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# Enhancing efficiency of ratio-type estimators of population variance by a combination of information on robust location measures

**F. Naz[a,*], M. Abid[b], T. Nawaz[b,c], and T. Pang[a]**

a. *Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou 310027, People's Republic of China.*
b. *Faculty of Physical Sciences, Department of Statistics, Government College University, Faisalabad, Pakistan.*
c. *School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China.*

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**Abstract.** Numerous ratio-type estimators of population variance have been proposed in the literature based on different characteristics of studies and auxiliary variables. However, the existing estimators are mostly based on the conventional measures of population characteristics and their efficiency is questionable in the presence of outliers in the data. This study presents improved families of variance estimators under simple random sampling without replacement (SRSWOR), assuming that the information on some robust non-conventional location parameters of the auxiliary variable, besides the usual conventional parameters, is known. The bias and mean square error of the proposed families of estimators were obtained and the efficiency conditions were derived mathematically. The theoretical results were supplemented with numerical illustrations by using real datasets, which indicated the supremacy of the suggested families of estimators.



## 1. Introduction

Estimation of population variance along with population mean is of interest in many practical situations such as business, manufacturing industry, services industry, pharmaceutical industry, medical sciences, biological sciences, and agriculture [1]. Therefore, development of efficient estimators of population variance constantly attracts the researchers. Commonly, in survey sampling, the information on an auxiliary variable, which is closely related to the study variable, is utilized to enhance efficiency of estimators of the population characteristic under investigation. When auxiliary information is available and the study variable positively correlates with the auxiliary variable, the ratio estimation method is frequently used to improve efficiency of the estimators of population characteristics [2]. The ratio-type estimators are frequently used in different fields such as process monitoring, designing of acceptance sampling plan for lot sentencing, environmental studies, and forestry, to name but some instances [3–8]. Generally, the information on conventional auxiliary parameters such as the coefficients of kurtosis, skewness, variation, and correlation is used in a linear combination with sample information of the study and auxiliary variables to design the estimators of population variance. Interested readers can refer to Isaki [9], Upadhyaya and Singh [10], Kadilar and Cingi [11], Subramani and Kumarapandiyan [12–14], Khan

*. *Corresponding author. Tel.: +86-13093790613*
*E-mail addresses: naz_farah25@zju.edu.cn (F. Naz); mabid@zju.edu.cn, mabid@gcuf.edu.pk (M. Abid); tahir.nawaz@sjtu.edu.cn, tahir.nawaz@gcuf.edu.pk (T. Nawaz); txpang@zju.edu.cn (T. Pang)*

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and Shabbir [15], Solanki et al. [1], Yaqub and Shabbir [16], Maqbool and Javaid [17], Adichwal et al. [18], Maji et al. [19], Abid et al. [20], Singh et al. [21], and Muneer et al. [22] as well as their cited references for more information on this subject. While the presence of outliers in the data can negatively affect efficiency of the ratio-type estimators, development of a ratio-type estimator that deals with such a condition is a neglected area in survey sampling. Therefore, there is a need for estimators that can accommodate the outliers without loss of efficiency.

The present study incorporates information on robust non-conventional location parameters of the auxiliary variables coupled to the conventional parameters under the ratio-type structure of estimation to propose new ratio-type estimators of population variance. The non-conventional measures used in this study are Tri-Mean (TM), Mid-Range (MR), Hodges-Lehmann estimator of the mean (HL), and Deciles Mean (DM). Although MR is not a robust measure against outliers, it is an effective estimator of the mean when the underlying distribution has excess kurtosis and the sample size ($n$) is small, i.e. $4 \leq n \leq 20$ [23–25]. Since the objective of the current study is to use non-conventional and robust measures of location for variance estimation, MR is included as a non-conventional measure of location. The advantage of using TM, HL, and DM lies in their ability to remain stable in the presence of outliers. These measures are used in a linear combination with other conventional measures to improve efficiency of the variance estimator in Simple Random Sampling Without Replacement (SRSWOR) scheme. The properties of these non-conventional measures can be found in studies conducted by Wang et al. [26], Ferrell [25], Hettmansperger and McKean [27], and Rana et al. [28].

The rest of the paper is structured as follows. The existing estimators of population variance that utilize the known information on conventional parameters of the auxiliary variable are described in Section 2. The proposed improved families of estimators of population variance, which combine the information on known conventional and non-conventional parameters of the auxiliary variable, are described in Section 3. In Section 4, the performance of the proposed estimators is compared with some of the existing estimators of population variance. Finally, the conclusion and some recommendations are presented in Section 5.

## 2. Existing estimators of variance

This section reviews some of the existing estimators of population variance under Simple Random Sampling (SRS), which are based on known auxiliary information of conventional parameters.

Suppose a finite population $V = \{V_1, V_2, \ldots, V_N\}$ has $N$ units and let $Y$ be a variable of interest with measurements $Y_i$ taken from each population unit for constituting a set of observations $Y = \{Y_1, Y_2, \ldots, Y_N\}$. The purpose of the measurement process is to estimate the population variance $S_Y^2 = N^{-1} \sum_{i=1}^{N} \left(Y_i - \bar{Y}\right)^2$ by drawing a random sample from the population. Assume that the information on auxiliary variable parameters (both conventional and non-conventional) is readily available or can be easily obtained without involving much time or cost. The additional information can be advantageously used in the ratio, the regression, the product, and the ratio-cum regression methods of estimation to define the estimators of population variance or other population characteristics such as mean. One such estimator has been defined by Isaki [9], which utilizes the information on the known variance of the auxiliary variable $\left(S_X^2\right)$ to estimate the population variance $\left(S_Y^2\right)$. It is also known as the traditional ratio-type estimator. The estimator along with its approximate Bias (B(.)) and Mean Square Error (MSE(.)) is given as:

$$\hat{S}_R^2 = \frac{s_y^2}{s_x^2} S_X^2,$$

$$B\left(\hat{S}_R^2\right) \approx \eta S_Y^2 \left(\left(\beta_{2(X)} - 1\right) - \left(\lambda_{22} - 1\right)\right),$$

$$MSE\left(\hat{S}_R^2\right) \approx \eta S_Y^4 \Big(\left(\beta_{2(Y)} - 1\right) + \left(\beta_{2(X)} - 1\right) - 2\left(\lambda_{22} - 1\right)\Big). \quad (1)$$

Upadhyaya and Singh [10] proposed a modified ratio-type estimator of population variance by incorporating information on the coefficient of kurtosis of the auxiliary variable. It is given as:

$$\hat{S}_1^2 = s_y^2 \left(\frac{S_X^2 + \beta_{2(X)}}{s_x^2 + \beta_{2(X)}}\right).$$

Kadilar and Cingi [11] used the coefficient of variation of the auxiliary variable as a linear combination with the variance of the auxiliary variable to define a ratio-type estimator as:

$$\hat{S}_2^2 = s_y^2 \left(\frac{S_X^2 + C_X}{s_x^2 + C_X}\right).$$

Subramani and Kumarapandiyan [14] used the median of the auxiliary variable to propose a modified ratio-type estimator of the population variance, which is defined as.

$$\hat{S}_3^2 = s_y^2 \left(\frac{S_X^2 + M_{d(X)}}{s_x^2 + M_{d(X)}}\right).$$

Subramani and Kumarapandiyan [13] suggested the following quartiles based estimators of the population variance.

$$\hat{S}_i^2 = s_y^2 \left( \frac{S_X^2 + Q_{j(X)}}{s_x^2 + Q_{j(X)}} \right)$$

for $(i = 4, 5, \dots, 8)$ and: $(Q_{j(X)} = Q_{1(X)}, Q_{3(X)}, Q_{r(X)}, Q_{d(X)}, Q_{a(X)})$, respectively.

Subramani and Kumarapandiyan [13] stated that these estimators outperformed some of the existing ratio-type estimators of population variance with smaller MSEs.

Subramani and Kumarapandiyan [12] used deciles of the auxiliary variable to suggest new ratio-type estimators of the population variance. The deciles-based estimators showed superior efficiency to the existing estimators. These estimators are defined as:

$$\hat{S}_i^2 = s_y^2 \left( \frac{S_X^2 + D_{j(X)}}{s_x^2 + D_{j(X)}} \right)$$

for $(i = 9, 10, \dots, 18)$ and $(D_{j(X)} = D_{1(X)}, D_{2(X)}, \dots, D_{10(X)})$, respectively.

Kadilar and Cingi [11] suggested some modified ratio-type estimators of population variance by integrating information on the population coefficient of variation into the population coefficient of kurtosis of the auxiliary variable and the variance of the auxiliary variable as:

$$\hat{S}_{19}^2 = s_y^2 \left( \frac{C_X S_X^2 + \beta_{2(X)}}{C_X s_x^2 + \beta_{2(X)}} \right),$$

$$\hat{S}_{20}^2 = s_y^2 \left( \frac{\beta_{2(X)} S_X^2 + C_X}{\beta_{2(X)} s_x^2 + C_X} \right).$$

Motivated by Kadilar and Cingi [11] and Subramani and Kumarapandiyan [14], a new ratio-type estimator of the population variance was introduced by Subramani and Kumarapandiyan [29], which utilized the population information on the coefficient of variation and the median of the auxiliary variable, given by:

$$\hat{S}_{21}^2 = s_y^2 \left( \frac{C_X S_X^2 + M_{d(X)}}{C_X s_x^2 + M_{d(X)}} \right).$$

Khan and Shabbir [15] suggested a similar estimator of population variance, which utilized the information on the coefficient of correlation between the study and auxiliary variables in addition to the information on upper quartile of the auxiliary variable as:

$$\hat{S}_{22}^2 = s_y^2 \left( \frac{\rho S_X^2 + Q_{3(X)}}{\rho s_x^2 + Q_{3(X)}} \right).$$

In general, the bias and MSEs of the estimators $\hat{S}_i^2\,(i = 1, \dots, 22)$ up to the first degree of approximation are given as:

$$B\left(\hat{S}_i^2\right) \approx \eta S_Y^2 \gamma_i \left\{ \gamma_i \left(\beta_{2(X)} - 1\right) - \left(\lambda_{22} - 1\right) \right\},$$

$$MSE\left(\hat{S}_i^2\right) \approx \eta S_Y^4 \Big\{ \left(\beta_{2(Y)} - 1\right) + \gamma_i^2 \left(\beta_{2(X)} - 1\right) - 2\gamma_i \left(\lambda_{22} - 1\right) \Big\}, \qquad (2)$$

where:

$$\gamma_1 = \frac{S_X^2}{S_X^2 + \beta_{2(X)}}, \qquad \gamma_2 = \frac{S_X^2}{S_X^2 + C_X},$$

$$\gamma_3 = \frac{S_X^2}{S_X^2 + M_{d(X)}}, \qquad \gamma_4 = \frac{S_X^2}{S_X^2 + Q_{1(X)}},$$

$$\gamma_5 = \frac{S_X^2}{S_X^2 + Q_{3(X)}}, \qquad \gamma_6 = \frac{S_X^2}{S_X^2 + Q_{r(X)}},$$

$$\gamma_7 = \frac{S_X^2}{S_X^2 + Q_{d(X)}}, \qquad \gamma_8 = \frac{S_X^2}{S_X^2 + Q_{a(X)}},$$

$$\gamma_9 = \frac{S_X^2}{S_X^2 + D_{1(X)}}, \qquad \gamma_{10} = \frac{S_X^2}{S_X^2 + D_{2(X)}},$$

$$\gamma_{11} = \frac{S_X^2}{S_X^2 + D_{3(X)}}, \qquad \gamma_{12} = \frac{S_X^2}{S_X^2 + D_{4(X)}},$$

$$\gamma_{13} = \frac{S_X^2}{S_X^2 + D_{5(X)}}, \qquad \gamma_{14} = \frac{S_X^2}{S_X^2 + D_{6(X)}},$$

$$\gamma_{15} = \frac{S_X^2}{S_X^2 + D_{7(X)}}, \qquad \gamma_{16} = \frac{S_X^2}{S_X^2 + D_{8(X)}},$$

$$\gamma_{17} = \frac{S_X^2}{S_X^2 + D_{9(X)}}, \qquad \gamma_{18} = \frac{S_X^2}{S_X^2 + D_{10(X)}},$$

$$\gamma_{19} = \frac{C_X S_X^2}{C_X S_X^2 + \beta_{2(X)}}, \qquad \gamma_{20} = \frac{\beta_{2(X)} S_X^2}{\beta_{2(X)} S_X^2 + C_X},$$

$$\gamma_{21} = \frac{C_X S_X^2}{C_X S_X^2 + M_{d(X)}}, \qquad \gamma_{22} = \frac{\rho S_X^2}{\rho S_X^2 + Q_{3(X)}}.$$

Upadhyaya and Singh [30] used the information on the population mean of the auxiliary variable as a ratio to define a new ratio-type estimator of variance. The estimator, its bias, and MSE are defined as:

$$\hat{S}_{23}^2 = s_y^2 \frac{\bar{X}}{\bar{x}},$$

$$B\left(\hat{S}_{23}^2\right) \approx \eta S_Y^2 \left(C_X^2 - \lambda_{21} C_X\right),$$

$$MSE\left(\hat{S}_{23}^2\right) \approx \eta S_Y^4 \left\{ \left(\beta_{2(Y)} - 1\right) + C_X^2 - 2\lambda_{21} C_X \right\}. \qquad (3)$$

Recently, Subramani and Kumarapandiyan [31] proposed a new class of modified ratio-type estimators of population variance by modifying the estimator proposed by Upadhyaya and Singh [30]. They showed that their new class of estimators outperformed the estimators proposed by Upadhyaya and Singh [10], Upadhyaya and Singh [30], Kadilar and Cingi [11], and Subramani and Kumarapandiyan [12–14,29] under certain efficiency conditions.

The general structure of Subramani and Kumarapandiyan's [31] class of estimators with its bias and MSE is:

$$\hat{S}^2_{SK(i)} = s^2_y \left(\frac{\bar{X}+\omega_i}{\bar{x}+\omega_i}\right); \qquad \text{for} \quad (i=1,2,\ldots,51),$$

$$B\left(\hat{S}^2_{SK(i)}\right) \approx \eta S^2_Y \left\{\gamma^2_{SK(i)} C^2_X - \gamma_{SK(i)}\lambda_{21} C_X\right\},$$

$$MSE\left(\hat{S}^2_{SK(i)}\right) \approx \eta S^4_Y \left\{\left(\beta_{2(Y)}-1\right) + \gamma^2_{SK(i)} C^2_X - 2\gamma_{SK(i)}\lambda_{21} C_X\right\}, \tag{4}$$

where:

$$\gamma_{SK(i)} = \frac{\bar{X}}{\bar{X}+\omega_i} \quad \omega_1 = C_X, \quad \omega_2 = \beta_{2(X)},$$

$$\omega_3 = \beta_{1(X)}, \quad \omega_4 = \rho, \quad \omega_5 = S_X,$$

$$\omega_6 = M_{d(X)}, \quad \omega_7 = Q_{1(X)}, \quad \omega_8 = Q_{3(X)},$$

$$\omega_9 = Q_{r(X)}, \quad \omega_{10} = Q_{d(X)}, \quad \omega_{11} = Q_{a(X)},$$

$$\omega_{12} = D_{1(X)}, \quad \omega_{13} = D_{2(X)}, \quad \omega_{14} = D_{3(X)},$$

$$\omega_{15} = D_{4(X)}, \quad \omega_{16} = D_{5(X)}, \quad \omega_{17} = D_{6(X)},$$

$$\omega_{18} = D_{7(X)}, \quad \omega_{19} = D_{8(X)}, \quad \omega_{20} = D_{9(X)},$$

$$\omega_{21} = D_{10(X)}, \quad \omega_{22} = \frac{C_X}{\beta_{2(X)}}, \quad \omega_{23} = \frac{\beta_{2(X)}}{C_X},$$

$$\omega_{24} = \frac{C_X}{\beta_{1(X)}}, \quad \omega_{25} = \frac{\beta_{1(X)}}{C_X}, \quad \omega_{26} = \frac{C_X}{\rho},$$

$$\omega_{27} = \frac{\rho}{C_X}, \quad \omega_{28} = \frac{C_X}{S_X}, \quad \omega_{29} = \frac{S_X}{C_X},$$

$$\omega_{30} = \frac{C_X}{M_{d(X)}}, \quad \omega_{31} = \frac{M_{d(X)}}{C_X}, \quad \omega_{32} = \frac{\beta_{2(X)}}{\beta_{1(X)}},$$

$$\omega_{33} = \frac{\beta_{1(X)}}{\beta_{2(X)}}, \quad \omega_{34} = \frac{\beta_{2(X)}}{\rho}, \quad \omega_{35} = \frac{\rho}{\beta_{2(X)}},$$

$$\omega_{36} = \frac{\beta_{2(X)}}{S_X}, \quad \omega_{37} = \frac{S_X}{\beta_{2(X)}}, \quad \omega_{38} = \frac{\beta_{2(X)}}{M_{d(X)}},$$

$$\omega_{39} = \frac{M_{d(X)}}{\beta_{2(X)}}, \quad \omega_{40} = \frac{\beta_{1(X)}}{\rho}, \quad \omega_{41} = \frac{\rho}{\beta_{1(X)}},$$

$$\omega_{42} = \frac{\beta_{1(X)}}{S_X}, \quad \omega_{43} = \frac{S_X}{\beta_{1(X)}}, \quad \omega_{44} = \frac{\beta_{1(X)}}{M_{d(X)}},$$

$$\omega_{45} = \frac{M_{d(X)}}{\beta_{1(X)}}, \quad \omega_{46} = \frac{\rho}{S_X}, \quad \omega_{47} = \frac{S_X}{\rho},$$

$$\omega_{48} = \frac{\rho}{M_{d(X)}}, \quad \omega_{49} = \frac{M_{d(X)}}{\rho}, \quad \omega_{50} = \frac{S_X}{M_{d(X)}},$$

$$\omega_{51} = \frac{M_{d(X)}}{S_X}.$$

All of the above-modified estimators of variance are biased. However, under certain conditions, these estimators are superior to the traditional ratio estimator of variance suggested by Isaki [9] with lower MSEs by utilizing more auxiliary information.

## 3. Proposed estimators of variance

This section presents three different families of the ratio-type estimators of population variance assuming that the information on some robust non-conventional measures of the location of the auxiliary variable is readily available. First, we use tri-mean ($TM_X$), which is the weighted average of the population median, upper, and lower quartiles of the auxiliary variable and defined as:

$$TM_X = \frac{Q_{2(\mathrm{X})} + \left(Q_{1(\mathrm{X})} + Q_{3(\mathrm{X})}\right)/2}{2}.$$

It is a robust measure of location and remains stable in the presence of outliers with a breakdown point of 0.25. That is, it remains bounded for data having nearly 25% contamination. The properties of $TM_X$ can be found in Wang et al. [26] and Abid et al. [32]. Second, estimators are included based on mid-range ($MR_X$), which is the average of the minimum and maximum values occurring in a data set. Although it is highly sensitive to outliers, for small sample sizes and excess kurtosis, it is a useful measure of location (see [23–25] for more details). Third, Hodges-Lehmann ($HL_X$) estimator, discussed in Hettmansperger and McKean [27], is employed, which is based on the median of the pair-wise Walsh averages. It is robust against outliers and has breakdown point of 0.29. It is defined as:

$$HL_X = median\left(\left(\frac{X(j)+X(k)}{2}\right); 1 \le j \le k \le N\right).$$

Finally, this study utilizes another robust measure of location suggested by Rana et al. [28], namely deciles mean ($DM_X$). It is simply the average of deciles and defined as:

$$DM_X = \frac{\sum_{j=1}^{9} D_{j(X)}}{9}.$$

Rana et al. [28] compared its performance with the mean, median, and the $TM_X$ and observed that it performed better than the other three measures of location as its MSE was lower.

All these non-conventional or robust measures are integrated with other conventional measures such as the coefficients of skewness, variation, correlation between the study and auxiliary variables, and kurtosis in the context of ratio method of estimation under SRS to estimate the population variance. Since the non-conventional and robust measures are being incorporated in this study, it is expected that efficiency of the ratio-type estimators of population variance will be enhanced.

### 3.1. The suggested estimators of class-I

Motivated by the Searls estimator [33], the proposed class of estimators for population variance is defined as:

$$\hat{S}^2_{p1-j} = Ks^2_y \left(\frac{\Phi\bar{X}+\psi}{\Phi\bar{x}+\psi}\right), \tag{5}$$

where $K$ is the Searls [33] constant and its value will be determined in a later stage, $s^2_y$ is the sample mean of the study variable, and $\bar{X}$ and $\bar{x}$ are the population and sample means of the auxiliary variable, respectively. It is worth mentioning that $\left(\Phi\bar{X}+\psi\right)>0$, $(\Phi\bar{x}+\psi)>0$, and $\Phi$ can either be a known real number or known conventional parameter of the auxiliary variable $X$, whereas $\psi$ is considered as the known non-conventional parameter of the auxiliary variable $X$.

To obtain the bias and MSE of $\hat{S}^2_{p1-j}$, in terms of relative errors, we can express $e_0 = \frac{s^2_y - S^2_y}{S^2_y}$ and $e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$ such that $s^2_y = S^2_y(1+e_0)$, $\bar{x} = \bar{X}(1+e_1)$, $E(e_0) = E(e_1) = 0$, $E\left(e^2_0\right) = \eta\left(\beta_{2(y)}-1\right)$, $E\left(e^2_1\right) = \eta C^2_x$, and $E(e_0e_1) = \eta\lambda_{21}C_x$.

Expressing $\hat{S}^2_{p1-j}$ in terms of $e$'s, we have:

$$\hat{S}^2_{p1-j} = KS^2_y(1+e_0)(1+\xi e_1)^{-1}, \tag{6}$$

where $\xi = \frac{\Phi\bar{X}}{\Phi\bar{X}+\psi}$.

Assuming $|\xi e_1|<1$ so that $(1+\xi e_1)^{-1}$ is expandable, expanding the right-hand side of Eq. (6), and neglecting the terms of $e$'s having power greater than two, we have:

$$\hat{S}^2_{p1-j} \approx K\left(S^2_y + S^2_y e_0 - S^2_y\xi e_1 - S^2_y\xi e_0e_1 + S^2_y\xi^2e^2_1\right),$$

or:

$$\hat{S}^2_{p1-j} - S^2_y \approx S^2_y\Big\{(K-1) + K\left(e_0 - \xi e_1 - \xi e_0e_1 + \xi^2e^2_1\right)\Big\}. \tag{7}$$

The bias of $\hat{S}^2_{p1-j}$ up to the first degree of approximation is obtained by applying expectation to both sides of Eq. (7) as:

$$B\left(\hat{S}^2_{p1-j}\right) \approx S^2_y\left\{(K-1)+\eta K\left(\xi^2C^2_x - \xi\lambda_{21}C_x\right)\right\}. \tag{8}$$

Squaring both sides of Eq. (7), keeping terms of $e$'s only up to the second order, and applying expectation, we get MSE of $\hat{S}^2_{p1-j}$ up to the first degree of approximation as:

$$MSE\left(\hat{S}^2_{p1-j}\right) \approx S^4_y\Big\{(K-1)^2 + \eta K^2\left(\beta_{2(y)}-1\right) + \eta C^2_x\left(3K^2\xi^2 - 2K\xi^2\right) + \eta\lambda_{21}C_x\left(2K\xi - 4K^2\xi\right)\Big\}. \tag{9}$$

Differentiating Equation (9) with respect to $K$ and equating it to zero, after simplification, we get the optimum value of $K$ as:

$$K = \frac{1+\eta C_x\xi\left(C_x\xi - \lambda_{21}\right)}{1+\eta\left\{\left(\beta_{2(y)}-1\right) + C_x\xi\left(3C_x\xi - 4\lambda_{21}\right)\right\}} = K_{(opt)}. \tag{10}$$

Let $K_1 = 1+\eta C_x\xi\left(C_x\xi-\lambda_{21}\right)$ and $K_2 = 1 + \eta\left\{\left(\beta_{2(y)}-1\right)+C_x\xi\left(3C_x\xi - 4\lambda_{21}\right)\right\}$, so that $K_{(opt)} = \frac{K_1}{K_2}$.

Substituting the above result in Eq. (9) and simplifying the minimum MSE of $\hat{S}^2_{p1-j}$ gives:

$$MSE\left(\hat{S}^2_{p1-j}\right)_{min} \approx S^4_y\left(1-\frac{K^2_1}{K_2}\right). \tag{11}$$

Similarly, substituting the optimal value of $K$, i.e. $K_{(opt)}$, in Eq. (8), we have the minimum bias of $\hat{S}^2_{p1-j}$ as:

$$B\left(\hat{S}^2_{p1-j}\right)_{min} \approx S^2_y\Big\{\left(K_{(opt)}-1\right) + \eta K_{(opt)}\left(\xi^2C^2_x - \xi\lambda_{21}C_x\right)\Big\}. \tag{12}$$

Many different types of estimators can be generated from $\hat{S}^2_{p1-j}$ by setting different values of $K$, $\Phi$, and $\psi$. For example, if $K=\Phi=1$ and $\psi=0$, then the estimator suggested by Upadhyaya and Singh [30] becomes a member of the proposed class-I of estimators. Similarly, if $K=\Phi=1$ and $\psi=\omega_i$, then the modified class of estimators proposed by Subramani and Kumarapandiyan [31] also becomes a member of the class $\hat{S}^2_{p1-j}$. Some new members of the class $\hat{S}^2_{p1-j}$ are summarized in Table 1.

#### 3.1.1. Efficiency conditions for class-I estimators

The proposed estimators of class-I perform better than

**Table 1.** Some new members of the proposed class-I estimators.

| Estimator | Value of Constant | |
|---|---|---|
| | $\Phi$ | $\psi$ |
| $\hat{S}^2_{p1-1}=Ks^2_y\left[\frac{\beta_{1(X)}\bar{X}+TM_X}{\beta_{1(X)}\bar{x}+TM_X}\right]$ | $\beta_{1(X)}$ | $TM_X$ |
| $\hat{S}^2_{p1-2}=Ks^2_y\left[\frac{\beta_{1(X)}\bar{X}+MR_X}{\beta_{1(X)}\bar{x}+MR_X}\right]$ | $\beta_{1(X)}$ | $MR_X$ |
| $\hat{S}^2_{p1-3}=Ks^2_y\left[\frac{\beta_{1(X)}\bar{X}+HL_X}{\beta_{1(X)}\bar{x}+HL_X}\right]$ | $\beta_{1(X)}$ | $HL_X$ |
| $\hat{S}^2_{p1-4}=Ks^2_y\left[\frac{\beta_{1(X)}\bar{X}+DM_X}{\beta_{1(X)}\bar{x}+DM_X}\right]$ | $\beta_{1(X)}$ | $DM_X$ |
| $\hat{S}^2_{p1-5}=Ks^2_y\left[\frac{C_X\bar{X}+TM_X}{C_X\bar{x}+TM_X}\right]$ | $C_X$ | $TM_X$ |
| $\hat{S}^2_{p1-6}=Ks^2_y\left[\frac{C_X\bar{X}+MR_X}{C_X\bar{x}+MR_X}\right]$ | $C_X$ | $MR_X$ |
| $\hat{S}^2_{p1-7}=Ks^2_y\left[\frac{C_X\bar{X}+HL_X}{C_X\bar{x}+HL_X}\right]$ | $C_X$ | $HL_X$ |
| $\hat{S}^2_{p1-8}=Ks^2_y\left[\frac{C_X\bar{X}+DM_X}{C_X\bar{x}+DM_X}\right]$ | $C_X$ | $DM_X$ |
| $\hat{S}^2_{p1-9}=Ks^2_y\left[\frac{\rho\bar{X}+TM_X}{\rho\bar{x}+TM_X}\right]$ | $\rho$ | $TM_X$ |
| $\hat{S}^2_{p1-10}=Ks^2_y\left[\frac{\rho\bar{X}+MR_X}{\rho\bar{x}+MR_X}\right]$ | $\rho$ | $MR_X$ |
| $\hat{S}^2_{p1-11}=Ks^2_y\left[\frac{\rho\bar{X}+HL_X}{\rho\bar{x}+HL_X}\right]$ | $\rho$ | $HL_X$ |
| $\hat{S}^2_{p1-12}=Ks^2_y\left[\frac{\rho\bar{X}+DM_X}{\rho\bar{x}+DM_X}\right]$ | $\rho$ | $DM_X$ |
| $\hat{S}^2_{p1-13}=Ks^2_y\left[\frac{\beta_{2(X)}\bar{X}+TM_X}{\beta_{2(X)}\bar{x}+TM_X}\right]$ | $\beta_{2(X)}$ | $TM_X$ |
| $\hat{S}^2_{p1-14}=Ks^2_y\left[\frac{\beta_{2(X)}\bar{X}+MR_X}{\beta_{2(X)}\bar{x}+MR_X}\right]$ | $\beta_{2(X)}$ | $MR_X$ |
| $\hat{S}^2_{p1-15}=Ks^2_y\left[\frac{\beta_{2(X)}\bar{X}+HL_X}{\beta_{2(X)}\bar{x}+HL_X}\right]$ | $\beta_{2(X)}$ | $HL_X$ |
| $\hat{S}^2_{p1-16}=Ks^2_y\left[\frac{\beta_{2(X)}\bar{X}+DM_X}{\beta_{2(X)}\bar{x}+DM_X}\right]$ | $\beta_{2(X)}$ | $DM_X$ |

Isaki's [9] traditional estimator of population variance if $MSE\left(\hat{S}^2_{p1-j}\right)_{min}\leq MSE\left(\hat{S}^2_R\right)$, i.e.:

$$S^4_y\left(1-\frac{K_1^2}{K_2}\right)\leq\eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+\left(\beta_{2(X)}-1\right)-2\left(\lambda_{22}-1\right)\Big\},$$

or:

$$K_1^2\geq K_2\left[1-\eta\left(\beta_{2(Y)}+\beta_{2(X)}-2\lambda_{22}\right)\right].$$

Similarly, the estimators contained in $\hat{S}^2_{p1-j}$ will achieve superior efficiency to the existing estimators $\hat{S}^2_i$ $(i=1,...,22)$ detailed in Section 2 if:

$$S^4_y\left(1-\frac{K_1^2}{K_2}\right)\leq\eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right)-2\gamma_i\left(\lambda_{22}-1\right)\Big\},$$

or:

$$K_1^2\geq K_2\Big[1-\eta\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right)-2\gamma_i\left(\lambda_{22}-1\right)\Big\}\Big].$$

The suggested class-I estimators will be more efficient than those of Upadhyaya and Singh [30] if:

$$S^4_y\left(1-\frac{K_1^2}{K_2}\right)\leq\eta S^4_Y\left\{\left(\beta_{2(Y)}-1\right)+C_X^2-2\lambda_{21}C_X\right\}$$

or

$$K_1^2\geq K_2\left[1-\eta\left\{\left(\beta_{2(Y)}-1\right)+C_X^2-2\lambda_{21}C_X\right\}\right].$$

The estimators envisaged in the proposed class $\hat{S}^2_{p1-j}$ will exhibit superior performance to Subramani and Kumarapandiyan's [31] modified class of estimators if:

$$S^4_y\left(1-\frac{K_1^2}{K_2}\right)\leq\eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+\gamma^2_{SK(i)}C_X^2-2\gamma_{SK(i)}\lambda_{21}C_X\Big\},$$

or:

$$K_1^2\geq K_2\Big[1-\eta\Big\{\left(\beta_{2(Y)}-1\right)+\gamma^2_{SK(i)}C_X^2-2\gamma_{SK(i)}\lambda_{21}C_X\Big\}\Big].$$

### *3.2. The suggested estimators of class-II*

This section deals with another proposed class of estimators of population variance under SRS, which is based on power transformation. The general structure of the estimators of class-II is defined as:

$$\hat{S}^2_{p2-l}=Ls^2_y\left(\frac{\varphi\bar{X}+\psi}{\varphi\bar{x}+\psi}\right)^{\frac{\delta\bar{X}}{\delta\bar{X}+\nu}},\qquad(13)$$

where $L$ is Searls [33] constant and its value will be determined later, $\varphi$ and $\delta$ can be real numbers or functions of a known conventional parameter of the auxiliary variable $X$, and $\psi$ and $\nu$ are the known functions of the non-conventional parameters of the auxiliary variable $X$.

**Table 2.** Some new members of the proposed class-II estimators.

| Estimator | Value of constant | | | |
|---|---|---|---|---|
| | $\varphi$ | $\psi$ | $\delta$ | $\nu$ |
| $\hat{S}^2_{p2-1} = Ls^2_y\left[\frac{\beta_{1(X)}\bar{X}+TM_X}{\beta_{1(X)}\bar{x}+TM_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+TM_X}}$ | $\beta_{1(X)}$ | $TM_X$ | $\beta_{1(X)}$ | $TM_X$ |
| $\hat{S}^2_{p2-2} = Ls^2_y\left[\frac{\beta_{1(X)}\bar{X}+MR_X}{\beta_{1(X)}\bar{x}+MR_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+MR_X}}$ | $\beta_{1(X)}$ | $MR_X$ | $\beta_{1(X)}$ | $MR_X$ |
| $\hat{S}^2_{p2-3} = Ls^2_y\left[\frac{\beta_{1(X)}\bar{X}+HL_X}{\beta_{1(X)}\bar{x}+HL_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+HL_X}}$ | $\beta_{1(X)}$ | $HL_X$ | $\beta_{1(X)}$ | $HL_X$ |
| $\hat{S}^2_{p2-4} = Ls^2_y\left[\frac{\beta_{1(X)}\bar{X}+DM_X}{\beta_{1(X)}\bar{x}+DM_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+DM_X}}$ | $\beta_{1(X)}$ | $DM_X$ | $\beta_{1(X)}$ | $DM_X$ |
| $\hat{S}^2_{p2-5} = Ls^2_y\left[\frac{C_X\bar{X}+TM_X}{C_X\bar{x}+TM_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+TM_X}}$ | $C_X$ | $TM_X$ | $C_X$ | $TM_X$ |
| $\hat{S}^2_{p2-6} = Ls^2_y\left[\frac{C_X\bar{X}+MR_X}{C_X\bar{x}+MR_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+MR_X}}$ | $C_X$ | $MR_X$ | $C_X$ | $MR_X$ |
| $\hat{S}^2_{p2-7} = Ls^2_y\left[\frac{C_X\bar{X}+HL_X}{C_X\bar{x}+HL_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+HL_X}}$ | $C_X$ | $HL_X$ | $C_X$ | $HL_X$ |
| $\hat{S}^2_{p2-8} = Ls^2_y\left[\frac{C_X\bar{X}+DM_X}{C_X\bar{x}+DM_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+DM_X}}$ | $C_X$ | $DM_X$ | $C_X$ | $DM_X$ |
| $\hat{S}^2_{p2-9} = Ls^2_y\left[\frac{\rho\bar{X}+TM_X}{\rho\bar{x}+TM_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+TM_X}}$ | $\rho$ | $TM_X$ | $\rho$ | $TM_X$ |
| $\hat{S}^2_{p2-10} = Ls^2_y\left[\frac{\rho\bar{X}+MR_X}{\rho\bar{x}+MR_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+MR_X}}$ | $\rho$ | $MR_X$ | $\rho$ | $MR_X$ |
| $\hat{S}^2_{p2-11} = Ls^2_y\left[\frac{\rho\bar{X}+HL_X}{\rho\bar{x}+HL_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+HL_X}}$ | $\rho$ | $HL_X$ | $\rho$ | $HL_X$ |
| $\hat{S}^2_{p2-12} = Ls^2_y\left[\frac{\rho\bar{X}+DM_X}{\rho\bar{x}+DM_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+DM_X}}$ | $\rho$ | $DM_X$ | $\rho$ | $DM_X$ |
| $\hat{S}^2_{p2-13} = Ls^2_y\left[\frac{\beta_{2(X)}\bar{X}+TM_X}{\beta_{2(X)}\bar{x}+TM_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+TM_X}}$ | $\beta_{2(X)}$ | $TM_X$ | $\beta_{2(X)}$ | $TM_X$ |
| $\hat{S}^2_{p2-14} = Ls^2_y\left[\frac{\beta_{2(X)}\bar{X}+MR_X}{\beta_{2(X)}\bar{x}+MR_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+MR_X}}$ | $\beta_{2(X)}$ | $MR_X$ | $\beta_{2(X)}$ | $MR_X$ |
| $\hat{S}^2_{p2-15} = Ls^2_y\left[\frac{\beta_{2(X)}\bar{X}+HL_X}{\beta_{2(X)}\bar{x}+HL_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+HL_X}}$ | $\beta_{2(X)}$ | $HL_X$ | $\beta_{2(X)}$ | $HL_X$ |
| $\hat{S}^2_{p2-16} = Ls^2_y\left[\frac{\beta_{2(X)}\bar{X}+DM_X}{\beta_{2(X)}\bar{x}+DM_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+DM_X}}$ | $\beta_{2(X)}$ | $DM_X$ | $\beta_{2(X)}$ | $DM_X$ |

Following the procedure described in Section 3.1, the minimum bias and minimum MSE of the proposed class-II estimators are given as:

$$B\left(\hat{S}^2_{p2-l}\right)_{min} \approx S^2_y\left[\left(L_{(opt)}-1\right)+\eta L_{(opt)}\theta_1 C_X \left\{\frac{\left(\theta_2^2+\theta_2\right)}{2}\theta_1 C_X-\theta_2\lambda_{21}\right\}\right], \tag{14}$$

$$MSE\left(\hat{S}^2_{p2-l}\right)_{min} \approx S^4_y\left(1-\frac{L_1^2}{L_2}\right), \tag{15}$$

where:

$$\theta_1=\frac{\varphi\bar{X}}{\varphi\bar{X}+\psi}, \quad \theta_2=\frac{\delta\bar{X}}{\delta\bar{X}+\nu}, \quad L_{(opt)}=\frac{L_1}{L_2},$$

$$L_1=1+\frac{\eta}{2}\theta_1 C_X\left\{\left(\theta_2^2+\theta_2\right)\theta_1 C_X-2\theta_2\lambda_{21}\right\},$$

$$L_2=1+\eta\left\{\left(\beta_{2(Y)}-1\right)+\left(2\theta_2^2+\theta_2\right)\theta_1^2 C_X^2 -4\theta_1\theta_2\lambda_{21}C_X\right\}.$$

Various ratio-type estimators can be obtained by assuming different values of $L$, $\varphi$, $\delta$, $\psi$, and $\nu$ in the proposed class $\hat{S}^2_{p2-l}$. For instance, if we set $L=\varphi=\delta=1$ and $\psi=\nu=0$, then the estimator suggested by Upadhyaya and Singh [30] is a member of class $\hat{S}^2_{p2-l}$ of estimators. On the other hand, if we set $L=\varphi=\delta=1$, $\psi=\omega_i$, and $\nu=0$, then Subramani and Kumarapandiyan [31] class of estimators is a member of the proposed class-II estimators. Similarly, if we set $L=K$, $\delta=1$, and $\nu=0$, the class of estimators defined in Section 3.1 becomes a member of class-II estimators.

Some new members of the proposed class-II are defined in Table 2, which are based on a combination of conventional and non-conventional parameters of the auxiliary variable.

*3.2.1. Efficiency conditions for class-II estimators*
The estimator $\hat{S}^2_{p2-l}$ performs better than the Isaki [9] traditional estimator of population variance if $MSE\left(\hat{S}^2_{p2-l}\right)_{min} \leq MSE\left(\hat{S}^2_R\right)$, i.e.:

$$S_y^4\left(1-\frac{L_1^2}{L_2}\right) \leq \eta S_Y^4\Big\{\left(\beta_{2(Y)}-1\right)+\left(\beta_{2(X)}-1\right) -2\left(\lambda_{22}-1\right)\Big\},$$

or:

$$L_1^2 \geq L_2\left[1-\eta\left(\beta_{2(Y)}+\beta_{2(X)}-2\lambda_{22}\right)\right].$$

The estimators defined in class-II will achieve greater efficiency than the existing estimators $\hat{S}_i^2\,(i=1,\ldots,22)$ defined in Section 2 if the following condition is satisfied.

$$S_y^4\left(1-\frac{L_1^2}{L_2}\right) \leq \eta S_Y^4\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right) -2\gamma_i\left(\lambda_{22}-1\right)\Big\},$$

or:

$$L_1^2 \geq L_2\Big[1-\eta\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right) -2\gamma_i\left(\lambda_{22}-1\right)\Big\}\Big].$$

The suggested class-II estimators will outperform Upadhyaya and Singh's [30] modified ratio-type estimator of population variance in terms of efficiency if:

$$S_y^4\left(1-\frac{L_1^2}{L_2}\right) \leq \eta S_Y^4\Big\{\left(\beta_{2(Y)}-1\right)+C_X^2-2\lambda_{21}C_X\Big\},$$

or:

$$L_1^2 \geq L_2\left[1-\eta\left\{\left(\beta_{2(Y)}-1\right)+C_X^2-2\lambda_{21}C_X\right\}\right].$$

The estimators envisaged in the proposed class $\hat{S}^2_{p2-l}$ will exhibit superior performance to Subramani and Kumarapandiyan's [31] modified class of estimators if:

$$S_y^4\left(1-\frac{L_1^2}{L_2}\right) \leq \eta S_Y^4\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_{SK(i)}^2C_X^2 -2\gamma_{SK(i)}\lambda_{21}C_X\Big\},$$

or:

$$L_1^2 \geq L_2\Big[1-\eta\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_{SK(i)}^2C_X^2 -2\gamma_{SK(i)}\lambda_{21}C_X\Big\}\Big].$$

### *3.3. The suggested estimators of class-III*

In this section, we present a new class of ratio-cum regression estimators of population variance. The proposed class of estimators $\left(\hat{S}^2_{p3-m}\right)$ is defined as:

$$\hat{S}^2_{p3-m} = M\left\{s_y^2\left(\frac{\alpha\bar{X}+\zeta}{\alpha\bar{x}+\zeta}\right)^{\frac{\tau\bar{X}}{\tau\bar{X}+\upsilon}}+b\left(\bar{X}-\bar{x}\right)\right\}, \quad (16)$$

where $M$ is Searls [33] constant and its value will be determined later, $\alpha$ and $\tau$ can be real numbers or functions of the known conventional parameter of the auxiliary variable $X$, $\zeta$ and $\upsilon$ are the known functions of the non-conventional parameters of the auxiliary variable $X$, and $b$ is the regression coefficient between the study and auxiliary variables.

The minimized bias and minimized MSE of class-III estimators are obtained by adopting the procedure detailed in Section 3.1 as follows:

$$B\left(\hat{S}^2_{p3-m}\right)_{min} \approx S_y^2\Big[\left(M_{(opt)}-1\right)+\eta M_{(opt)}\theta_1 C_X \Big\{\frac{\left(\theta_2^2+\theta_2\right)}{2}\theta_1C_X-\theta_2\lambda_{21}\Big\}\Big], \quad (17)$$

$$MSE\left(\hat{S}^2_{p3-m}\right)_{min} \approx \left(S_y^4-\frac{M_1^2}{M_2}\right), \quad (18)$$

where:

$$\theta_1=\frac{\alpha\bar{X}}{\alpha\bar{X}+\zeta}, \quad \theta_2=\frac{\tau\bar{X}}{\tau\bar{X}+\upsilon}, \quad M_{(opt)}=\frac{M_1}{M_2},$$

$$M_1=S_y^4\left[1+\frac{\eta}{2}\theta_1C_X\left\{\left(\theta_2^2+\theta_2\right)\theta_1C_X-2\theta_2\lambda_{21}\right\}\right],$$

and:

$$M_2=S_y^4+\eta S_y^4\Big\{\left(\beta_{2(Y)}-1\right)+\left(2\theta_2^2+\theta_2\right)\theta_1^2C_X^2 -4\theta_1\theta_2\lambda_{21}C_X\Big\}-2\eta bS_y^2\bar{X}\left\{\lambda_{21}C_X-\theta_1\theta_2C_X^2\right\} +\eta b\bar{X}^2C_X^2.$$

Numerous ratio and regression type estimators can easily be obtained by choosing different values of $M,\alpha,\tau,\zeta,\upsilon$, and $b$ in the proposed class $\hat{S}^2_{p3-m}$. For instance, if we set $M=\alpha=\tau=1$, $\zeta=\upsilon=b=0$, then the estimator suggested by Upadhyaya and Singh [30] is a member of class $\hat{S}^2_{p3-m}$ of estimators. Similarly, if we set $M=\alpha=\tau=1$, $\zeta=\omega_i$, and $\upsilon=b=0$ then Subramani and Kumarapandiyan [31] class of estimators becomes a member of the proposed class $\hat{S}^2_{p3-m}$ of estimators. If we set $M=K$, $\alpha=1$, and $\upsilon=b=0$, the class of estimators defined in Section 3.1 becomes a member of class $\hat{S}^2_{p3-m}$ of estimators.

**Table 3.** Some new members of the proposed class-III estimators.

| Estimator | Value of constant | | | |
|---|---|---|---|---|
| | $\alpha$ | $\zeta$ | $\tau$ | $\upsilon$ |
| $\hat{S}^2_{p3-1} = Ms^2_y\left[\frac{\beta_{1(X)}\bar{X}+TM_X}{\beta_{1(X)}\bar{x}+TM_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+TM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{1(X)}$ | $TM_X$ | $\beta_{1(X)}$ | $TM_X$ |
| $\hat{S}^2_{p3-2} = Ms^2_y\left[\frac{\beta_{1(X)}\bar{X}+MR_X}{\beta_{1(X)}\bar{x}+MR_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+MR_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{1(X)}$ | $MR_X$ | $\beta_{1(X)}$ | $MR_X$ |
| $\hat{S}^2_{p3-3} = Ms^2_y\left[\frac{\beta_{1(X)}\bar{X}+HL_X}{\beta_{1(X)}\bar{x}+HL_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+HL_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{1(X)}$ | $HL_X$ | $\beta_{1(X)}$ | $HL_X$ |
| $\hat{S}^2_{p3-4} = Ms^2_y\left[\frac{\beta_{1(X)}\bar{X}+DM_X}{\beta_{1(X)}\bar{x}+DM_X}\right]^{\frac{\beta_{1(X)}\bar{X}}{\beta_{1(X)}\bar{X}+DM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{1(X)}$ | $DM_X$ | $\beta_{1(X)}$ | $DM_X$ |
| $\hat{S}^2_{p3-5} = Ms^2_y\left[\frac{C_X\bar{X}+TM_X}{C_X\bar{x}+TM_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+TM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $C_X$ | $TM_X$ | $C_X$ | $TM_X$ |
| $\hat{S}^2_{p3-6} = Ms^2_y\left[\frac{C_X\bar{X}+MR_X}{C_X\bar{x}+MR_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+MR_X}} + b\left(\bar{X}-\bar{x}\right)$ | $C_X$ | $MR_X$ | $C_X$ | $MR_X$ |
| $\hat{S}^2_{p3-7} = Ms^2_y\left[\frac{C_X\bar{X}+HL_X}{C_X\bar{x}+HL_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+HL_X}} + b\left(\bar{X}-\bar{x}\right)$ | $C_X$ | $HL_X$ | $C_X$ | $HL_X$ |
| $\hat{S}^2_{p3-8} = Ms^2_y\left[\frac{C_X\bar{X}+DM_X}{C_X\bar{x}+DM_X}\right]^{\frac{C_X\bar{X}}{C_X\bar{X}+DM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $C_X$ | $DM_X$ | $C_X$ | $DM_X$ |
| $\hat{S}^2_{p3-9} = Ms^2_y\left[\frac{\rho\bar{X}+TM_X}{\rho\bar{x}+TM_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+TM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\rho$ | $TM_X$ | $\rho$ | $TM_X$ |
| $\hat{S}^2_{p3-10} = Ms^2_y\left[\frac{\rho\bar{X}+MR_X}{\rho\bar{x}+MR_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+MR_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\rho$ | $MR_X$ | $\rho$ | $MR_X$ |
| $\hat{S}^2_{p3-11} = Ms^2_y\left[\frac{\rho\bar{X}+HL_X}{\rho\bar{x}+HL_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+HL_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\rho$ | $HL_X$ | $\rho$ | $HL_X$ |
| $\hat{S}^2_{p3-12} = Ms^2_y\left[\frac{\rho\bar{X}+DM_X}{\rho\bar{x}+DM_X}\right]^{\frac{\rho\bar{X}}{\rho\bar{X}+DM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\rho$ | $DM_X$ | $\rho$ | $DM_X$ |
| $\hat{S}^2_{p3-13} = Ms^2_y\left[\frac{\beta_{2(X)}\bar{X}+TM_X}{\beta_{2(X)}\bar{x}+TM_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+TM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{2(X)}$ | $TM_X$ | $\beta_{2(X)}$ | $TM_X$ |
| $\hat{S}^2_{p3-14} = Ms^2_y\left[\frac{\beta_{2(X)}\bar{X}+MR_X}{\beta_{2(X)}\bar{x}+MR_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+MR_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{2(X)}$ | $MR_X$ | $\beta_{2(X)}$ | $MR_X$ |
| $\hat{S}^2_{p3-15} = Ms^2_y\left[\frac{\beta_{2(X)}\bar{X}+HL_X}{\beta_{2(X)}\bar{x}+HL_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+HL_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{2(X)}$ | $HL_X$ | $\beta_{2(X)}$ | $HL_X$ |
| $\hat{S}^2_{p3-16} = Ms^2_y\left[\frac{\beta_{2(X)}\bar{X}+DM_X}{\beta_{2(X)}\bar{x}+DM_X}\right]^{\frac{\beta_{2(X)}\bar{X}}{\beta_{2(X)}\bar{X}+DM_X}} + b\left(\bar{X}-\bar{x}\right)$ | $\beta_{2(X)}$ | $DM_X$ | $\beta_{2(X)}$ | $DM_X$ |

Moreover, for regression coefficient, with $b = 0$, the class of estimators defined in Section 3.2 becomes a member of the proposed class $\hat{S}^2_{p3-m}$.

Table 3 contains some new members of the proposed class-III to estimate population variance based on the auxiliary information.

*3.3.1. Efficiency conditions for class-III estimators*

The estimators in class-III $\hat{S}^2_{p3-m}$ perform better than the traditional ratio estimator of population variance proposed by Isaki [9] if $MSE\left(\hat{S}^2_{p3-m}\right)_{min} \leq MSE\left(\hat{S}^2_R\right)$, i.e.:

$$\left(S^4_y - \frac{M_1^2}{M_2}\right) \leq \eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+\left(\beta_{2(X)}-1\right) -2\left(\lambda_{22}-1\right)\Big\},$$

or:

$$M_1^2 \geq M_2 S^4_Y\left[1-\eta\left(\beta_{2(Y)}+\beta_{2(X)}-2\lambda_{22}\right)\right].$$

The estimators defined in class-III are superior in efficiency to the estimators defined in Section 2, i.e.: $\hat{S}^2_i\,(i = 1,\ldots,22)$ if:

$$\left(S^4_y - \frac{M_1^2}{M_2}\right) \leq \eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right) -2\gamma_i\left(\lambda_{22}-1\right)\Big\},$$

or:

$$M_1^2 \geq M_2 S^4_y\Big[1-\eta\Big\{\left(\beta_{2(Y)}-1\right)+\gamma_i^2\left(\beta_{2(X)}-1\right) -2\gamma_i\left(\lambda_{22}-1\right)\Big\}\Big].$$

The estimators suggested in class-III outperform Upadhyaya and Singh's [30] modified ratio-type estimator of population variance in terms of efficiency if:

$$\left(S^4_y - \frac{M_1^2}{M_2}\right) \leq \eta S^4_Y\Big\{\left(\beta_{2(Y)}-1\right)+C^2_X-2\lambda_{21}C_X\Big\},$$

or:

$$M_1^2 \geq M_2 S_y^4 \left[1 - \eta \left\{ \left(\beta_{2(Y)} - 1\right) + C_X^2 - 2\lambda_{21} C_X \right\} \right].$$

The estimators envisaged in the proposed class-III $\hat{S}^2_{p3-m}$ will exhibit superior performance to Subramani and Kumarapandiyan's [31] modified class of estimators if:

$$\left(S_y^4 - \frac{M_1^2}{M_2}\right) \leq \eta S_Y^4 \left\{ \left(\beta_{2(Y)} - 1\right) + \gamma_{SK(i)}^2 C_X^2 - 2\gamma_{SK(i)} \lambda_{21} C_X \right\},$$

or:

$$M_1^2 \geq M_2 S_y^4 \left[1 - \eta \left\{ \left(\beta_{2(Y)} - 1\right) + \gamma_{SK(i)}^2 C_X^2 - 2\gamma_{SK(i)} \lambda_{21} C_X \right\} \right].$$

From the theoretical results, it is established that the proposed families of estimators have superior efficiency under certain conditions. To further support these theoretical findings, an empirical study is conducted in the following section.

## 4. Empirical study

To gauge the performance of the proposed classes of estimators against their competing estimators of variance, two real datasets (population-I and population-II) are taken from Singh and Chaudhary [34], page 177. These datasets have also been considered by Subramani and Kumarapandiyan [31] and some other studies [35–38]. In population-I, $Y$ denotes the area under wheat in 1974 and $X$ denotes the area under wheat in 1971. In population-II, $Y$ represents the area under wheat in 1974 and $X$ is the area under wheat in 1973. The original populations do not contain many outlier observations. To compare the performances of the proposed and existing estimators using a population crippled with outlier observations, some outliers are introduced into population-I and it is named population-III. It is expected that the performance of the proposed classes of estimators will remain relatively stable in the presence of outliers in the data. Figures 1-3 depict the nature of the population data considered in this study.

To compare the performances of the proposed and the existing estimators, the Percentage Relative Efficiencies (PREs) of the proposed classes of estimators with respect to the traditional ratio estimator suggested by Isaki [9] are obtained as:

$$PRE_{(p)} = \frac{MSE_{(TR)}}{MSE_{(p)}} \times 100,$$

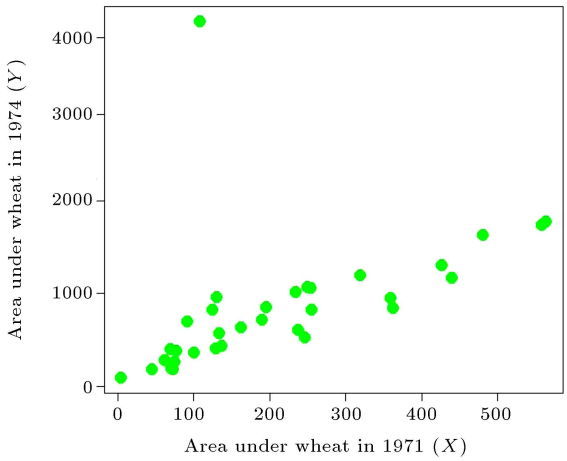


**Figure 1.** Scatter plot of population-I .

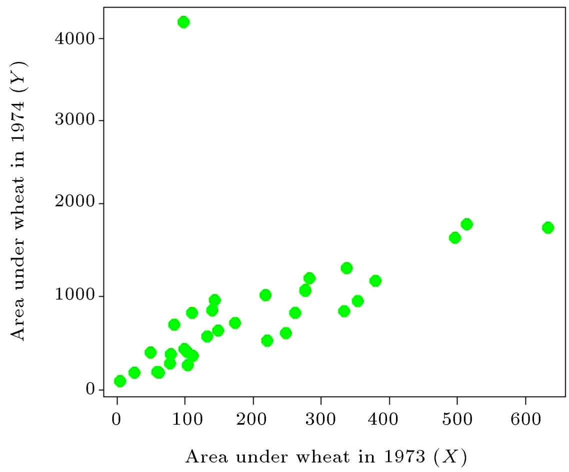


**Figure 2.** Scatter plot of population-II.

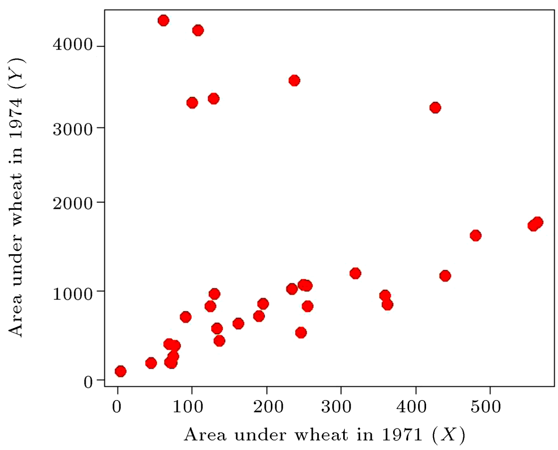


**Figure 3.** Scatter plot of population-III.

where $PRE_{(p)}$ denotes the PRE of the proposed estimators as compared to the traditional ratio estimator, $MSE_{(TR)}$ is the mean square error of the traditional ratio estimator, and $MSE_{(p)}$ is the mean square error of the proposed estimator. It is worth mentioning that for the sake of brevity, we have taken only the most efficient estimator from the class of estimators proposed by Subramani and Kumarapandiyan [31], which is based on $D_{10(X)}$, for comparison purpose. The population characteristics are summarized in Table 4.

The computed PREs of the existing estimators are compared with the traditional ratio estimator of Isaki [9] in Tables 5-7 and PREs of the proposed estimators as compared to the traditional ratio estimator are given in Tables 8-10.

From the results reported in Tables 5-10, the main findings are summarized as follows:

1. In general, the estimators proposed in this study have greater PREs than the existing estimators of Isaki [9], Upadhyaya and Singh [10,30], Kadilar and Cingi [11], Khan and Shabbir [15], and Subramani and Kumarapandiyan [12–14,29,31] for all the populations considered. For example, Subramani and Kumarapandiyan [31] estimator has the maximum PRE among the existing estimators, i.e. 111.56, 117.10, and 178.49 for populations I, II, and III, respectively. On the other hand, the maximum levels of PRE of the proposed estimators are 141.59, 148.42, and 202.40 for populations I, II and III, respectively. This indicates better performance of the proposed classes of estimators in terms of PRE;
2. PREs of the existing and the proposed estimators increase as the number of outliers in the data increases (cf. Tables 5-6 and Tables 9-10). For instance, the PRE of Subramani and Kumarapandiyan [31] (best among the existing estimators considered in this study) increases from 111.56 to 178.49 and the maximum increase in PRE of the proposed estimators is from 148.42 to 202.40;
3. When the number of outliers in the data increases, the majority of the proposed estimators still remain more efficient than all the existing estimators

**Table 4.** Values of the population characteristics.

| Characteristic | POP-I | POP-II | Pop-III | Characteristic | POP-I | POP-II | POP-III |
|---|---|---|---|---|---|---|---|
| $N$ | 34 | 34 | 34 | $Q_{d(X)}$ | 8.03 | 8.94 | 8.03 |
| $n$ | 20 | 20 | 20 | $Q_{a(X)}$ | 17.45 | 18.86 | 17.45 |
| $\bar{Y}$ | 85.64 | 85.64 | 129.76 | $D_{1(X)}$ | 7.03 | 6.06 | 7.03 |
| $\bar{X}$ | 20.89 | 19.94 | 20.89 | $D_{2(X)}$ | 7.68 | 8.30 | 7.68 |
| $S_Y$ | 73.31 | 73.31 | 120.76 | $D_{3(X)}$ | 10.82 | 10.27 | 10.82 |
| $C_Y$ | 0.86 | 0.86 | 0.93 | $D_{4(X)}$ | 12.94 | 11.12 | 12.94 |
| $S_X$ | 15.05 | 15.02 | 15.05 | $D_{5(X)}$ | 15.00 | 14.25 | 15.00 |
| $C_X$ | 0.72 | 0.75 | 0.72 | $D_{6(X)}$ | 22.72 | 21.02 | 22.72 |
| $\beta 1(X)$ | 0.87 | 1.28 | 0.87 | $D_{7(X)}$ | 25.04 | 26.45 | 25.04 |
| $\beta 2(X)$ | 2.91 | 3.73 | 2.91 | $D_{8(X)}$ | 33.56 | 30.44 | 33.56 |
| $\beta 2(Y)$ | 13.37 | 13.37 | 3.54 | $D_{9(X)}$ | 43.61 | 37.32 | 43.61 |
| $\lambda_{22}$ | 1.15 | 1.22 | 0.78 | $D_{10(X)}$ | 56.40 | 63.40 | 56.40 |
| $\lambda_{21}$ | -0.31 | -0.29 | -0.43 | $TM_X$ | 16.23 | 16.56 | 16.23 |
| $M_{d(X)}$ | 15.00 | 14.25 | 15.00 | $MR_X$ | 28.45 | 32.00 | 28.45 |
| $Q_{1(X)}$ | 9.43 | 9.93 | 9.43 | $HL_X$ | 18.95 | 18.25 | 18.95 |
| $Q_{3(X)}$ | 25.48 | 27.80 | 25.48 | $DM_X$ | 19.82 | 18.36 | 19.82 |
| $Q_{r(X)}$ | 16.05 | 17.88 | 16.05 | | | | |

**Table 5.** PREs of the existing estimators vs. traditional ratio estimator for population-I.

| Estimator | PRE | Estimator | PRE | Estimator | PRE | Estimator | PRE |
|---|---|---|---|---|---|---|---|
| $\hat{S}_1^2$ | 100.32 | $\hat{S}_2^2$ | 100.08 | $\hat{S}_3^2$ | 101.53 | $\hat{S}_4^2$ | 100.99 |
| $\hat{S}_5^2$ | 102.47 | $\hat{S}_6^2$ | 101.63 | $\hat{S}_7^2$ | 100.85 | $\hat{S}_8^2$ | 101.76 |
| $\hat{S}_9^2$ | 100.75 | $\hat{S}_{10}^2$ | 100.82 | $\hat{S}_{11}^2$ | 101.13 | $\hat{S}_{12}^2$ | 101.34 |
| $\hat{S}_{13}^2$ | 101.53 | $\hat{S}_{14}^2$ | 102.23 | $\hat{S}_{15}^2$ | 102.43 | $\hat{S}_{16}^2$ | 103.12 |
| $\hat{S}_{17}^2$ | 103.85 | $\hat{S}_{18}^2$ | 104.69 | $\hat{S}_{19}^2$ | 100.44 | $\hat{S}_{20}^2$ | 100.03 |
| $\hat{S}_{21}^2$ | 102.06 | $\hat{S}_{22}^2$ | 104.71 | $\hat{S}_{23}^2$ | 104.81 | $\hat{S}_{SK}^2$ | **111.56** |

Note: Bold value indicates maximum PRE.

**Table 6.** PREs of the proposed classes of estimators vs. traditional ratio estimator for population-I.

| **Proposed Class-I** | | **Proposed Class-II** | | **Proposed Class-III** | |
|---|---|---|---|---|---|
| **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** |
| $\hat{S}^2_{p1-1}$ | 139.59 | $\hat{S}^2_{p2-1}$ | 140.83 | $\hat{S}^2_{p3-1}$ | 140.78 |
| $\hat{S}^2_{p1-2}$ | 140.26 | $\hat{S}^2_{p2-2}$ | 141.30 | $\hat{S}^2_{p3-2}$ | 141.25 |
| $\hat{S}^2_{p1-3}$ | 139.78 | $\hat{S}^2_{p2-3}$ | 140.98 | $\hat{S}^2_{p3-3}$ | 140.93 |
| $\hat{S}^2_{p1-4}$ | 139.84 | $\hat{S}^2_{p2-4}$ | 141.02 | $\hat{S}^2_{p3-4}$ | 140.97 |
| $\hat{S}^2_{p1-5}$ | 139.83 | $\hat{S}^2_{p2-5}$ | 141.02 | $\hat{S}^2_{p3-5}$ | 140.96 |
| $\hat{S}^2_{p1-6}$ | 140.46 | $\hat{S}^2_{p2-6}$ | 141.40 | $\hat{S}^2_{p3-6}$ | 141.36 |
| $\hat{S}^2_{p1-7}$ | 140.01 | $\hat{S}^2_{p2-7}$ | 141.15 | $\hat{S}^2_{p3-7}$ | 141.10 |
| $\hat{S}^2_{p1-8}$ | 140.07 | $\hat{S}^2_{p2-8}$ | 141.18 | $\hat{S}^2_{p3-8}$ | 141.13 |
| $\hat{S}^2_{p1-9}$ | 140.37 | $\hat{S}^2_{p2-9}$ | 141.36 | $\hat{S}^2_{p3-9}$ | 141.31 |
| $\hat{S}^2_{p1-10}$ | 140.88 | $\hat{S}^2_{p2-10}$ | **141.59** | $\hat{S}^2_{p3-10}$ | 141.55 |
| $\hat{S}^2_{p1-11}$ | 140.53 | $\hat{S}^2_{p2-11}$ | 141.44 | $\hat{S}^2_{p3-11}$ | 141.39 |
| $\hat{S}^2_{p1-12}$ | 140.57 | $\hat{S}^2_{p2-12}$ | 141.46 | $\hat{S}^2_{p3-12}$ | 141.41 |
| $\hat{S}^2_{p1-13}$ | 138.17 | $\hat{S}^2_{p2-13}$ | 139.22 | $\hat{S}^2_{p3-13}$ | 139.14 |
| $\hat{S}^2_{p1-14}$ | 138.78 | $\hat{S}^2_{p2-14}$ | 140.03 | $\hat{S}^2_{p3-14}$ | 139.96 |
| $\hat{S}^2_{p1-15}$ | 138.32 | $\hat{S}^2_{p2-15}$ | 139.44 | $\hat{S}^2_{p3-15}$ | 139.37 |
| $\hat{S}^2_{p1-16}$ | 138.37 | $\hat{S}^2_{p2-16}$ | 139.51 | $\hat{S}^2_{p3-16}$ | 139.43 |

Note: Bold value indicates maximum PRE.

**Table 7.** PREs of the existing estimators vs. traditional ratio estimator for population-II.

| **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** |
|---|---|---|---|---|---|---|---|
| $\hat{S}^2_1$ | 100.55 | $\hat{S}^2_2$ | 100.11 | $\hat{S}^2_3$ | 102.00 | $\hat{S}^2_4$ | 101.43 |
| $\hat{S}^2_5$ | 103.65 | $\hat{S}^2_6$ | 102.47 | $\hat{S}^2_7$ | 101.29 | $\hat{S}^2_8$ | 102.59 |
| $\hat{S}^2_9$ | 100.89 | $\hat{S}^2_{10}$ | 101.20 | $\hat{S}^2_{11}$ | 101.47 | $\hat{S}^2_{12}$ | 101.59 |
| $\hat{S}^2_{13}$ | 102.00 | $\hat{S}^2_{14}$ | 102.86 | $\hat{S}^2_{15}$ | 103.50 | $\hat{S}^2_{16}$ | 103.95 |
| $\hat{S}^2_{17}$ | 104.68 | $\hat{S}^2_{18}$ | 107.07 | $\hat{S}^2_{19}$ | 100.73 | $\hat{S}^2_{20}$ | 100.03 |
| $\hat{S}^2_{21}$ | 102.60 | $\hat{S}^2_{22}$ | 106.99 | $\hat{S}^2_{23}$ | 109.46 | $\hat{S}^2_{SK}$ | **117.10** |

Note: Bold value indicates maximum PRE.

considered in this study (cf. Tables 9 and 10). In general, some of the proposed estimators have lower PREs than the Subramani and Kumarapandiyan [31] estimator. These estimators are more efficient than the rest of the existing estimators;

4. Among the proposed classes of estimators in this study, the estimators proposed in class-II exhibit the best performance in terms of PRE (cf. Tables 6, 8, and 10);

5. In each of the proposed classes of estimators, the estimators which integrate the information of the correlation coefficient of the study and auxiliary variables with the non-conventional parameters of an auxiliary variable are superior in terms of PRE to other proposed estimators (cf. Tables 6, 8, and 10);

6. The proposed estimator of population variance that combines the mid-range ($MR_X$) and the correlation coefficient between the study and auxiliary variables turns out to be the most efficient estimator among all the proposed classes (cf. Tables 6, 8, and 10);

7. In the presence of outliers, the performances of the majority of the existing estimators are almost similar to the traditional ratio estimator (cf. Tables 5, 7, and 9). However, the suggested estimators perform very well as compared to the existing and

**Table 8.** PREs of the proposed classes of estimators vs. traditional ratio estimator for population-II.

| **Proposed Class-I** | | **Proposed Class-II** | | **Proposed Class-III** | |
|---|---|---|---|---|---|
| **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** |
| $\hat{S}^2_{p1-1}$ | 145.80 | $\hat{S}^2_{p2-1}$ | 147.19 | $\hat{S}^2_{p3-1}$ | 147.13 |
| $\hat{S}^2_{p1-2}$ | 146.69 | $\hat{S}^2_{p2-2}$ | 147.91 | $\hat{S}^2_{p3-2}$ | 147.86 |
| $\hat{S}^2_{p1-3}$ | 145.93 | $\hat{S}^2_{p2-3}$ | 147.32 | $\hat{S}^2_{p3-3}$ | 147.26 |
| $\hat{S}^2_{p1-4}$ | 145.94 | $\hat{S}^2_{p2-4}$ | 147.33 | $\hat{S}^2_{p3-4}$ | 147.27 |
| $\hat{S}^2_{p1-5}$ | 146.52 | $\hat{S}^2_{p2-5}$ | 147.79 | $\hat{S}^2_{p3-5}$ | 147.74 |
| $\hat{S}^2_{p1-6}$ | 147.30 | $\hat{S}^2_{p2-6}$ | 148.24 | $\hat{S}^2_{p3-6}$ | 148.20 |
| $\hat{S}^2_{p1-7}$ | 146.65 | $\hat{S}^2_{p2-7}$ | 147.88 | $\hat{S}^2_{p3-7}$ | 147.83 |
| $\hat{S}^2_{p1-8}$ | 146.65 | $\hat{S}^2_{p2-8}$ | 147.89 | $\hat{S}^2_{p3-8}$ | 147.83 |
| $\hat{S}^2_{p1-9}$ | 147.16 | $\hat{S}^2_{p2-9}$ | 148.17 | $\hat{S}^2_{p3-9}$ | 148.13 |
| $\hat{S}^2_{p1-10}$ | 147.76 | $\hat{S}^2_{p2-10}$ | **148.42** | $\hat{S}^2_{p3-10}$ | 148.38 |
| $\hat{S}^2_{p1-11}$ | 147.26 | $\hat{S}^2_{p2-11}$ | 148.22 | $\hat{S}^2_{p3-11}$ | 148.18 |
| $\hat{S}^2_{p1-12}$ | 147.27 | $\hat{S}^2_{p2-12}$ | 148.23 | $\hat{S}^2_{p3-12}$ | 148.18 |
| $\hat{S}^2_{p1-13}$ | 144.51 | $\hat{S}^2_{p2-13}$ | 145.57 | $\hat{S}^2_{p3-13}$ | 145.48 |
| $\hat{S}^2_{p1-14}$ | 145.25 | $\hat{S}^2_{p2-14}$ | 146.59 | $\hat{S}^2_{p3-14}$ | 146.52 |
| $\hat{S}^2_{p1-15}$ | 144.60 | $\hat{S}^2_{p2-15}$ | 145.71 | $\hat{S}^2_{p3-15}$ | 145.63 |
| $\hat{S}^2_{p1-16}$ | 144.61 | $\hat{S}^2_{p2-16}$ | 145.72 | $\hat{S}^2_{p3-16}$ | 145.64 |

Note: Bold value indicates maximum PRE.

**Table 9.** PREs of the existing estimators vs. traditional ratio estimator for population-III.

| **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** |
|---|---|---|---|---|---|---|---|
| $\hat{S}^2_1$ | 101.11 | $\hat{S}^2_2$ | 100.28 | $\hat{S}^2_3$ | 105.56 | $\hat{S}^2_4$ | 103.54 |
| $\hat{S}^2_5$ | 109.19 | $\hat{S}^2_6$ | 105.93 | $\hat{S}^2_7$ | 106.43 | $\hat{S}^2_8$ | 103.03 |
| $\hat{S}^2_9$ | 102.66 | $\hat{S}^2_{10}$ | 102.90 | $\hat{S}^2_{11}$ | 104.05 | $\hat{S}^2_{12}$ | 104.82 |
| $\hat{S}^2_{13}$ | 105.56 | $\hat{S}^2_{14}$ | 108.26 | $\hat{S}^2_{15}$ | 109.05 | $\hat{S}^2_{16}$ | 111.86 |
| $\hat{S}^2_{17}$ | 115.03 | $\hat{S}^2_{18}$ | 118.82 | $\hat{S}^2_{19}$ | 101.54 | $\hat{S}^2_{20}$ | 100.10 |
| $\hat{S}^2_{21}$ | 107.60 | $\hat{S}^2_{22}$ | 140.23 | $\hat{S}^2_{23}$ | 133.23 | $\hat{S}^2_{SK}$ | **178.49** |

Note: Bold value indicates maximum PRE.

traditional ratio-type estimators (cf. Tables 6, 8, and 10).

These findings highlight the significance of the incorporation of non-conventional and robust measures of the auxiliary variable in the ratio method of estimation for estimating population variance in the presence of outliers. The inclusion of these measures results in higher PREs with the proposed estimators than with the existing estimators, which are only based on conventional measures.

## 5. Conclusion and recommendations

The present study introduces three new classes of ratio-type estimators of population variance under Simple Random Sampling (SRS) by integrating the information of non-conventional measures of an auxiliary variable with the conventional measures. The algebraic expressions for the bias and Mean Square Errors (MSEs) were theoretically obtained. Moreover, the efficiency conditions under which the proposed estimators performed better than the existing estimators were derived. An empirical study was conducted to supplement the theoretical results. The empirical results revealed that the proposed estimators outperformed the existing estimators of Isaki [9], Upadhyaya and Singh [10,30], Kadilar and Cingi [11], Khan and Shabbir [15], and Subramani and Kumarapandiyan [12–14,29,31] in terms of MSE and PRE when outliers

**Table 10.** PREs of the proposed classes of estimators vs. traditional ratio estimator for population-III.

| Proposed Class-I | | Proposed Class-II | | Proposed Class-III | |
|---|---|---|---|---|---|
| **Estimator** | **PRE** | **Estimator** | **PRE** | **Estimator** | **PRE** |
| $\hat{S}^2_{p1-1}$ | 172.96 | $\hat{S}^2_{p2-1}$ | 187.08 | $\hat{S}^2_{p3-1}$ | 186.97 |
| $\hat{S}^2_{p1-2}$ | 180.71 | $\hat{S}^2_{p2-2}$ | 194.20 | $\hat{S}^2_{p3-2}$ | 194.09 |
| $\hat{S}^2_{p1-3}$ | 175.17 | $\hat{S}^2_{p2-3}$ | 189.36 | $\hat{S}^2_{p3-3}$ | 189.24 |
| $\hat{S}^2_{p1-4}$ | 175.82 | $\hat{S}^2_{p2-4}$ | 189.98 | $\hat{S}^2_{p3-4}$ | 189.87 |
| $\hat{S}^2_{p1-5}$ | 177.65 | $\hat{S}^2_{p2-5}$ | 191.67 | $\hat{S}^2_{p3-5}$ | 191.56 |
| $\hat{S}^2_{p1-6}$ | 185.08 | $\hat{S}^2_{p2-6}$ | 197.23 | $\hat{S}^2_{p3-6}$ | 197.13 |
| $\hat{S}^2_{p1-7}$ | 179.85 | $\hat{S}^2_{p2-7}$ | 193.53 | $\hat{S}^2_{p3-7}$ | 193.42 |
| $\hat{S}^2_{p1-8}$ | 180.48 | $\hat{S}^2_{p2-8}$ | 194.02 | $\hat{S}^2_{p3-8}$ | 193.92 |
| $\hat{S}^2_{p1-9}$ | 194.36 | $\hat{S}^2_{p2-9}$ | 201.53 | $\hat{S}^2_{p3-9}$ | 201.44 |
| $\hat{S}^2_{p1-10}$ | 197.71 | $\hat{S}^2_{p2-10}$ | **202.40** | $\hat{S}^2_{p3-10}$ | 202.31 |
| $\hat{S}^2_{p1-11}$ | 195.47 | $\hat{S}^2_{p2-11}$ | 201.86 | $\hat{S}^2_{p3-11}$ | 201.77 |
| $\hat{S}^2_{p1-12}$ | 195.77 | $\hat{S}^2_{p2-21}$ | 201.94 | $\hat{S}^2_{p3-12}$ | 201.85 |
| $\hat{S}^2_{p1-13}$ | 159.99 | $\hat{S}^2_{p2-13}$ | 169.68 | $\hat{S}^2_{p3-13}$ | 169.55 |
| $\hat{S}^2_{p1-14}$ | 165.92 | $\hat{S}^2_{p2-14}$ | 178.52 | $\hat{S}^2_{p3-14}$ | 178.40 |
| $\hat{S}^2_{p1-15}$ | 161.51 | $\hat{S}^2_{p2-15}$ | 172.09 | $\hat{S}^2_{p3-15}$ | 171.97 |
| $\hat{S}^2_{p1-16}$ | 161.98 | $\hat{S}^2_{p2-16}$ | 172.81 | $\hat{S}^2_{p3-16}$ | 172.69 |

Note: Bold value indicates maximum PRE.

were present in the data. Based on the findings of the present study, it is strongly recommended that if the information on robust non-conventional measures of the auxiliary variable is available together with the information on conventional measures of the auxiliary variable, the proposed estimators should be used to efficiently estimate the population variance under SRS after making necessary adjustments to the bias. This study was limited to the use of non-conventional location or robust measures of location and the proposed estimators were based on only a single auxiliary variable in the context of ratio method of estimation. The scope of the present study can be extended by using non-conventional dispersion measures and inclusion of additional auxiliary variables. Moreover, the robust measures can be employed to enhance efficiency of the variance estimators under different sampling schemes such as stratified random sampling, two-phase sampling, and ranked set sampling.

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## Appendix

| | |
|---|---|
| $N$ | Population size |
| $n$ | Sample size |
| $f=n/N$ | Sampling fraction |
| $\eta=\frac{1-f}{n}$ | A constant |
| $Y$ | Study variable |
| $X$ | Auxiliary variable |
| $\bar{Y},\bar{X}$ | Population means |
| $\bar{y},\bar{x}$ | Sample means |
| $S_Y,S_X$ | Population standard deviations |
| $s_y,s_x$ | Sample standard deviations |
| $C_Y,C_X$ | Population coefficients of variation |
| $\rho$ | Population correlation coefficient |
| $\beta_{2(Y)}=\frac{\mu_{40}}{\mu_{20}^2}$ | Population coefficient of kurtosis of the study variable |
| $\beta_{2(X)}=\frac{\mu_{04}}{\mu_{02}^2}$ | Population coefficient of kurtosis of the auxiliary variable |
| $\beta_{1(X)}=\frac{\mu_{03}^2}{\mu_{02}^3}$ | Population coefficient of skewness of the auxiliary variable |
| $M_{d(X)}$ | Population median of the auxiliary variable |
| $Q_{1(X)}$ | Population lower quartile of the auxiliary variable |
| $Q_{3(X)}$ | Population upper quartile of the auxiliary variable |
| $Q_{r(X)}$ | Population inter-quartile range of the auxiliary variable |
| $Q_{a(X)}$ | Population inter-quartile average of the auxiliary variable |
| $Q_{d(X)}$ | Population semi-inter-quartile range of the auxiliary variable |
| $D_{i(X)}$ | $i$th $(i=1,2,...,10)$ population decile of the auxiliary variable |
| $TM_X$ | Population tri-mean of the auxiliary variable $X$ |
| $MR_X$ | Population mid-range of the auxiliary variable $X$ |
| $HL_X$ | Hodges-Lehmann mean of the auxiliary variable $X$ |
| $DM_X$ | Population deciles mean of the auxiliary variable $X$ |
| $B(.)$ | Bias of the estimator |
| $MSE(.)$ | Mean square error of the estimator |
| $\hat{S}_R^2$ | Traditional ratio estimator |
| $\hat{S}_i^2$ | Existing ratio estimator |
| $\hat{S}_{SK(i)}^2$ | Subramani and Kumarapandiyan (2015) class of estimators |
| $\hat{S}_{p1-j}^2$ | Proposed class-I estimators |
| $\hat{S}_{p2-l}^2$ | Proposed class-II estimators |
| $\hat{S}_{p3-m}^2$ | Proposed class-III estimators |

$$\mu_{rs}=\frac{1}{N}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^r\left(X_i-\bar{X}\right)^s;$$

$$\lambda_{rs}=\frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}};$$

$$\lambda_{22}=\frac{\mu_{22}}{\mu_{20}\mu_{02}};$$

$$\lambda_{21}=\frac{\mu_{21}}{\mu_{20}\sqrt{\mu_{02}}}.$$

## Biographies

**Farah Naz** earned her MSc and Mphil degrees in Statistics from the Islamia University Bahawalpur, Pakistan, and Government College University Faisalabad, Pakistan, respectively. Currently, she is pursuing PhD in Statistics in the School of Mathematical Sciences, Institute of Statistics, Zhejiang University, Hangzhou, People's Republic of China, under the Chinese Government Scholarship Program (2017). Her research interest is survey sampling and distribution theory.

**Muhammad Abid** obtained his MSc and MPhil degrees in Statistics from Quaid-i-Azam University, Islamabad, Pakistan, in 2008 and 2010, respectively. He received his PhD in Statistics from the School of Mathematical Sciences, Institute of Statistics, Zhejiang University, Hangzhou, People's Republic of China, in 2017. He served as a statistical officer in National Accounts Wing, Pakistan Bureau of Statistics (PBS), during 2010-2011. He has been serving as an Assistant Professor in the Department of Statistics, Government College University Faisalabad, Pakistan, since 2017. He has published more than 15 research papers in research journals. His research interests include statistical quality control, Bayesian statistics, non-parametric techniques, and survey sampling.

**Tahir Nawaz** obtained his MSc and MPhil degrees in Statistics from the Islamia University Bahawalpur, Pakistan, and Government College University Lahore, Pakistan, respectively. He served as a statistical officer in Punjab Bureau of Statistics, Pakistan, during May 2007 to November 2009. Also, he served as a lecturer in Statistics at Islamia University Bahawalpur, Pakistan, during November 2009 to October 2013. He has been working as a lecturer in the Department of Statistics, Government College University Faisalabad, Pakistan, since November 2013. He is also pursuing his PhD in Statistics in the School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, People's Republic of China, under the Chinese Government Scholarship Program (2016). His research interest includes statistical process control, distribution theory, and survey sampling.

**Tianxiao Pang** earned his PhD in Probability and Mathematical Statistics from the Department of Mathematics, Zhejiang University, Hangzhou, People's Republic of China in 2005 and is now an Associate Professor of Statistics at the same university. More than 30 refereed publications in various reputed international journals are to his credit. His research interest is in probability limit theory, large sample theory in statistics, and econometrics.