Numerical study of hydrothermal characteristics in nanofluid using KKL model with Brownian motion

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Abstract: Finite element method (FEM) is used to study the hydrothermal characteristics of the nano-fluid subjected to Brownian motion. For effective thermal conductivity and effective, viscosity Koo-Kleinstreuer-Li (KKL) model is used. It is observed that the dispersion of nano-particles in Newtonian liquid causes a significant increase in the effective thermal conductivity. The results based on the dispersion of nano-particles help engineers to design an efficient thermal system. A significant role of viscous dissipation on diffusion of momentum of wall into the fluid is observed. Therefore, dissipations effects cannot be ignored while designing thermal systems. The buoyant force is responsible for the effect of electromagnetic thermal radiations on the velocity of fluid convectively heated surface enhances the rate of generation of entropy. This study also recommends that nano-fluids are the best coolants as compare to the base fluids. Imposition of magnetic field causes more entropy generation.

Keywords: KKL-model, effective diffusion coefficients, entropy generation, Brownian motion, thermal radiation.

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**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$u,v$</td>
<td>Velocity components</td>
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<tr>
<td>$Gr$</td>
<td>Grashof number</td>
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<tr>
<td>$Ha$</td>
<td>Hartmann number</td>
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<tr>
<td>$Ec$</td>
<td>Eckert number</td>
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<tr>
<td>$B_i$</td>
<td>Biot number</td>
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<tr>
<td>$Nr$</td>
<td>radiation parameter</td>
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<tr>
<td>$Pr$</td>
<td>Prandtl number</td>
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<tr>
<td>$Re$</td>
<td>Reynold number</td>
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<td>$Br$</td>
<td>Brinkman number</td>
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<tr>
<td>$N_s$</td>
<td>entropy generation number</td>
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<tr>
<td>$\bar{q}$</td>
<td>radiative heat flux</td>
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<tr>
<td>$T$</td>
<td>Fluid temperature</td>
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<tr>
<td>$k$</td>
<td>thermal conductivity</td>
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<tr>
<td>$\beta_T$</td>
<td>volumetric expansion coefficient</td>
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<tr>
<td>$\kappa_b$</td>
<td>Boltzmann constant</td>
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<tr>
<td>$k^*$</td>
<td>absorption coefficient</td>
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<tr>
<td>$\sigma^*$</td>
<td>Stefan-Boltzmann constant</td>
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<td>$c_p$</td>
<td>specific heat</td>
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<td>$B_0$</td>
<td>magnetic field</td>
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<tr>
<td>$Q$</td>
<td>heat generation/absorption coefficient</td>
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<tr>
<td>$\rho$</td>
<td>density</td>
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<tr>
<td>$\sigma$</td>
<td>electrical conductivity</td>
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<tr>
<td>$\varphi$</td>
<td>volume fraction</td>
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<tr>
<td>$\mu$</td>
<td>dynamic viscosity</td>
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<tr>
<td>$\eta$</td>
<td>independent similarity function</td>
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<tr>
<td>$\lambda$</td>
<td>unsteadiness parameter</td>
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<td>$\Omega$</td>
<td>temperature difference number</td>
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**Greek Symbols**

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**Subscripts**

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<tr>
<td>$f$</td>
<td>fluid</td>
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<tr>
<td>$nf$</td>
<td>nanofluid</td>
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<td>$eff$</td>
<td>effective</td>
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<td>$p$</td>
<td>particles</td>
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<td>$s$</td>
<td>solid particles</td>
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<td>$w$</td>
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**Introduction**

The enhancement of the efficiency of thermal systems has been a major concern of technologists. Several techniques are used for this purpose. Besides conventional methods, some novel and efficient techniques to enhance the thermal efficiency are preferred because of number of reasons.

One of the classical methods for the enhancement of heat transfer is to extend the surface of cooling fin but this requires an increase in the size of cooling system which is not desirable in many cases. Therefore, engineers and scientists have introduced the technique of inclusion of nanosize metallic particles in the pure fluid. This causes the enhancement of thermal conductivity of the constituent fluid and consequently, thermal system works in an efficient way. This inclusion of nanosize metallic particles in a pure fluid give rise a fluid mixture with totally different
thermo-physical properties as compare to the pure fluid (the base fluid). This concept of dispersion of metallic nanosized particles has addressed many challenges of extension of surface of cooling fins. Invention of such fluids at industry level motivated the researchers to introduce imperial/mathematical models for the thermo-physical properties of nanofluids and, theoretically, studied the effects of nano-size metallic particles on the enhancement of thermal conductivity of the resulting fluids. Such kind of improvements in the process of heat transfer may cause a great revolution in the thermal and cooling systems.

**Comparative analysis of models for effective diffusion coefficients:** In order to analyze the effects of dispersion of nano-size metallic particles on the effectiveness of viscosity and thermal conductivity of the pure fluid, different models have been used. Every model has some limitations. However, some models lead to less thermal performance as compare to the thermal performance of others. These models include, Einstein model [1], Brink model [2], Batchelor model [3], Graham model [4], Wang et al. [5] model, Avsec and Oblak model [6], Masoumi et al. [7] model etc. These models forecast the effectiveness of viscosity of mixture of nano-particles and fluid. These models give correlations for viscosities (viscosity of base fluid, viscosity of nano-particles and viscosity of nanofluid) and say nothing about the effective thermal conductivity of the mixture of nano-particles and base fluid. The use of effective viscosity models without considering theoretical models for effective thermal conductivity leads to erroneous results. Due to this reason, theoretical models for effective viscosity and effective thermal conductivity are used simultaneously. The theoretical correlations for effective thermal conductivity can be found in refs. [8-15]. However, these models predict that the effective viscosity is a function of volume fraction. The role of Brownian motion of nano-particles in base fluid put a bar on the above mentioned models which do not exhibit the effect of Brownian motion of nano-particles on the
effective viscosity. This motivated researchers to propose new models, capable of exhibiting the
effects of Brownian motion. Some new correlations considering the effects of Brownian motion,
Effectiveness of viscosity are proposed/used in Refs. [16-24].
However, theoretical models used in [8-15] do not consider Brownian motion. Theoretical models
for effective viscosity and thermal conductivity, Brownian motion effect are discussed in refs.
[16-24]. However, the latest model is by Koo, Kleinstreuer and Li (KKL) [25] and is given by

\[ \rho_{nf} = (1-\varphi) \rho_f + \varphi \rho_s, \quad (\rho c_p)_{nf} = (1-\varphi)(\rho c_p)_f + \varphi(\rho c_p)_s, \quad (1) \]

\[ \sigma_{nf} = \sigma_f \left( 1+ \frac{3(\sigma - 1\varphi)}{\sigma + 2 - (\sigma - 1\varphi)} \right), \quad \sigma = \frac{\sigma_s}{\sigma_f}, \quad (2) \]

Correlations for \( k_{eff} \) and \( \mu_{nf} \) in the presence of Brownian motion are given

\[ k_{static} = 1+ \frac{3\varphi}{(k_f/k_s-1)(k_s/k_f+2-(k_s/k_f-1)\varphi)}, \quad k_{eff} = k_{static} + k_{Brownian}, \]

\[ k_{eff} = 1+ \frac{3\varphi(k_s/k_f-1)}{(k_s/k_f+2-(k_s/k_f-1)\varphi)} + 5 \times 10^4 g'(\varphi, T, d_p) \nu \rho_f (c_p)_f \sqrt{\frac{k_s T}{d_p \rho_p}} \]

\[ R_f = 4 \times 10^{-8} km^2 / W, \quad R_f = -d_p (1 / k_p - 1 / k_{p, eff}). \]

\[ g'(\varphi, T, d_p) = \left[ \begin{array}{c} a_1 + a_3 Ln(\varphi) + a_2 Ln(d_p) + a_5 Ln(d_p)^2 \\ + a_3 Ln(d_p) Ln(\varphi) \\ + \left( a_6 + a_5 Ln(\varphi) + a_2 Ln(d_p) + a_5 Ln(d_p)^2 \right) \\ + a_2 Ln(\varphi) Ln(d_p) \end{array} \right] \]

\[ k_{Brownian} = 5 \times 10^4 g'(\varphi, T, d_p) \nu \rho_f (c_p)_f \sqrt{\frac{k_s T}{d_p \rho_p}}, \quad 300 K < T < 325 K. \]

\[ \mu_{nf} = \frac{\mu_f}{(1-\varphi)^3} + \frac{k_{Brownian}}{k_f} \times \frac{\mu_f}{Pr_f}, \]
Thermo-physical properties of water and two types of metallic nano-particles which are used by J. Li and M. Sheikholeslami [26] are given in the Table 1 and coefficient values Copper Oxide and Aluminum Oxide nanofluids are shown in Table 2.

This model is used in some recent studies. For example, Sheikholeslami et al. [26] studied an enhancement of heat transfer in turbulent flow caused by twisted tape turbulators using KKL model. Sheikholeslami [27] analyzed the effect of nano-particles and Brownian motion on the transfer of heat in a square closure. In this study, Sheikholeslami [27] augmented nanofluid characteristics by using KKL model for effective viscosity and effective thermal conductivity. Another study was conducted by Sheikholeslami [28] in order to explore the process of solidification in nanofluid by considering KKL-method. Sheikholeslami and Rokni [29] used control volume finite element approach (CVFEA) to investigate the effect of Lorentz force on the flow of nanofluid in a porous enclosure using non-equilibrium model together with KKL-model for effective viscosity and thermal conductivity. Sheikholeslami et al. [30] studied the effects of Darcy porous medium on the MHD flow in a complex shaped container containing $Al_2O_3$ -nanofluid while estimating the effective viscosity and thermal conductivity through KKL-model. Sheikholeslami with other contributors used KKL-model for effective viscosity and thermal conductivity. These studies are referenced in [31-38]. Some latest studies on nanofluid flows are mentioned in Refs. [39-57].

Minimization of the entropy generation in the thermal system is a major concern as wastage of energy causes a great disorder. Therefore, the control of entropy generation during the heat transfer has been investigated extensively in the last few years. Bejan [58] was first to work on the minimization of entropy generation. After this pioneering work, several studies have been published. But, here, some recent investigations are described. For instance, Bhatti et al. [59]
investigated the effects of magnetic field on the entropy generation of nonlinear transport of heat and mass in the boundary layer flow. Numerical investigation of entropy generation during the heat transfer in the cavity flow was carried out by Armaghani et al. [60]. Vincenzo et al. [61] analyzed the effect of entropy generation due to temperature difference and viscous losses/friction loses in the flow.

Literature review reveals that no FEM study on hydro-thermal characteristics in nano-fluid subjected to Lorentz force, thermal radiation, buoyancy force using KKL-model is investigated so far. A comprehensive literature review is given in section one. Modeling of unsteady flow of nanofluid in the presence of buoyancy force and Joule heating is given in section two. Galerkin weak formulation and coefficients of stiffness matrix are given in section three. Section four is designated for results and discussion. Entropy analysis is also given in section four. Key points of this study are listed in section five.

**Mathematical models and modeling**

**Problem statement:** Let us investigate the effects of dispersion of nano-particles (CuO and Al₂O₃) on the performance of thermal conductivity and viscosity using KKL-model. The flow over the sheet is due to unsteady motion of sheet moving with velocity \( U_w(x,t) = ax / (1-ct) \) and is subjected to the magnetic field. Nanofluid is assumed to exhibit thermal radiation and generates heat during thermal changes. Buoyant force is considerable order of magnitude. Thermo-physical properties (viscosity, density, thermal conductivity, specific heat etc.) are constant. The transport of heat nanofluid (occupying half space \( y > 0 \) ) is due to convection from the hot fluid (occupying half space \( y < 0 \) ) of temperature \( T_w(x,t) = T_o + ax / (1-ct) \). The buoyant force under Boussinesq approximation is significant as shown in Fig. 1.

**Diffusion models:** Mass, momentum and thermal diffusion models under boundary layer
approximations are \([51, 52]\)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho_{\text{of}}} \left( \mu_{\text{of}} \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma_{\text{of}} B_0^2 u}{\rho_{\text{of}}} + \beta_f g (T - T_0), \tag{4}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{(\rho c_p)_{\text{nf}}} \left[ k_{\text{eff}} \frac{\partial^2 T}{\partial y^2} + Q(T - T_\infty) \right] + \frac{\sigma_{\text{nf}} B_0^2}{\rho c_p} u^2 - \nabla \bar{q} \tag{5}
\]

Stefan Boltzmann law is defined by 
\[\bar{q} = -\frac{4\sigma^*}{\pi^*} \nabla \left( T^4 - T_\infty^4 \right).\]

The initial and boundary conditions are

\[u = v = 0, \ T = T_\infty, \ \forall \ x, y, \ t < 0 \tag{6}\]

\[u(x, y) = U_w(x, t), \ v = 0, -k_f \frac{\partial T}{\partial y} = h_f \left[ T_w(x, t) - T \right], \ \text{at} \ y = 0, \ \left\{ \begin{array}{l} u \rightarrow 0, \ T \rightarrow T_\infty \ \text{as} \ y \rightarrow \infty. \end{array} \right. \tag{7}\]

(d) Normalization of equations: Diffusion Eqs. \((3) - (5)\) and initial, boundary conditions \((6)\) and \((7)\) are made dimensionless by the use of following transformations

\[u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}, \ \psi(x, y) = \sqrt{\frac{aw}{(1-\epsilon t)}} \chi \left( \eta \right), \ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \ \eta = \sqrt{\frac{aw}{\nu (1-\epsilon t)}} y, \tag{8}\]

Hence we get

\[f'''' - \frac{A_5}{A_3} M^2 f' + \frac{A_5}{A_3} Gr \theta - \frac{A_4}{A_3} \left[ f'' - \frac{1}{2} \eta f'' + \lambda \left( f' + \frac{1}{2} \eta f'' \right) \right] = 0, \tag{9}\]

\[f(0) = 0, f'(0) = 1, f' \rightarrow 0,\]

\[f'(0) = 0, \beta \theta + \frac{A_5 A_4}{A_4} M^2 Pr Ecf f'' + \frac{A_4}{A_4} Pr \left[ f \theta' - f' \theta - \lambda \left( 2 \theta + \frac{1}{2} \eta \theta \right) \right] = 0. \tag{10}\]
\[ \theta'(\eta) = -B_i(1-\theta(\eta)), \theta \to 0 \text{ as } \eta \to \infty, \]  
(11)

where

\[ A_1 = \frac{\rho_{nf}}{\rho_f}, A_2 = \frac{(\rho_{nf})_f}{(\rho_f)_f}, A_3 = \frac{\mu_{nf}}{\mu_f}, A_4 = \frac{k_{nf}}{k_f}, A_5 = \frac{\sigma_{nf}}{\sigma_f}. \]  
(12)

and

\[ M^2 = \frac{\sigma_f B_o^2 (1-ct)}{a \rho_f}, \quad Gr = \frac{\beta_f (T_w - T_\infty)(1-ct)^2}{a^2 x}, \quad \lambda = \frac{c}{a}, \quad N_r = \frac{k_{nf} k^*}{4\sigma^* T_\infty^3}, \]
\[ Pr = \frac{\mu_f (c_p)_f}{k_f}, \quad Ec = \frac{(\alpha c_p)_f (T_w - T_\infty)^2}{(c_p)_f (T_w - T_\infty)}, B_i = \frac{h_f}{k_f} \sqrt{\frac{1-ct}{(T_w - T_\infty)}}, \beta = \frac{Q_0}{a (\rho c_p)_f}, \]  
(13)

are, respectively, the Hartmann number, the Grashof number, the unsteadiness parameter, the radiation parameter, the Prandtl number, the Eckert number, the Biot number and heat generation/absorption parameter. The prescribed wall temperature case can be recovered as \( B_i \to \infty \). Also note that \( k_{Brownian} = 0 \) and \( \varphi = 0 \) is the case when fluid is pure and nano-particles are not dispersed, [the case of Butt and Ali 62] and for \( Gr = 0, N_r = 0 \) and \( Ec = 0 \), the problem reduces to the case of Das et al. [63] with heat generation/absorption. The case of \( M^2 = 0, \lambda = 0, k_{Brownian} = 0, \varphi = 0 \) and \( B_i \to \infty \) is also considered by Abolbashari et al. [64] and Das et al [63].

**Numerical Method**

Galerkin finite element method (GFEM) is implemented to carry out the simulations for the transfer heat through dimensionless conservation. As a part of procedure following steps are done.

**Domain discretization:** The physical domain (after dimensional analysis) \([0, \infty]\) is divided into line elements with two nodes per elements.

**Selection of weight and interpolation functions:** As there are two nodes per element, therefore,
weight and shape functions interpolation functions) are selected in linear form. Further, as suggested by the Galerkin approach, weight functions are taken equal to the interpolation functions. The following linear interpolation functions are defined as

\[
\psi_j = (-1)^{j-1} \left( \frac{\eta_{j+1} - \eta_j}{\eta_{j+1} - \eta_j} \right), \quad j = 1, 2.
\]

**Constructions of residual equations:** Residual equations are defined and are multiplied by weights. The resulting weighted residuals are integrated over a typical element \([\eta_e, \eta_{e+1}]\).

**Weak form of weighted residuals:** The weighted integral residual are integrated over the line element to convert strong form of weighted residual into weak form.

**Derivation of stiffness coefficients:** The dependent unknowns are approximated over element \([\eta_e, \eta_{e+1}]\) by the finite element approximations. The unknown nodal values are computed. The approximations in weak formulation of weighted residuals, one obtains the finite element model of the form

\[
\begin{bmatrix} K^e \{\pi^e\} \end{bmatrix} \{\pi^e\} = \{F^e\} + \{Q^e\}
\]

where \([ K^e \{\pi]\) is the stiffness matrix for typical element, \{\pi^e\} are unknown nodal values, \{F^e\} is the boundary vector and \{Q^e\} is the source vector.

**Assembly process:** Following the assembly procedure of finite element approach, we get the system of nonlinear algebraic equations of the form

\[
\begin{bmatrix} K(\pi) \end{bmatrix} \{\pi\} = \{F\}
\]

where \([ K(\pi) \) is global coefficient matrix, It is important to note that the coefficients of stiffness matrix \([ K(\pi) \) are also functions of unknown nodal values. Therefore, systems of algebraic equations (12) are being solved numerical by an iterative procedure. Here in this study, Picard's
linearization procedure is used which works in the following way

\[
\begin{bmatrix}
K(\pi)^{-1}
\end{bmatrix} \{\pi\} = \mathbf{F}
\]

(15)

where \(\{\pi\}^{r-1}\) are nodal values computed at \((r-1)^{th}\) iteration and \(\{\pi\}^{r}\) are the nodal values being computed at the \(r^{th}\) iteration.

**Computer implementation:** The linearized system of algebraic equations (15) is solved iteratively using Guass-Siedal approach. For computational procedure described above is implemented using Matlab. The developed computer code works with tolerance \(10^{-5}\) several computational experiments were done to search the value of \(\eta\) where asymptotic boundary conditions are satisfied. Simulations carried out in this study reveal that the asymptotic boundary conditions are satisfied when \(\eta = 6\). So \(\eta_{\text{max}} = 6\) is taken as infinity i.e. the computational domain is \([0, 6]\).

**Convergence and error analysis:** The error in the simulated results is calculated using

\[
\text{error} = |\{\pi\}^r - \{\pi\}^{r-1}|
\]

and convergence criteria is set the following form

\[
\max_{i} |\{\pi\}^r_i - \{\pi\}^{r-1}_i| < \xi
\]

where \(\xi\) is the tolerance and it is taken equal \(10^{-5}\) to in this coming analysis.

**Validation:**

To validate the results, the numerical values of \(-\Theta'(0)\) for special case (for different Pr when \(M^2 = \lambda = \varphi = k_{\text{brownian}} = 0,\) and \(B_i \to \infty\).) are compared to already published benchmarks by Das et al. [63] and Albashari et al. [64]. This comparison is displayed in Table 3. This table guarantees an excellent agreement between present and already published work.
Results and discussion

In order to analyze the physics of the described flow situation, the computed field variables are displayed versus the physical parameters. Fig. 2 displays the influence of magnetic field on the flow of CuO-nanoliquid (solid curves) and Al₂O₃-nanoliquid (dotted curves). Hartmann number is the ratio of Lorentz force to inertial force and an increase in Hartmann number corresponds to an increase in magnetic field intensity. Hence, it is observed that flow experiences retardation when magnetic field intensity is increased. It is also noted that Al₂O₃-nanoliquid experiences more Lorentz force as compared to the CuO-nanoliquid (see Fig. 2). Behavior of flow of Al₂O₃ and CuO-nanofluids under an increase in the buoyant force is represented by Fig. 3. During simulations, it is also observed that flow is assisted by positive buoyant force (Gr > 0) whereas it is opposed by negative buoyant force (Gr < 0). This impact of positive buoyant force on CuO-nanoliquid is more than the impact of buoyant force on the flow of Al₂O₃-nanoliquid as shown by Fig. 3. The velocity field is noted to decrease by increasing the unsteadiness parameter λ as represented by Fig. 4. The velocity of steady flow is greater than the velocity of unsteady flow. The boundary layer thickness is decreasing function of unsteadiness parameter λ. The intensity and heat dissipates due to Joule heating are directly proportional. Therefore, an increase in the intensity of magnetic field enhances the rate of dissipation of heat in the liquid regime and so temperature increases. This effect of intensity of magnetic field on the temperature is displayed in Fig. 5 for both Al₂O₃-nanoliquid and CuO-nanoliquid. From Fig. 5, it can also be observed that the rate of dissipation of heat to Ohmic heating (heating due to magnetic field) in Al₂O₃-nanoliquid regime is greater than that in CuO-nanoliquid when Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, φ = 0.04, B_i = 0.1, β = 0.9 and λ = 0.6. The Eckert number
$Ec$ appears as a coefficient of Joule heating term in the dimensionless form of energy equation (see Eq. (10)) and an increase in it causes an enhancement in rate of Joule heating. Consequently the temperature increases. This fact is displayed in Fig. 6. The comparison of Figs. 5 and 6 shows that the temperature of fluids varies in a similar fashion, in qualitative sense, when Hartmann and $Ec$ are increased. The temperature field for both $Al_2O_3$-nanoliquid (dotted curves) and $CuO$-nanoliquid (solid curves) under variation of unsteadiness parameter $\lambda$ is reflected by Fig. 7. This Fig. shows the declining behavior of temperature with respect to unsteadiness parameter $\lambda$. Fig. 8 displays the temperatures curves for $CuO$ and $Al_2O_3$. These temperature curves show that $CuO$ nanoparticles are responsible more heat transfer than $Al_2O_3$ nanoparticles. The temperature distribution for both $CuO$-nanoliquid and $Al_2O_3$-nanoliquid due to heat generation in liquid regime is given by Fig. 9. A significant increase in temperature due to internal heat generation is observed. The role of convective boundary in the enhancement of transfer of heat from heated wall to the nanoliquid regime is revealed by Fig. 10. This Fig. 10 clearly exposes that heat transfer process speeds up if the convection parameter (Biot number) is increased. However, this convection is more significant in $CuO$-nanoliquid as compare to the convection in $Al_2O_3$-nanoliquid. Thermal radiation effects of the temperature of $Al_2O_3$-nanoliquid and $CuO$-nanoliquid is shown in Fig. 11. It can be easily noted that heat transfer process increases if the radiation parameter ($N_r$) increases.

**Entropy Generation**

**Entropy analysis:** The entropy generation due to temperature gradient, viscous dissipation and Joule heating is defined by
\[ E_G = \frac{k_{ef}}{T^2} \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right] + \frac{\mu_{nf}}{T^2} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma_{nf} B^2}{T} u^2 \]  

(16)

Using the similarity transformations given in Eq. (8), one obtains the following dimensionless form of entropy generation.

\[ N_s = Re \frac{k_{ef}}{k_f} \theta^2 + \frac{Br}{(1 - \varphi)^2 + \frac{k_{f_{nm}}}{k_f} \times \frac{1}{\nu_f}} \left[ f''^2 + \varphi_1 M^2 f'' \right], \]  

(17)

where,

\[ N_s = \frac{T_f^2 a^2 E_G}{k_f (T_w - T_e)^2}, \quad Re = \frac{U_w a}{v_f x}, \quad Br = \frac{\mu_f U_w^2}{k_f (T_w - T_e)^2}, \quad \Omega = \frac{T_w - T_e}{T_f} \]  

(18)

and, respectively, are called dimensionless entropy generation number, the Reynolds number, the Brinkman number and the non-dimensionless temperature difference number.

**Entropy generation profiles:**

The effect of an increase of the intensity of magnetic field on the entropy generation is (shown in Fig. 12). This graphical behavior of entropy generation verses of Hartmann number reveals that external magnetic field enhances the rate of entropy generation. This predicts that the imposition of magnetic field to thermal system causes more energy loses. Thus, we conclude that magnetic field should not be imposed if energy loses are to be minimized as it is required at industrial processes. It also noted from Fig. 12 that entropy generation in \( Al_2O_3 \)-nanoliquid is greater than the entropy generation \( CuO \)-nanoliquid. An increasing trend in entropy generation is observed when temperature difference parameter is increased (see Fig. 13). The entropy production is greatly influenced by an increase in the Reynolds number. An increasing trend in entropy generation due to an increase in Reynolds can be seen in Fig. 14. Fig. 15 depicts that entropy
generation increases when $Ec$ is increased.

**Conclusion**

The effects of Effective viscosity and thermal conductivity on enhancement of heat transfer due to the dispersion of nano-sized metallic particles ($CuO$ and $Al_2O_3$) are studied through KKL-model for Brownian motions. A significant increase in thermal conductivity is observed. The $Al_2O_3$ nanoliquid experiences more resistive force due to the magnetic field as compare to the resistance experienced by $CuO$-nanoliquid. The effect of four types of nano-particles ($Cu, Ag, Al_2O_3$ and $TiO_2$) on the transport of heat in unsteady two-dimensional boundary layer flow of a radiative fluid over a convectively heated surface in the presence of Joule heating, heat absorption/generation and buoyant force are investigated. It is observed that dispersion of nano-particles in the pure fluid increases the thermal conductivity of the resulting mixture which may play a vital role in the thermal systems. For favorable buoyant force the velocity of the mixture (mixture of nanoparticles and radiative fluid) increases which causes an increase in the thermal and momentum boundary layer thicknesses. However, in case of opposing buoyant force, a reverse mechanism regarding momentum and thermal boundary layer thicknesses is observed. The magnetic field intensity and Ohmic dissipation are directly proportional with each other. Hence an increase in the intensity of the magnetic field converts more electrical energy into heat (due to Ohmic dissipation process). It is also observed that an increase in the intensity of the magnetic field retards the flow and reduces the momentum boundary thicknesses. Therefore, it is advised that an external magnetic field may be applied to control the flow and momentum boundary layer thickness. However, it should be in mind that an increase in the imposition of external magnetic field has opposite effect on the thermal boundary layer thicknesses due to Joule heating mechanism. It is also important to mention that momentum boundary layer thickness for
hydrodynamic flow is higher than that of the magnetohydrodynamic flow. However, thermal boundary layer thickness of hydrodynamic flow is less than that of the magnetohydrodynamic flow. It is also observed that the effective thermal conductivity of $Al_2O_3$ nano liquid is greater than that of $CuO$-nanoliquid. Therefore, the use $Al_2O_3$ is recommended if more transportation of heat is required. Convection at the surface plays a notable effect on the temperature of nanoliquids ($Al_2O_3$-liquid and $CuO$-liquid). Further, the heat generation in $CuO$-liquid is less than the heat generation in $Al_2O_3$ near the vicinity of the wall. However, away from the wall, opposite trend is observed.

A significant rise in the temperature due to an increase in intensity is observed. Therefore, control of Joule heating in the design of thermal system is necessary. However, this dissipation of heat may be desirable in some biological fluid flows. Moreover, an increase in the intensity of the magnetic field causes an increase in the entropy generation. The positive buoyancy force enhances the entropy generation. However, opposing buoyancy force reduces energy losses. Energy losses in steady flow are high as compare to the unsteady flow. The key observations are listed below:

- The buoyant force is responsible for the influence of thermal radiations on the flow of nanofluid. It is observed that if buoyant force is not considered, then there is no effect of thermal radiations on the flow and hence momentum boundary layer thickness. As the buoyant force is significant in vertical flows, therefore, it is recommended that horizontal arrangement of physical model (sheet) should be taken if no impact of thermal radiations on the flow of nanofluid is desired.
- The magnetic field decelerates the fluid motion due to hindrance caused by the Lorentz force. Therefore, it is recommended to apply external magnetic field perpendicular to the plane of sheet if momentum boundary layer thickness is to be controlled.
Convectively heated surface causes more the entropy generation. Therefore, it is recommended not to use the convectively heated surface.

Imposition of external magnetic field increases the entropy generation and is responsible of great energy loses. Therefore, thermal systems work efficiently without loses of energy if external magnetic field is not imposed.

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7) Masoumi, N. Sohrabi, N. and Behzadmehr, A. “A new model for calculating the effective


16) Xuan, Y. Li, Q. and Hu, W. “Aggregation structure and thermal conductivity of nanofluids”, J. of American Institute of Chemical Engineers (AIChE), 49(4), pp. 1038-1043 (2003).


36) Sheikholeslami, M. Shehzad, S. A. Abbasi, F. M. and Li, Z. “Nanofluid flow and forced convection heat transfer due to Lorentz forces in a porous lid driven cubic enclosure with hot


52) Ghadikolaei, S. S. Hosseinzadeh, Kh. and Ganji, D. D. “Investigation on three dimensional
squeezing flow of mixture base fluid (ethylene glycol-water) suspended by hybrid nanoparticle
(Fe₃O₄-Ag) dependent on shape factor”, Journal of Molecular Liquids, (2018).


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Fig. 1. Physical model and coordinates system.

Fig. 2. Flow behavior for verses of $M$ when $Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, B_i = 0.1, \varphi = 0.04, \beta = 0.9$ and $\lambda = 0.6$.

Fig. 3. Flow behavior for verses of $Gr$ when $Pr = 2.73, Ec = 0.1, N_r = 0.2, M = 0.8, B_i = 0.1, N_r = 0.2, \varphi = 0.04, \beta_i = 0.9$ and $\lambda = 0.6$.

Fig. 4. Unsteadiness behavior for fluid when $Pr = 2.73, Ec = 0.1, N_r = 0.2, M = 0.8, B_i = 0.1, N_r = 0.2, \varphi = 0.04, \beta = 0.9$ and $Gr = 4$.

Fig. 5. Temperature curves verses $M$ when $Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, \varphi = 0.04, B_i = 0.1, \beta = 0.9$ and $\lambda = 0.6$.

Fig. 6. Temperature curves verses $Ec$ when $Gr = 4, Pr = 2.73, N_r = 0.2, B_i = 0.1, M = 0.8, \varphi = 0.04, \beta = 0.9$ and $\lambda = 0.6$.

Fig. 7. Temperature curves verses $\lambda$ when $Gr = 4, \beta = 0.9, Pr = 2.73, Ec = 0.1, N_r = 0.2, \varphi = 0.04, M = 0.8$ and $B_i = 0.1$.

Fig. 8. Temperature curves verses $\varphi$ when $Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, B_i = 0.1, M = 0.8, \beta = 0.9$ and $\lambda = 0.6$. 
Fig. 9. Temperature curves verses $\beta$ when $Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, B_i = 0.1, \varphi = 0.04, M = 0.8$ and $\lambda = 0.6$.

Fig. 10. Temperature curves verses $B_i$ when $Gr = 4, Pr = 2.73, Ec = 0.1, N_r = 0.2, \varphi = 0.04, M = 0.8$ and $\lambda = 0.6$.

Fig. 11. Temperature curves verses $N_r$ when $Gr = 4, Pr = 2.73, Ec = 0.1, B_i = 0.1, \beta = 0.9, \varphi = 0.04, M = 0.8$ and $\lambda = 0.6$.

Fig. 12. The effect of $M$ on the entropy generation when $Gr = 4, Pr = 2.73, Ec = 0.1, B_i = 0.1, \beta = 0.9, N_r = 0.2, \lambda = 0.9, Re = 1, Br = 1$ and $\Omega = 1$.

Fig. 13. The effect of $\Omega$ on the entropy generation when $Gr = 4, Pr = 2.73, N_r = 0.2, B_i = 0.1, \beta = 0.9, M = 0.8, B_i = 0.1, \lambda = 0.9, Re = 1, Br = 1$ and $Ec = 0.1$.

Fig. 14. The effect of $Re$ on the entropy generation when $Gr = 4, Pr = 2.73, Ec = 0.1, B_i = 0.1, \beta = 0.9, M = 0.8, N_r = 0.2, \lambda = 0.9, Br = 1$ and $\Omega = 1$.

Fig. 15. The effect of $Ec$ on the entropy generation when $Gr = 4, Pr = 2.73, N_r = 0.2, B_i = 0.1, \beta = 0.9, M = 0.8, B_i = 0.1, \lambda = 0.9, Re = 1, Br = 1$ and $\Omega = 1$. 
Fig. 1

\[ U_w = \frac{ax}{1 - ct} \]

\[ T_w = T_\infty + \frac{ax}{1 - ct} \]

Fig. 2

Graph showing the function \( f(\eta) \) with different curves for various values of \( M \). The x-axis is labeled \( \eta \) and the y-axis is labeled \( f(\eta) \). The curves are labeled for \( M = 0.5, 1.0, 2.4, 4.0 \).
Fig. 3

Fig. 4
Fig. 11

Fig. 12
**Fig. 13**

**Fig. 14**
Fig. 15
Table 1. Thermo-physical properties of water and nanoparticles used by [25, 28].

Table 2. The values of coefficients for CuO and Al₂O₃ nanofluids [25, 28].

Table 3. Validation of present results by comparing with the results published by Albashari et al. [64] and Das et al. [63] of \(-\theta'(0)\) for different Pr when \(M^2 = \lambda = \varphi = k_{brownian} = 0\), and \(B_i \to \infty\).

<table>
<thead>
<tr>
<th>Physical property</th>
<th>water/base fluid</th>
<th>CuO</th>
<th>Al₂O₃</th>
</tr>
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<tbody>
<tr>
<td>(\rho/(Kg.m^{-3}))</td>
<td>997.1</td>
<td>8933</td>
<td>3970</td>
</tr>
<tr>
<td>(c_p/(J\cdot kg^{-1} \cdot K^{-1}))</td>
<td>4179</td>
<td>385</td>
<td>765</td>
</tr>
<tr>
<td>(k/(W\cdot m^{-1} \cdot K^{-1}))</td>
<td>0.613</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>(d_p (nm))</td>
<td>-</td>
<td>47</td>
<td>29</td>
</tr>
<tr>
<td>(\sigma/(s.m^{-2}))</td>
<td>5.5\times10^{-6}</td>
<td>1\times10^{-10}</td>
<td>35\times10^{-6}</td>
</tr>
</tbody>
</table>

Table 1.
<table>
<thead>
<tr>
<th>Coefficient Values</th>
<th>$CuO – water$</th>
<th>$Al_2O_3 – water$</th>
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<tr>
<td>$a_1$</td>
<td>-26.593310846</td>
<td>52.813488759</td>
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<tr>
<td>$a_2$</td>
<td>-0.403818333</td>
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<td>$a_4$</td>
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<tr>
<td>$a_5$</td>
<td>6.42185846658×10^{-2}</td>
<td>0.176919300241</td>
</tr>
<tr>
<td>$a_6$</td>
<td>48.40336955</td>
<td>-298.19819084</td>
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<tr>
<td>$a_7$</td>
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<td>$a_9$</td>
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</tr>
<tr>
<td>$a_{10}$</td>
<td>-0.7200</td>
<td>-0.999063481</td>
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Table 2.

<table>
<thead>
<tr>
<th>Pr</th>
<th>Abolbashari et al. [64]</th>
<th>Das et al. [63]</th>
<th>Present study</th>
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<tr>
<td>0.72</td>
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<td>0.80876122</td>
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<td>10.0</td>
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<td>3.72055436</td>
<td>3.61534147</td>
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Table 3.