,11], pattern recognition, medical diagnosis, financial services, etc. Recently, various researchers have proposed different types of distance measures for different types of sets, viz. fuzzy sets, IFSs, and PFSs [12]. The Hamming distance, Euclidean distance, and Hausdorff distance measures are some popular and widely utilized distance measures in the application and research world of soft computing [13–19]. Zhang and Xu [6] provided a new measure for Pythagorean fuzzy numbers and applied it to an MCDM problem. Furthermore, Li and Zeng [20] stated that the strength and direction of commitment would play an important role in well describing PFNs.

They considered four fundamental parameters, namely membership, non-membership, strength, and direction of commitment of PFNs, and provided some distance measures. Most recently, Liu et al. [21] introduced important distance measures for IVPFNs along with their generalized, weighted, and ordered weighted versions. In addition, they proposed some generalized probabilistic distance measures and various important operators and used them for MCDM problems. This study incorporates a new accuracy function for IVPFNs to overcome the limitations of the score/accuracy function in the existing methodologies and employs IVPFNs for solving an MCDM problem.

Different parts of this paper are organized as follows. First, the basic notation for Pythagorean fuzzy numbers is presented along with their corresponding scores and accuracy functions in Section 2. Then, the notion of IVPFNs is discussed and the shortcomings in the existing score/accuracy functions of IVPFNs are investigated in Section 3. Also, in this section, a new accuracy function for IVPFNs will be proposed to handle the stated shortcomings. In Section 4, first, some exiting distance measures and weighted averaging operators are listed for IVPFNs and then, a new interval-valued Pythagorean fuzzy p-distance measure for IVPFNs is introduced in order to overcome the shortcomings in the existing methodology. In Section 5, procedural steps of the proposed algorithm are provided for an MCDM problem. Subsequently, a numerical example is solved to validate the algorithm as well as the applicability of the proposed accuracy function, IVPF p-distance measure, and the weighted averaging operators. Based on the illustrative example of the MCDM problem, some important remarks on the limitations of the existing methods are given in Section 6. Finally, Section 7 provides the concluding remarks and the scope for future work.

2. Pythagorean fuzzy numbers and their score and accuracy functions

Some fundamentals related to PFSs/numbers along with their score/accuracy functions are presented in this section.

A PFS over U (domain) is defined by Yager [4] as:

$$P = \{< u,\, \mu_P(u),\, \nu_P(u) > |u \in U\},$$

where $\mu_P : U \to [0, 1]$ and $\nu_P : U \to [0, 1]$ are the membership and non-membership functions such that $0 \leq (\mu_P(u))^2 + (\nu_P(u))^2 \leq 1$. The numbers $\mu_P(u)$ and $\nu_P(u)$ denote the degrees of membership and non-membership of $u \in U$ in P, respectively. For each PFS $P \in U$, the quantity $\pi_P(u) = \sqrt{1 - \mu_P^2(u) - \nu_P^2(u)}$ represents the degree of indeterminacy of $u \in U$.

For simplicity, Zhang and Xu [6] called the pair $(\mu_P(u), \nu_P(u))$ as a Pythagorean fuzzy number denoted by $p = (\mu_P, \nu_P)$. Peng and Yang [7] defined an interval-valued PFS in U given by:

$$P = \Big\{u,\, \mu_p(u) = [\underline{\mu}_p(u),\, \overline{\mu}_p(u)],$$

$$\nu_p(u) = [\underline{\nu}_p(u),\, \overline{\nu}_p(u)]|u \in U\Big\},$$

where $[\underline{\mu}_p(u), \overline{\mu}_p(u)]$ and $[\underline{\nu}_p(u), \overline{\nu}_p(u)]$ denote the membership and non-membership degrees of u in P, respectively, under the condition $0 \leq \left(\overline{\mu}_p(u)\right)^2 + \left(\overline{\nu}_p(u)\right)^2 \leq 1$. Here, $\underline{\mu}_p(u) = \inf \mu_p(u)$, $\overline{\mu}_p(u) = \sup \mu_p(u)$, $\underline{\nu}_p(u) = \inf \nu_p(u)$, $\overline{\nu}_p(u) = \sup \nu_p(u)$ for all $u \in U$. The degree of indeterminacy $\pi_p(u) = [\underline{\pi}_p(u), \overline{\pi}_p(u)]$ for all $u \in U$ is called the interval-valued

Pythagorean fuzzy index of u in P, where:

$$\underline{\pi}_p(u) = \sqrt{1 - \left(\overline{\mu}_p(u)\right)^2 - \left(\overline{\nu}_p(u)\right)^2},$$

$$\overline{\pi}_p(u) = \sqrt{1 - (\underline{\mu}_p(u))^2 - \left(\underline{\nu}_p(u)\right)^2}.$$

Yager and Abbasov [5] used another representation of PFN as $p = (r_P, d_P)$, where r_p, and $d_p \in [0,1]$ are called the strength and direction of the commitment of p, respectively. There is a one-to-one correspondence between (μ_p, ν_p) and (r_p, d_p), given by $\mu_p = r_p \cos(\theta_p)$, $\nu_p = r_p \sin(\theta_p)$, where $\theta_p = \arccos(\mu_p/r_p)$ and $d_p = 1 - \frac{2\theta_p}{\pi}$.

Furthermore, Yager and Abbasov [5] showed that the Pythagorean membership degrees were contained in the class of complex numbers, denoted by $\prod -i$. Therefore, they presented PFN $p = (\mu_p, \nu_p)$ as $p = re^{-i\theta}$, where $\mu_p = r_p \cos(\theta)$ and $\nu_p = r_p \sin(\theta)$.

In order to compare two PFNs, Yager [4] proposed the following formula:

$$V(p) = \frac{1}{2} + r_p\left(d_p - \frac{1}{2}\right) = \frac{1}{2} + r_p\left(\frac{1}{2} - \frac{2\theta_p}{\pi}\right). \quad (1)$$

Let $p_1 = (r_{p_1}, d_{p_1})$ and $p_2 = (r_{p_2}, d_{p_2})$ be two PFNs, then:

- If $V(p_1) > V(p_2)$, then $p_1 \succ p_2$;
- If $V(p_1) = V(p_2)$, then $p_1 \sim p_2$.

Furthermore, for comparing two Pythagorean fuzzy numbers, Zhang and Xu [6] proposed a score function of $p = (\mu_p, \nu_p)$ given as:

$$s(p) = (\mu_p)^2 - (\nu_p)^2, \quad (2)$$

where $s(p) \in [-1, 1]$.

Based on the score function (Eq. (2)), Zhang and Xu [6] gave the comparison rule of:

Let $p_1 = (\mu_{p_1}, \nu_{p_1})$ and $p_2 = (\mu_{p_2}, \nu_{p_2})$ be two PFNs, then:

- $s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2$;
- $s(p_1) > s(p_2) \Rightarrow p_1 \succ p_2$;
- $s(p_1) = s(p_2) \Rightarrow p_1 \sim p_2$.

It has also been pointed out by Peng and Yang [22] that the score function defined by Zhang and Xu [6] is not reasonable in some cases. For instance, suppose that for two PFNs, $p_1 = (0.6, 0.6)$ and $p_2 = (0.7, 0.7)$; then, by using score function (Eq. (2)), we have $p_1 \sim p_2$. However, p_1 and p_2 are different. Thus, with regard to the shortcoming of score function (Eq. (2)), the idea of accuracy function of a PFN has been proposed in the literature. The revised rules for comparison are as follows:

- Let $p = (\mu_p, \nu_p)$ be a PFN, then the accuracy function of p is given by

  $$a(p) = (\mu_p)^2 + (\nu_p)^2, \quad (3)$$

  where $a(p) = [0, 1]$. Based on the above accuracy function (Eq. (3)), the following comparison rules are provided.
- Let $p_1 = (\mu_{p_1}, \nu_{p_1})$ and $p_2 = (\mu_{p_2}, \nu_{p_2})$ be two PFNs, then

1. $s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2$;
2. If $s(p_1) = s(p_2)$, then:
   - $a(p_1) < a(p_2) \Rightarrow p_1 \prec p_2$;
   - $a(p_1) = a(p_2) \Rightarrow p_1 \sim p_2$.

3. Proposed score and accuracy functions

Here, the basics of interval-valued fuzzy numbers along with their score and accuracy functions are discussed. Also, a new accuracy function for the interval-valued fuzzy numbers is proposed and studied in contrast with the existing accuracy functions.

For an interval valued PFS P, consider the pair:

$$\left([\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)]\right),$$

as an IVPFN denoted by:

$$p = \left([\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)]\right).$$

Then, for convenience, we represent IVPFN as $p = \left((\underline{r}_p, \overline{r}_p), (\underline{d}_p, \overline{d}_q)\right)$, where the pair $(\underline{r}_p, \overline{r}_p)$ is called the lower and upper strength of p and the pair $(\underline{d}_p, \overline{d}_p)$ is called the lower and upper directions of the lower and upper strength of p, respectively. Moreover, the pairs $(\underline{r}_p, \overline{r}_p)$ and $(\underline{d}_p, \overline{d}_p)$ are connected with interval-valued membership degree $[\underline{\mu}_p(u), \overline{\mu}_p(u)]$ and non-membership degrees $[\underline{\nu}_p(u), \overline{\nu}_p(u)]$, indicating the support for membership/belongingness and the support against membership of $u \in P$, respectively.

The relationship between $([\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)])$ and $\left((\underline{r}_p, \overline{r}_p), (\underline{d}_p, \overline{d}_q)\right)$ is as follows:

$$\underline{\mu}_p(u) = \underline{r}_p \cos(\underline{\theta}_p), \ \underline{\nu}_p(u) = \underline{r}_p \sin(\underline{\theta}_p);$$

$$\overline{\mu}_p(u) = \overline{r}_p \cos(\overline{\theta}_p), \ \overline{\nu}_p(u) = \overline{r}_p \sin(\overline{\theta}_p),$$

and:

$$(\underline{d}_p, \overline{d}_p) = \left(1 - \frac{2\underline{\theta}_p}{\pi}, 1 - \frac{2\overline{\theta}_p}{\pi}\right),$$

where:

$$\underline{\theta}_p = \arccos(\underline{\mu}_p(u)/\underline{r}_p), \qquad \overline{\theta}_p = \arccos(\overline{\mu}_p(u)/\overline{r}_p).$$

Furthermore, we can easily show that the Pythagorean membership and non-membership degrees of IVPFN, $p = \left([\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)]\right)$, can be viewed as a radial length, $p = [\underline{r}_p e^{i\underline{\theta}_p}, \overline{r}_p e^{i\overline{\theta}_p}]$, in the complex plane, where:

$$\underline{r}_p = \sqrt{(\underline{\mu}_p(u))^2 + \left(\underline{\nu}_p(u)\right)^2},$$

$$\overline{r}_p = \sqrt{\left(\overline{\mu}_p(u)\right)^2 + (\overline{\nu}_p(u))^2}.$$

To resolve the issue of the comparison of IVPFNs, Peng and Yang [7] utilized the notion of score and accuracy functions, which are given as follows:

$$s(p) = \frac{1}{2}\left((\underline{\mu}_p(u))^2 + \left(\overline{\mu}_p(u)\right)^2 - \left(\underline{\nu}_p(u)\right)^2 - (\overline{\nu}_p(u))^2\right), \tag{4}$$

$$a(p) = \frac{1}{2}\left((\underline{\mu}_p(u))^2 + \left(\overline{\mu}_p(u)\right)^2 + \left(\underline{\nu}_p(u)\right)^2 + (\overline{\nu}_p(u))^2\right), \tag{5}$$

where $s(p) \in [-1, 1]$ and $a(p) \in [0, 1]$.

Based on these functions, they provided the comparison rules of: Let p_1 and p_2 be two IVPFNs, then:

1. $s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2$;
2. $s(p_1) > s(p_2) \Rightarrow p_1 \succ p_2$;
3. If $s(p_1) = s(p_2)$, then:
   - $a(p_1) < a(p_2) \Rightarrow p_1 \prec p_2$;
   - $a(p_1) > a(p_2) \Rightarrow p_1 \succ p_2$;
   - $a(p_1) = a(p_2) \Rightarrow p_1 \sim p_2$.

In some cases, it may be observed that the score and accuracy functions given by Eqs. (4) and (5) are not able to rank IVPFNs accurately. In the following example, it is shown that the existing score and the accuracy functions are not sufficiently appropriate to set the correct order of preference of the objects involved in the MCDM problem.

Example 1. *Consider two IVPFNs given by:*

$$p_1 = ([0.3, 0.6], [0.4, 0.8]),$$

and:

$$p_2 = \left([\sqrt{0.20}, \sqrt{0.25}], [\sqrt{0.35}, \sqrt{0.45}]\right).$$

Using Eq. (4), we get $s(p_1) = -0.1750$ *and* $s(p_2) = -0.1750$. *Now, we compute the value of accuracy function by using Eq. (5) and get* $a(p_1) = 0.6250$ *and* $a(p_2) = 0.6250$. *Therefore, based on the comparison rule, we get* $p_1 \sim p_2$. *However, it may clearly be noted that* $p_1 \neq p_2$. *Hence, the existing score and accuracy functions of the IVPFNs are not capable enough to give the correct order of preference.*

Furthermore, to overcome the shortcomings of the accuracy function given by Eq. (5), Garg [23] proposed a new improved accuracy function by considering the hesitation degree in the formulation.

For any IVPFN, $p = ([\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)])$, the improved accuracy function $K(p)$ of p is defined as:

$$K(p) = \frac{1}{2}\Bigg((\underline{\mu}_p(u))^2 + \left(\overline{\mu}_p(u)\right)^2 \sqrt{1 - (\underline{\mu}_p(u))^2 - \left(\underline{\nu}_p(u)\right)^2} + (\overline{\mu}_p(u))^2 + (\underline{\mu}_p(u))^2 \sqrt{1 - (\overline{\mu}_p(u))^2 - (\overline{\nu}_p(u))^2}\Bigg). \tag{6}$$

Garg [23] gave the comparison rule on the basis of Eq. (6): Let p_1 and p_2 be two IVPFNs:

- $K(p_1) < K(p_2) \Rightarrow p_1 \prec p_2$;
- If $K(p_1) > K(p_2) \Rightarrow p_1 \succ p_2$;
- If $K(p_1) = K(p_2) \Rightarrow p_1 \sim p_2$.

If we apply Eq. (6) to Example 1, then we get $K(p_1) = 0.3809$ and $K(p_2) = 0.3636$. Since $K(p_1) > K(p_2)$, then p_1 has higher preference than p_2.

Example 2. *Let* $p_1 = \left([0.1, 0.2], [\sqrt{0.05}, 0.6]\right)$ *and* $p_2 = \left([0.1, 0.2], [\sqrt{0.04}, \sqrt{0.37}]\right)$ *be two IVPFNs; then, by using Eqs. (4) and (5), we obtain the values* $s(p_1) = -0.1400, s(p_2) = -0.1400$ *and* $a(p_1) = 0.2700, a(a_2) = 0.2700$. *This shows that* p_1 *and* p_2 *are equivalent. However, in fact, they are different. On the other hand, by using Eq. (6), we get* $K(p_1) = 0.04826$ *and* $K(p_2) = 0.04833$, *i.e.* $K(p_1) < K(p_2)$, *which implies that* $p_1 \prec p_2$. *If we take the value of accuracy function up to four decimal places, then the improved accuracy function is unable to differentiate between* p_1 *and* p_2. *Hence, both existing accuracy functions (Eqs. (5) and (6)) are unable to differentiate between the IVPFNs.*

Hence, to resolve the above-stated comparison issues between IVPFNs, we extend Yager's [4] accuracy function (Eq. (1)) of PFNs to IVPFNs as follows.

For each IVPFN, $p = ([\underline{r}_p, \overline{r}_p], [\underline{d}_p, \overline{d}_p])$, we define a new accuracy function as:

$$T(p) = \frac{2(\underline{r}_p\underline{d}_p + \overline{r}_p\overline{d}_p) - (\underline{r}_p + \overline{r}_p) + 2}{4}, \tag{7}$$

where:

$$\underline{r}_p = \sqrt{(\underline{\mu}_p(u))^2 + \left(\underline{\nu}_p(u)\right)^2},$$

$$\overline{r}_p = \sqrt{\left(\overline{\mu}_p(u)\right)^2 + (\overline{\nu}_p(u))^2},$$

and:

$$(\underline{d}_p, \overline{d}_p) = \left(1 - \frac{2\underline{\theta}_p}{\pi}, 1 - \frac{2\overline{\theta}_p}{\pi}\right).$$

Based on the proposed accuracy function (Eq. (7)), we give the comparison rules of: Let p_1 and p_2 be two IVPFNs, then:

- $T(p_1) < T(p_2) \Rightarrow p_1 \prec p_2$;
- $T(p_1) > T(p_2) \Rightarrow p_1 \succ p_2$;
- $T(p_1) = T(p_2) \Rightarrow p_1 \sim p_2$.

If we apply the proposed accuracy function (Eq. (7)) to the above two examples, we obtain the following values:

- In Example 1, $T(p_1) = 0.4322$, $T(p_2) = 0.4288$;
- In Example 2, $T(p_1) = 0.3782$, $T(p_2) = 0.3818$;
- In light of the revised comparison rules for the proposed accuracy function (Eq. (7)), in Example 1, we have $T(p_1) > T(p_2)$, which indicate that $p_1 \succ p_2$, and in Example 2, we have $T(p_1) < T(p_2)$, which indicate that $p_1 \prec p_2$.

In order to demonstrate effectiveness of the proposed accuracy function, the findings and observations of this study regarding the preference order are presented in contrast with the existing accuracy functions in Table 1.

From Table 1, it is clear that for Example 1, Peng and Yang [7] score and accuracy functions are not appropriate for setting the correct preference order of IVPFNs and the preference ordering obtained using the proposed accuracy functions in this study is consistent with Garg [23] accuracy functions. For Example 2, both existing accuracy functions of Peng and Yang [7] and Garg [23] are not sufficiently appropriate for setting the correct preference order of IVPFNs, but the proposed accuracy function in this study is capable to set the correct ordering. Therefore, the ranking results obtained by the proposed accuracy function are better than those of the existing score and the accuracy functions.

Example 3. *Consider two IVPNFs given by:*

$$p_1 = \left([\sqrt{0.3}, \sqrt{0.6}], [\sqrt{0.2}, \sqrt{0.3}]\right),$$

and:

$$p_2 = \left([\sqrt{0.3}, \sqrt{0.4}], [\sqrt{0.1}, \sqrt{0.2}]\right).$$

We compute $\underline{r}_{p_1} = \sqrt{0.5}$*,* $\overline{r}_{p_1} = \sqrt{0.9}$*,* $\underline{d}_{p_1} = 0.5641$*,* $\overline{d}_{p_1} = 0.6082$*,* $\underline{r}_{p_2} = \sqrt{0.4}$*,* $\overline{r}_{p_2} = \sqrt{0.6}$*,* $\underline{d}_{p_2} = 0.6082$*,* $\overline{d}_{p_2} = 0.6667$*,* $s(p_1) = 0.2$*,* $s(p_2) = 0.2$*, and* $a(p_1) = 0.07$*,* $a(p_2) = 0.05$*. In view of the rules for comparison given in [7], we obtain* $p_1 \succ p_2$*. Also, by the comparison rules offered by Garg [23], we get:* $K(p_1) = 0.7096$*,* $K(p_2) = 0.5998$*, i.e.* $K(p_1) > K(p_2)$*, which implies that* $p_1 \succ p_2$*. However, by Eq. (7), we get* $T(p_1) = 0.5740$*,* $T(p_2) = 0.5946$*, i.e.,* $T(p_1) < T(p_2)$ *which implies that* $p_1 \prec p_2$*. Hence, the comparison rules of the proposed accuracy function (Eq. (7)), lead to contradictory results.*

In the above example, the direction of commitment $\underline{d}_{p_1}(> \pi/4)$ is greater than the direction of commitment $\overline{d}_{p_2}(< \pi/4)$ and $\overline{d}_{p_1} = \underline{d}_{p_2} = 0.6082 < \pi/4$, which indicates that p_2 supports the membership better than p_1, i.e., p_2 is preferred to p_1 (refer to Table 2). Since the proposed accuracy function (Eq. (7)) takes the direction of commitment of IVPFN as an input, its ranking result may be better than the existing score and the accuracy functions.

Table 1. Comparative analysis for Examples 1 and 2.

	Methodology	Score and accuracy functions		Preference order
Example 1	Peng and Yang [7]	$s(p_1) = -0.175$, $a(p_1) = 0.625$,	$s(p_2) = -0.17$ $a(p_2) = 0.625$	$p_1 \sim p_2$
	Garg [23]	$K(p_1) = 0.381$,	$K(p_2) = 0.364$	$p_1 \succ p_2$
	Proposed method	$T(p_1) = 0.432$,	$T(p_2) = 0.429$	$p_1 \succ p_2$
Example 2	Peng and Yang [7]	$s(p_1) = -0.140$ $a(p_1) = 0.270$,	$s(p_2) = -0.140$, $a(p_2) = 0.270$	$p_1 \sim p_2$
	Garg [23]	$K(p_1) = 0.0483$,	$K(p_2) = 0.0483$	$p_1 \sim p_2$
	Proposed method	$T(p_1) = 0.378$,	$T(p_2) = 0.382$	$p_1 \prec p_2$

Table 2. Comparative analysis for Example 3.

	Methodology	Score and accuracy functions		Preference order
Example 3	Peng and Yang [7]	$s(p_1) = 0.2$, $a(p_1) = 0.07$,	$s(p_2) = 0.2$, $a(p_2) = 0.05$	$p_1 \succ p_2$
	Garg [23]	$K(p_1) = 0.7096$,	$K(p_2) = 0.5998$	$p_1 \succ p_2$
	Proposed method	$T(p_1) = 0.5740$,	$T(p_2) = 0.5946$	$p_1 \prec p_2$

4. IVPF distance measures and weighted averaging operators

In this section, first, we present the distance measures and weighted averaging operators proposed by Liu et al. [21] for IVPFNs. Then, a new distance measure is given for IVFFNs and the shortcomings in the existing distance measure proposed by Liu et al. [21] are discussed.

Liu et al. [21] proposed the following IVPF p-distance measure, generalized IVPF weighted distance measure, generalized probabilistic IVPF Ordered Weighted Averaging (OWA) distance operators, and further proceedings for IVPFNs.

Definition 1. IVPF p-distance measure: *Consider any two IVPFNs:*

$$\xi_1 = (\mu_{\xi_1}, \nu_{\xi_1}) = \left([\underline{\mu}_{\xi_1}, \overline{\mu}_{\xi_1}],\ [\underline{\nu}_{\xi_1}, \overline{\nu}_{\xi_1}]\right),$$

and:

$$\xi_2 = (\mu_{\xi_2}, \nu_{\xi_2}) = \left([\underline{\mu}_{\xi_2}, \overline{\mu}_{\xi_2}],\ [\underline{\nu}_{\xi_2}, \overline{\nu}_{\xi_2}]\right).$$

The IVPF p-distance between ξ_1 and ξ_2 is denoted by $d_p(\xi_1, \xi_2)$ and defined as follows:

$$\begin{aligned} d^p(\xi_1, \xi_2) =& \frac{1}{4}\Big[|(\underline{\mu}_{\xi_1})^2 - (\underline{\mu}_{\xi_2})^2|^p + |(\overline{\mu}_{\xi_1})^2 - (\overline{\mu}_{\xi_2})^2|^p \\ &+ |(\underline{\nu}_{\xi_1})^2 - (\underline{\nu}_{\xi_2})^2|^p + |(\overline{\nu}_{\xi_1})^2 - (\overline{\nu}_{\xi_2})^2|^p \\ &+ |(\underline{\pi}_{\xi_1})^2 - (\underline{\pi}_{\xi_2})^2|^p + |(\overline{\pi}_{\xi_1})^2 \\ &- (\overline{\pi}_{\xi_2})^2|^p\Big]. \end{aligned} \tag{8}$$

Definition 2. Generalized IVPF Weighted Distance measure: *Let $A = (\xi_1, \xi_2, \ldots, \xi_n)$ and $B = (\eta_1, \eta_2, \ldots, \eta_n)$ be two n-tuples of IVPFNs, where $(\mu_{\xi_i}, \nu_{\xi_i}) = \left([\underline{\mu}_{\xi_i}, \overline{\mu}_{\xi_i}],\ [\underline{\nu}_{\xi_i}, \overline{\nu}_{\xi_i}]\right)$ and $(\mu_{\eta_i}, \nu_{\eta_i}) = \left([\underline{\mu}_{\eta_i}, \overline{\mu}_{\eta_i}], [\underline{\nu}_{\eta_i}, \overline{\nu}_{\eta_i}]\right)$. Then, the generalized IVPF Weighted Distance (GIVPFWD) measure is a function of $\Phi : IVPFN^n \times IVPFN^n \to \mathbb{R}$ and defined as:*

$$\Phi(A, B) = \left(\sum_{i=1}^{n} \omega_i d^p(\xi_i, \eta_i)\right)^{1/p}, \tag{9}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector with $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$ and $d^p(\xi_i, \eta_i)$ is the IVPF p-distance between IVPFNs ξ_i and η_i defined by Eq. (8).

In addition, the following points should be mentioned:

- If $d^p(\xi_i, \eta_i)$ is the ith largest value of $d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n$ in Eq. (9), then the distance measure (Eq. (9)) is called the generalized IVPF ordered weighted distance (GIVPFOWD) measure for IVPFNs.
- If we take $p = 1$ in Eq. (9), then it becomes the IVPF Weighted Averaging Distance (IVPFWAD) measure given by:

$$\Phi(A, B) = \sum_{i=1}^{n} \omega_i d^1(\xi_i, \eta_i). \tag{10}$$

- If we take $p = 2$ in Eq. (9), then it becomes the IVPF Weighted Euclidean Distance (IVPFWED) measure given by:

$$\Phi(A, B) = \left(\sum_{i=1}^{n} \omega_i d^2(\xi_i, \eta_i)\right)^{1/2}. \tag{11}$$

Definition 3. Generalized probabilistic IVPF-OWA distance operators. *Let $A = (\xi_1, \xi_2, \ldots, \xi_n)$ and $B = (\eta_1, \eta_2, \ldots, \eta_n)$ be two n-tuples of IVPFNs, where*

$$(\mu_{\xi_i}, \nu_{\xi_i}) = \left([\underline{\mu}_{\xi_i}, \overline{\mu}_{\xi_i}],\ [\underline{\nu}_{\xi_i}, \overline{\nu}_{\xi_i}]\right),$$

and

$$(\mu_{\eta_i}, \nu_{\eta_i}) = \left([\underline{\mu}_{\eta_i}, \overline{\mu}_{\eta_i}],\ [\underline{\nu}_{\eta_i}, \overline{\nu}_{\eta_i}]\right).$$

Then, the Probabilistic generalized IVPF Weighted Averaging Distance (P-GIVPFWAD) operator is a function of $\Phi : IVPFN^n \times IVPFN^n \to \mathbb{R}$ and defined as:

$$\Phi(A, B) = \left(\sum_{i=1}^{n} \rho_i d^p(\xi_i, \eta_i)\right)^{1/p}, \quad (12)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is a weight vector with* $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1; \rho_i = \lambda_i w_i + (1-\lambda_i)p_i$ *and* p_i *is the associated probability of IVPF p-distance* $d^p(\xi_i, \eta_i)$. $\lambda_i \in [0, 1]$ *and* $(1-\lambda_i)$ *represent the degree of weight and the degree of probabilistic information, respectively.*

Remarks

- If $\lambda_i = 0$, then GPWIVPF distance measure (Eq. (12)) is called the probabilistic generalized IVPF distance measure.
- If $\lambda_i = 1$, then it reduces to GIVPFWD distance measure (Eq. (9)).
- If we take $p = 1$ in Eq. (12), then it becomes P-IVPFWAD operator, given by:

$$\Phi(A, B) = \sum_{i=1}^{n} \rho_i d^1(\xi_i, \eta_i). \quad (13)$$

- If we take $p = 2$ in Eq. (12), then it becomes the Probabilistic Interval-Valued Pythagorean Fuzzy Weighted Euclidean Distance (P-IVPFWED) operator, given by:

$$\Phi(A, B) = \left(\sum_{i=1}^{n} \rho_i d^2(\xi_i, \eta_i)\right)^{1/2}. \quad (14)$$

- If $d^p(\xi_i, \eta_i)$ is the ith largest $d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n$ and p_i is its corresponding probability then the distance measure (Eq. (12)) is called P-GIVPFOWAD operator.
- If we take $\rho_i = \frac{w_i p_i}{\sum_{i=1}^{n} w_i p_i}$ and $d^p(\xi_i, \eta_i)$ as the ith largest $d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n$ with p_i as the corresponding probability of the ith largest $d^p(\xi_j, \eta_j)$, then the distance measure (Eq. (12)) is called IP-GIVPFOWAD operator.

It may be observed that the above distance measures or operators sometimes lead to some unreasonable output. For instance:

Example 4. *Let* $\xi_1 = ([0.5, 0.6], [0.6, 0.7])$, $\xi_2 = ([0.6, 0.7], [0.5, 0.6])$ *and* $\xi_3 = ([0.3, 0.4], [0.4, 0.5])$ *be three IVPFNs then, by using Eq. (8) for* $p = 1$, *we have:*

$$d(\xi_1, \xi_3) = d(\xi_2, \xi_3) = 0.40$$

$$\underline{r}_{\xi_1} = \underline{r}_{\xi_2} = 0.7810; \qquad \overline{r}_{\xi_1} = \overline{r}_{\xi_2} = 0.9220,$$

$$\underline{\theta}_{\xi_1} = 0.8761, \qquad \overline{\theta}_{\xi_1} = 0.8622;$$

$$\underline{\theta}_{\xi_2} = 0.6947, \qquad \overline{\theta}_{\xi_2} = 0.7086;$$

$$\underline{d}_{\xi_1} = 0.4423, \qquad \overline{d}_{\xi_1} = 0.4511;$$

$$\underline{d}_{\xi_2} = 0.5577, \qquad \overline{d}_{\xi_2} = 0.5489.$$

It is observed that ξ_1 *and* ξ_2 *have the same lower and upper lengths, but different lower and upper directions of commitment (strength). Thus, the distance between* ξ_1 *and* ξ_3 *should be different from the distance between* ξ_2 *and* ξ_3. *Hence, in this case IVPF p-distance measure (Eq. (8)) is not appropriate to use.*

Since the strength and direction of commitment are important parameters of IVPFNs, ignoring them may lead to inappropriate results. Hence, by taking all the four parameters into account for IVPFNs,

$$[\underline{\mu}_p(u), \overline{\mu}_p(u)], [\underline{\nu}_p(u), \overline{\nu}_p(u)], (\underline{r}_p, \overline{r}_p), \text{ and } (\underline{d}_p, \overline{d}_p),$$

we propose the IVPF p-distance measure between two IVPFNs, ξ_1 and ξ_2, as follows:

$$d_T^p(\xi_1, \xi_2) = \frac{1}{4}\Big[|\underline{\mu}_{\xi_1} - \underline{\mu}_{\xi_2}|^p + |\overline{\mu}_{\xi_1} - \overline{\mu}_{\xi_2}|^p + |\underline{\nu}_{\xi_1} - \underline{\nu}_{\xi_2}|^p + |\overline{\nu}_{\xi_1} - \overline{\nu}_{\xi_2}|^p + |\underline{r}_{\xi_1} - \underline{r}_{\xi_2}|^p + |\overline{r}_{\xi_1} - \overline{r}_{\xi_2}|^p + |\underline{d}_{\xi_1} - \underline{d}_{\xi_2}|^p + |\overline{d}_{\xi_1} - \overline{d}_{\xi_2}|^p\Big]. \quad (15)$$

By using the proposed IVPF p-distance measure (Eq. (15)) for $p = 1$, for Example 4, we have $d_T(\xi_1, \xi_3) = 0.3542$ and $d_T(\xi_2, \xi_3) = 0.4075$.

Based on Table 3, we can state that the proposed IVPF p-distance measure (Eq. (15)) is a better measure than Eq. (8).

5. Application to a MCDM problem

In this section, we employ the proposed accuracy function, distance measures, and OWA operators to deal with MCDM problem with interval-valued

Table 3. Comparative analysis for Example 4.

	Methodology	IVPF p-distance measure
Example 4	Liu et al. [21]	$d(\xi_1, \xi_3) = d(\xi_2, \xi_3) = 0.40$
	Proposed method	$d_T(\xi_1, \xi_3) = 0.3542$, $d_T(\xi_2, \xi_3) = 0.4075$

Pythagorean fuzzy information. Consider the following setup of an MCDM problem in IVPFNs environment.

Consider the set of m possible alternatives, say, $X = \{x_1, x_2, \ldots, x_m\}$, and the set of n-criteria by which the performance of the alternatives is evaluated, say, $C = \{C_1, C_2, \ldots, C_n\}$. Let $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be the weight vector of all criteria such that $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^{n} \omega_i = 1$. Assume that the performance of an alternative $x_i\,(i = 1, 2, \ldots, m)$ with respect to the criteria C_j, $j = 1, 2, \ldots, n$ is measured by IVPFNs $C_j(x_i) = ([\mu_{i,j}, \mu_{i,j}], [\nu_{i,j}, \nu_{i,j}])$, $j = 1, 2, \ldots, n; i = 1, 2. \ldots, m$; here, $[\mu_{i,j}, \mu_{i,j}]$ represents the degree that alternative x_i satisfies the criterion C_j and $[\nu_{i,j}, \nu_{i,j}]$ represents the degree that alternative x_i does not satisfy the criterion C_j. Let $D_{m\times n} = C_j(x_i)_{m\times n}$ be an interval-valued Pythagorean fuzzy decision matrix. The procedural steps of the proposed MCDM algorithm are as follows:

Step 1: Compute the accuracy of each IVPFN of the obtained decision matrix $M_{m\times n} = C_j(x_i)_{m\times n}$ by applying the proposed accuracy function (Eq. (7)).

Step 2: Determine the IVPF-Positive Ideal Solution (PIS):

$$x^{+} = \{\langle x_i, \max_i T(C_j(x_i))\rangle\},$$

and the IVPF-Negative Ideal Solution (NIS):

$$x^{-} = \{\langle x_i, \min_i T(C_j(x_i))\rangle\},$$

for $j = 1, 2, \ldots, n; i = 1, 2, \ldots, m$ with the help of the accuracies of IVPFNs obtained in Step 1.

Step 3: Evaluate the distance of each alternative $x_i, i = 1, 2, \ldots, m$ from x^+ and x^- using the proposed IVPF p-distance measure (Eq. (15)).

Step 4: Using the values obtained in Step 3 and rearranging the probability weight, evaluate the new weights by using:

$$\rho_i = \lambda_i w_i + (1-\lambda_i)p_i \quad \text{or} \quad \rho_i = \frac{w_i p_i}{\sum\limits_{i=1}^{n} w_i p_i}.$$

Step 5: Determine the P-GIVPFOWAD or IP-GIVPFOWAD of the alternative x_i by the positive ideal IVPFN solution x^+ and the negative ideal IVPFN solution x^-.

Step 6: Compute the coefficient of relative closeness for each alternative x_i as follows:

$$r(x_i) = \frac{D(x_i, x^-)}{D(x_i, x^+) + D(x_i, x^-)},$$

$$i = 1, 2, \ldots, m, \tag{16}$$

where $D(\cdot)$ is an IP-GIVPFOWAD or P-GIVPFOWAD.

Step 7: Rank all the alternatives based on the coefficient of relative closeness $r(x_i)$ and choose the optimal alternative.

5.1. Numerical example

In order to illustrate the implementation of the steps of the proposed algorithm stated above, we consider the following MCDM problem of the selection of the strategy for an optimal production referring to the related literature and undertakings completed in [21,24,25].

Suppose that a firm desires to manufacture a new product and looks for the optimal target of having the maximum benefit. Based upon a survey analysis of the market, they lay down the following five possible strategies (alternatives):

- x_1: Creating a new product aligned with rich customers;
- x_2: Creating a new product aligned with mid-level customers;
- x_3: Creating a new product aligned with low-level customers;
- x_4: Creating a new product suited to all customers;
- x_5: No manufacturing of any product.

After a detailed investigation into the information received from sources, the decision makers go for the following general criteria for the adaptability of strategies to production:

- C_1: Short-term benefits;
- C_2: Mid-term benefits;
- C_3: Long-term benefits;
- C_4: Production risk;
- C_5: Various other factors.

Construct the decision matrix:

$$D_{5\times 5} = C_j(x_i)_{5\times 5} = ([\mu_{i,j}, \mu_{i,j}], [\nu_{i,j}, \nu_{i,j}]),$$

$$j = 1, 2, ..., 5; \quad i = 1, 2, ..., 5,$$

as shown in Table 4.

Suppose that with reference to the problem under consideration, the decision makers find the probabilistic information $p = (0.3, 0.3, 0.2, 0.1, 0.1)$ and weight vector $w = (0.2, 0.25, 0.15, 0.3, 0.1)$, which represents the degree of importance/weightage of each criterion. Then, to get the most desirable alternative, we apply the steps of the proposed algorithm. First, we compute the accuracy of each IVPFN of the decision matrix as shown in Table 4 by applying the proposed accuracy function (Eq. (7)). The computed values are tabulated in Table 5.

Table 4. Interval-valued Pythagorean fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5
x_1	([0.6,0.7], [0.5,0.6])	([0.4,0.5], [0.5,0.6])	([0. 2,0.6], [0.3,0.6])	([0.5,0.6], [0.4,0.5])	([0.2,0.7], [0.3,0.7])
x_2	([0.5,0.6], [0.6,0.7])	([0.5,0.6], [0.4,0.5])	([0.3,0.6], [0.2,0.6])	([0.4,0.5], [0.5,0.6])	([0.3,0.7], [0.2,0.7])
x_3	([0.3,0.6], [0.4,0.8])	([0.6,0.7], [0.4,0.5])	([0.6,0.8], [0.3,0.4])	([0.4,0.5], [0.3,0.6])	([0.3,0.4], [0.7,0.8])
x_4	([0.5,0.7], [0.5,0.6])	([0.4,0.5], [0.1,0.3])	([0.1,0.2], [0.4,0.6])	([0.4,0.5], [0.1,0.2])	([0.5,0.7], [0.4,0.6])
x_5	([0.3,0.4], [0.1,0.2])	([0.4,0.6], [0.4,0.5])	([0.2,0.3], [0.5,0.6])	([0.4,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4])

Table 5. Results of applying accuracy function.

	C_1	C_2	C_3	C_4	C_5
x_1	0.5451	0.4549	0.4773	0.5451	0.4773
x_2	0.4549	0.5451	0.5227	0.4549	0.5227
x_3	0.4322	0.5905	0.6603	0.5000	0.3162
x_4	0.5225	0.6164	0.3357	0.6403	0.5451
x_5	0.5925	0.5225	0.3619	0.4774	0.6603

Based on the accuracies obtained in Table 5, we find the IVPF-PIS x^+ and the IVPF-NIS x^-, respectively, as follows:

$$x^+ = \Big\{\langle C_1, ([0.3, 0.4], [0.1, 0.2])\rangle, \langle C_2, ([0.4, 0.5], [0.1, 0.3])\rangle, \langle C_3, ([0.6, 0.8], [0.3, 0.4])\rangle, \langle C_4, ([0.4, 0.5], [0.1, 0.2])\rangle, \langle C_5, ([0.6, 0.8], [0.3, 0.4])\rangle\Big\}, \tag{17}$$

and:

$$x^- = \{\langle C_1, ([0.3, 0.6], [0.4, 0.8])\rangle, \langle C_2, ([0.4, 0.5], [0.5, 0.6])\rangle, \langle C_3, ([0.1, 0.2], [0.4, 0.6])\rangle, \langle C_4, ([0.4, 0.5], [0.5, 0.6])\rangle, \langle C_5, ([0.3, 0.4], [0.7, 0.8])\rangle\}. \tag{18}$$

Furthermore, we evaluate the P-GIVPFOWAD or IP-GIVPFOWAD for each alternative x_i, $i = 1, 2, \ldots, m$ from the IVPF-PIS x^+ and the IVPF-NIS x^-. The respective results are shown in Tables 6 to 9. Then, we find the relative closeness coefficient by Eq. (16) for each alternative x_i. The results are given in Tables 10 and 11 and illustrated in Figures 1 and 2. In Tables 10 to 12, we observe that for different values of parameter p, the ranking order of alternatives remains unchanged by applying either P-GIVPFOWAD or IP-GIVPFOWAD and all the results show that x_5 is the optimal alternative.

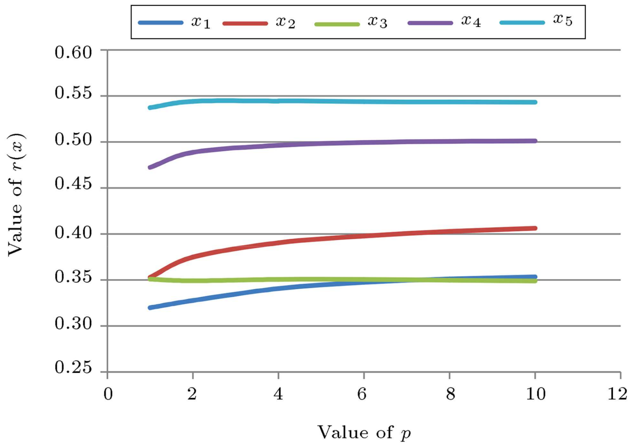

Figure 1. Ranking order based on P-GIVPFOWAD.

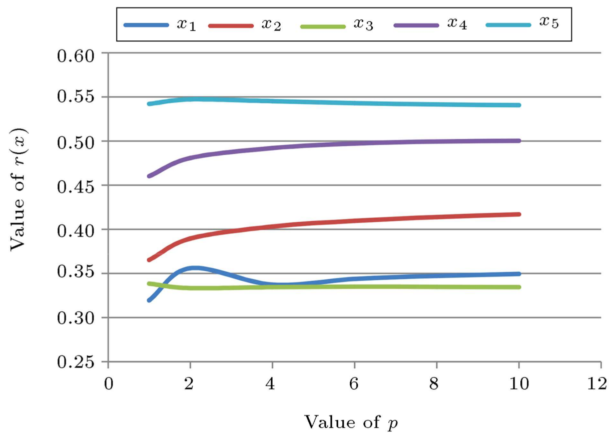

Figure 2. Ranking order based on IP-GIVPFOWAD.

6. Remarks on limitations of existing methods

Based on the numerical example and the values obtained in the previous section, we put forward some remarks on the limitations of the existing methods:

- If we deal with the MCDM problem with the proposed accuracy function (Eq. (7)) and Liu et al.'s [21] distance measure (Eq. (8)), Based on Table 9, we observe that the distances of x_1 and x_2 from IVPF-PIS x^+ for $p = 1$ are the same. However, the alternatives x_1 and x_2 with respect to all criteria $C_j, j = 1, 2, \ldots, n$, take different IVPFNs;

Table 6. Distances between x_i and x^+ obtained by P-GIVPFOWAD.

Value of p	$D(x_1, x^+)$	$D(x_2, x^+)$	$D(x_3, x^+)$	$D(x_4, x^+)$	$D(x_5, x^+)$
1	0.4934	0.4783	0.4152	0.3468	0.2807
2	0.3924	0.3717	0.3331	0.2635	0.2234
4	0.3681	0.3462	0.3138	0.2421	0.2125
6	0.3690	0.3452	0.3150	0.2405	0.2145
8	0.3728	0.3470	0.3183	0.2417	0.2171
10	0.3766	0.3490	0.3214	0.2434	0.2194

Table 7. Distances between x_i and x^- obtained by P-GIVPFOWAD

Value of p	$D(x_1, x^-)$	$D(x_2, x^-)$	$D(x_3, x^-)$	$D(x_4, x^-)$	$D(x_5, x^-)$
1	0.2320	0.2609	0.2245	0.3103	0.3258
2	0.1912	0.2229	0.1787	0.2519	0.2666
4	0.1902	0.2217	0.1695	0.2385	0.2539
6	0.1965	0.2280	0.1700	0.2398	0.2555
8	0.2017	0.2339	0.1712	0.2423	0.2583
10	0.2057	0.2386	0.1723	0.2444	0.2607

Table 8. Distances between x_i and x^+ obtained by IP-GIVPFOWAD.

Value of p	$D(x_1, x^+)$	$D(x_2, x^+)$	$D(x_3, x^+)$	$D(x_4, x^+)$	$D(x_5, x^+)$
1	0.4715	0.4568	0.3903	0.3662	0.2453
2	0.3815	0.3510	0.3122	0.2750	0.1963
4	0.3619	0.3281	0.2918	0.2501	0.1871
6	0.3647	0.3279	0.2910	0.2475	0.1889
8	0.3694	0.3301	0.2929	0.2483	0.1914
10	0.3738	0.3325	0.2951	0.2499	0.1935

Table 9. Distances between x_i and x^- obtained by IP-GIVPFOWAD.

Value of p	$D(x_1, x^-)$	$D(x_2, x^-)$	$D(x_3, x^-)$	$D(x_4, x^-)$	$D(x_5, x^-)$
1	0.2213	0.2629	0.1995	0.3122	0.2902
2	0.1842	0.2238	0.1561	0.2545	0.2372
4	0.1843	0.2215	0.1466	0.2423	0.2241
6	0.1910	0.2275	0.1466	0.2445	0.2243
8	0.1965	0.2331	0.1475	0.2476	0.2260
10	0.2008	0.2377	0.1483	0.2501	0.2277

Table 10. Relative closeness coefficients obtained by P-GIVPFOWAD.

Value of p	$r(x_1)$	$r(x_2)$	$r(x_3)$	$r(x_4)$	$r(x_5)$	Ranking order
1	0.3198	0.3529	0.3509	0.4722	0.5372	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
2	0.3276	0.3749	0.3492	0.4887	0.5441	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
4	0.3407	0.3904	0.3507	0.4963	0.5444	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
6	0.3475	0.3978	0.3505	0.4993	0.5436	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
8	0.3511	0.4027	0.3497	0.5006	0.5433	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$
10	0.3533	0.4061	0.3490	0.5010	0.5430	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$

Table 11. Relative closeness coefficients obtained by IP-GIVPFOWAD.

Value of p	$r(x_1)$	$r(x_2)$	$r(x_3)$	$r(x_4)$	$r(x_5)$	Ranking order
1	0.3194	0.3653	0.3383	0.4602	0.5419	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
2	0.3256	0.3894	0.3333	0.4806	0.5472	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_3 \trianglerighteq x_1$
4	0.3374	0.4030	0.3344	0.4921	0.5450	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$
6	0.3437	0.4096	0.3350	0.4970	0.5428	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$
8	0.3472	0.4139	0.3349	0.4993	0.5414	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$
10	0.3495	0.4169	0.3345	0.5002	0.5406	$x_5 \trianglerighteq x_4 \trianglerighteq x_2 \trianglerighteq x_1 \trianglerighteq x_3$

Table 12. Distances of x_1 and x_2 from x^+.

$D(x_1, x^+)$	0.5800	0.2550	0.3000	0.2800	0.3250
$D(x_2, x^+)$	0.5800	0.2550	0.3000	0.2800	0.3250

- Also:

$$P - GIVPFOWAD(x_1, x^+)$$

$$= P - GIVPFOWAD(x_2, x^+) = 0.3561,$$

and:

$$IP - GIVPFOWAD(x_1, x^+)$$

$$= IP - GIVPFOWAD(x_2, x^+) = 0.33340,$$

indicate that x_1 and x_2 are of the same preference. Hence, in such cases, Liu et al.'s [21] distance measure (Eq. (8)) is not proper choice for application.

- Scoring of alternatives x_1 and x_2 with respect to criteria C_3 and C_4 by using Peng and Yang's [7] score and accuracy functions (Eqs. (4) and (5)) is as follows: $s(C_3(x_1)) = s(C_3(x_2)) = -0.0250$. However, it should be noted that $C_3(x_1)$ and $C_3(x_2)$ are represented by different IVPFNs. Also, while $s(C_5(x_1)) = s(C_5(x_2)) = 0.0250$, $C_5(x_1)$ and $C_5(x_2)$ are represented by different IVPFNs;
- Regarding accuracies, we have $a(C_3(x_1)) = s(C_5(x_1)) = 0.4250$ and $a(C_3(x_2)) = s(C_5(x_2)) = 0.5550$, but $C_3(x_1)) \neq C_5(x_1)$ and $C_3(x_2)) \neq C_5(x_2)$. Hence, we can not proceed to make decisions in the right direction through the Liu et al. [21] approach;
- Representing the performance of alternatives x_3 and x_4 by IVPFNs $C_3(x_4) = ([0.1, 0.2], [0.04, 0.37])$ and $C_3(X_5) = [(0.2, 0.3], [0.05, 0.6])$, respectively, in terms of the criterion C_4 in the decision matrix as shown in Table 4 are by using Peng and Yang's [7] score and accuracy functions (Eqs. (4) and (5)), we have $s(C_3(x_4)) = -01400; s(C_3(x_5)) = -0.1400$ and $a(C_3(x_4)) = 0.2700; a(C_3(x_5)) = 0.2700$. This result shows that $C_3(x_4)$ and $C_3(x_5)$ are equivalent while, in fact, they are different;
- The above statement shows that by the Peng and Yang's [7] score and accuracy functions, we cannot determine the scoring and accuracy of $C_3(x_4)$ and $C_3(x_5)$, and we may fail to find the positive or negative ideal solution. Hence, we cannot proceed to make decisions by Liu et al. [21] approach.

7. Conclusions and scope of future work

In order to overcome the existing shortcomings in the literature, we successfully incorporated four important parameters of membership, non-membership, strength, and direction of commitment and introduced a new accuracy function for interval valued Pythagorean Fuzzy Sets (PFSs). Furthermore, a new IVPF p-distance measure for interval-valued Pythagorean fuzzy numbers was proposed and used along with the existing weighted averaging operators to deal with an example of Multi-Criteria Decision Making (MCDM) problem. In any field of model evaluation and assessment of the quality of prediction, the estimator score and accuracy functions may be utilized in the future research. Also, the proposed distance measure may be used in multi-label ranking metrics, regression metrics, and clustering metrics.

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