Septic B-spline method for solving nonlinear singular boundary value problems arising in physiological models

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Abstract

In this paper, we present a numerical collocation method for nonlinear singular two-point boundary value problems of second-order based on septic B-spline function, which depend on different physiological processes as steady-state oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis–Menten and heat sources distribution in the human head. The proposed method has uniform convergence for the exact solution. We display some physiological models which appear that our method is very effective and the maximum absolute errors and absolute residual errors are acceptable.


Keywords: Septic B-spline collocation method; Singular boundary value problems; Absolute residual errors; Error estimation; Uniform convergence.

1. Introduction

We examine the nonlinear singular boundary value problems of the form

$$z''(t) + \frac{k}{t} z(t) = g(t, z(t)), \quad t \in [0, 1],$$

(1)

under the boundary conditions

$$z'(0) = 0,$$

(2)

and
\[ \alpha z(1) + \beta z'(1) = \gamma, \]  

(3)

where \( \alpha > 0, \beta \geq 0, k \geq 1, \gamma \in \mathbb{R} \) and \( g(t, z(t)) \) is nonlinear continuous function has partial derivative with respect to \( z \), where \( \frac{\partial g}{\partial z} \geq 0 \), and continuous on the domain \([0,1]\). In this paper, we introduce three models arising in physiology. The solutions of Eqs. (1)-(3) have existence and uniqueness in \([1-3]\). For \( k = 1 \) and \( g(t, z) = le^{z(t)} \), where \( l \) is a physical parameter, the first model arises in electro-hydrodynamics \([4]\) and the above Eq. (1) under boundary conditions in Eqs. (2) and (3) with \( \alpha = 1, \beta = 0, \gamma = 0 \), arising in thermal explosions \([5-8]\). For different values of \( k = 0, 1, 2, 3 \) and \( g(t, z) = \frac{m z(t)}{z(t) + c} \), where \( m > 0, c > 0 \) in Eq. (1), the second model describes the steady state oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis–Menten \([9-12]\). The existence and uniqueness of the second model is proved in \([13]\). Many authors studied the oxygen diffusion problem when \( m = 0.76129, c = 0.03119 \) and subjected to the boundary conditions in Eqs. (2) and (3) with \( \alpha = \gamma = 5, \beta = 1 \) in \([4-7]\). The last model studies heat sources distribution in the human head in \([14-16]\), at \( k = 2 \) and \( g(t, z) = -d e^{-p z(t)} \), where \( d > 0, p > 0 \) in Eq. (1) with boundary conditions in Eqs. (2) and (3) such that \( \alpha = \gamma = 0 \), while \( \beta \) takes three different cases as \([4-6]\) and \([8,17]\). There are many authors introduced methods for solving a class of linear and nonlinear singular boundary value problems arising in applied science and engineering \([18-25]\). Also, many applications of ordinary and partial differential equations arising in mechanical engineering, computer science and electrical engineering are investigated in \([26-33]\).

Eq. (1) has singular point at \( t = 0 \), so we first use L’Hopital’s rule to modify Eq. (1) at \( t = 0 \), then the boundary value problems (1)-(3) convert to

\[
\begin{align*}
z''(t) + r(t)z'(t) &= q(t, z(t)), \\
\alpha z(1) + \beta z'(1) &= \gamma,
\end{align*}
\]

(4)

where

\[
r(t) = \begin{cases} 
0, & t = 0, \\
k, & t \neq 0,
\end{cases} \quad q(t, z(t)) = \begin{cases} 
g(t, z(t)), & t = 0, \\
k + 1, & t \neq 0.
\end{cases}
\]

This paper is organized as follows: In section 2, we analyze the proposed collocation method. In section 3, we derive the uniform convergence of the septic B-spline collocation method. In
section 4, we discuss some numerical models. In section 5, we introduce the conclusion of our results.

2. Septic B-spline collocation method

Let $[0,1]$ be the domain of the proposed problem which it is divided into $n$ subintervals $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$ with equal step size $h = \frac{1}{n}$ by the knots $t_j = jh, \{j = 0, 1, \ldots, n\}$, where $0 = t_0 < t_1 < \ldots < t_n = 1$. We find additional knots $t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}$ and $t_{n+3}$ outside the domain. We introduce the septic B-spline function $B_j(t)$ to find a numerical solution of the nonlinear singular boundary value problem (4) as follows:

$$B_j(t) = \frac{1}{h} \begin{cases} 
(t-t_{j-4})^7, & t \in [t_{j-4}, t_{j-3}] \\
(t-t_{j-4})^7 - 8(t-t_{j-3})^7, & t \in [t_{j-3}, t_{j-2}] \\
(t-t_{j-4})^7 - 8(t-t_{j-3})^7 + 28(t-t_{j-2})^7, & t \in [t_{j-2}, t_{j-1}] \\
(t-t_{j-4})^7 - 8(t-t_{j-3})^7 + 28(t-t_{j-2})^7 - 56(t-t_{j-1})^7, & t \in [t_{j-1}, t_j] \\
(t_{j+4} - t)^7 - 8(t_{j+3} - t)^7 + 28(t_{j+2} - t)^7 - 56(t_{j+1} - t)^7, & t \in [t_j, t_{j+1}] \\
(t_{j+4} - t)^7 - 8(t_{j+3} - t)^7 + 28(t_{j+2} - t)^7, & t \in [t_{j+1}, t_{j+2}] \\
(t_{j+4} - t)^7 - 8(t_{j+3} - t)^7, & t \in [t_{j+2}, t_{j+3}] \\
(t_{j+4} - t)^7, & t \in [t_{j+3}, t_{j+4}] \\
0, & \text{otherwise.}
\end{cases}$$

(j = -3, -2, -1, 0, \ldots, N + 2, N + 3), \quad (5)

consider $s(t)$ be the septic B-spline approximate solution to the exact solution $z(t)$ of the problem (4), given by

$$s(t) = \sum_{j=3}^{N+3} \delta_j B_j(t), \quad (6)$$

where $\delta_j$’s are constants found it from the collocation points $t_j, j = 0, 1, \ldots, n$ and the boundary conditions. The septic B-spline function $B_j(t)$ and it is six derivatives $B'_j(t), B''_j(t), B'''_j(t), B^{(4)}_j(t), B^{(5)}_j(t)$ and $B^{(6)}_j(t)$ values at the knots are summarized in Table 1.
From Eqs. (5) and (6), the values of \( s(t_j) \) at the nodal points and their six derivatives are

\[
s(t_j) = \delta_{j-3} + 120\delta_{j-2} + 1191\delta_{j-1} + 2416\delta_j + 1191\delta_{j+1} + 120\delta_{j+2} + \delta_{j+3},
\]

\[
s'(t_j) = \frac{1}{h}(-7\delta_{j-3} - 392\delta_{j-2} - 1715\delta_{j-1} + 1715\delta_{j+1} + 392\delta_{j+2} + 7\delta_{j+3}),
\]

\[
s''(t_j) = \frac{1}{h^2}(42\delta_{j-3} + 1008\delta_{j-2} + 630\delta_{j-1} - 3360\delta_j + 630\delta_{j+1} + 1008\delta_{j+2} + 42\delta_{j+3}),
\]

\[
s'''(t_j) = \frac{1}{h^3}(-210\delta_{j-3} - 1680\delta_{j-2} + 3990\delta_{j-1} - 3990\delta_{j+1} + 1680\delta_{j+2} + 210\delta_{j+3}),
\]

\[
s^{(4)}(t_j) = \frac{1}{h^4}(840\delta_{j-3} - 7560\delta_{j-1} + 13440\delta_j - 7560\delta_{j+1} + 840\delta_{j+3}),
\]

\[
s^{(5)}(t_j) = \frac{1}{h^5}(-2520\delta_{j-3} + 10080\delta_{j-2} - 12600\delta_{j-1} + 12600\delta_{j+1} - 10080\delta_{j+2} + 2520\delta_{j+3}),
\]

\[
s^{(6)}(t_j) = \frac{1}{h^6}(5040\delta_{j-3} - 30240\delta_{j-2} + 75600\delta_{j-1} - 100800\delta_j + 75600\delta_{j+1} - 30240\delta_{j+2} + 5040\delta_{j+3}),
\]

\[, j = 0, 1, ..., n. \tag{7}\]

Replacing from Eq. (7) in Eq. (4), we find

\[
\frac{1}{h^2}(42\delta_{j-3} + 1008\delta_{j-2} + 630\delta_{j-1} - 3360\delta_j + 630\delta_{j+1} + 1008\delta_{j+2} + 42\delta_{j+3}) = \frac{g(t_j, z(t_j))}{k + 1}, j = 0, \tag{8a}
\]

\[
\frac{1}{h^2}(42\delta_{j-3} + 1008\delta_{j-2} + 630\delta_{j-1} - 3360\delta_j + 630\delta_{j+1} + 1008\delta_{j+2} + 42\delta_{j+3}) + \frac{k}{h\delta_{j}}(-7\delta_{j-3} - 392\delta_{j-2} - 1715\delta_{j-1} + 1715\delta_{j+1} + 392\delta_{j+2} + 7\delta_{j+3}) = g(t_j, z(t_j)), j = 1, 2, ..., n. \tag{8b}
\]

After simplifying the two above equations, we get

\[
42\delta_{j-3} + 1008\delta_{j-2} + 630\delta_{j-1} - 3360\delta_j + 630\delta_{j+1} + 1008\delta_{j+2} + 42\delta_{j+3} = h^2 g_0, \tag{9a}
\]

\[
\xi_1(t_j)\delta_{j-3} + \xi_2(t_j)\delta_{j-2} + \xi_3(t_j)\delta_{j-1} + \xi_4(t_j)\delta_j + \xi_5(t_j)\delta_{j+1} + \xi_6(t_j)\delta_{j+2} + \xi_7(t_j)\delta_{j+3} = h^2 t_j g_j, j = 1, 2, ..., n, \tag{9b}
\]

where
\[ g_0 = \frac{g(t_0, z(t_0))}{k + 1}, \]

\[ \xi_1(t_j) = 42t_j - 7hk, \quad \xi_2(t_j) = 1008t_j - 392hk, \quad \xi_3(t_j) = 630t_j - 1715hk, \]

\[ \xi_4(t_j) = -3360t_j, \quad \xi_5(t_j) = 630t_j + 1715hk, \quad \xi_6(t_j) = 1008t_j + 392hk, \]

\[ \xi_7(t_j) = 42t_j + 7hk, \quad g_j = g(t_j, z(t_j)), \quad j = 1, 2, ..., n, \]

and the boundary conditions as follows:

\[ -7\delta_{-3} - 392\delta_{-2} - 1715\delta_{-1} + 1715\delta_{1} + 392\delta_{2} + 7\delta_{3} = 0, \]  

(10)

and

\[ \lambda_{n+3} \delta_{n+3} + \lambda_{n+2} \delta_{n+2} + \lambda_{n+1} \delta_{n+1} + \lambda_{n} \delta_{n} + \lambda_{n-1} \delta_{n-1} + \lambda_{n-2} \delta_{n-2} + \lambda_{n-3} \delta_{n-3} = h\gamma, \]  

(11)

where

\[ \lambda_{1}(t_n) = h\alpha - 7\beta, \quad \lambda_{2}(t_n) = 120h\alpha - 392\beta, \quad \lambda_{3}(t_n) = 1191h\alpha - 1715\beta, \quad \lambda_{4}(t_n) = 2416h\alpha, \]

\[ \lambda_{5}(t_n) = 1191h\alpha + 1715\beta, \quad \lambda_{6}(t_n) = 120h\alpha + 392\beta, \quad \lambda_{7}(t_n) = h\alpha + 7\beta. \]

However, still four equations are required. By differentiating Eq. (4) with respect to \( t \) fifth times, we get

\[ z^{(5)}(t) + r(t)z^{(4)}(t) + 3r'(t)z'''(t) + 3r''(t)z''(t) + r'''(t)z'(t) = q_1(t, z(t)), \]  

(12)

where

\[ r'(t) = \begin{cases} 0, & t = 0, \\ -\frac{k}{t^2}, & t \neq 0, \end{cases} \quad r''(t) = \begin{cases} 0, & t = 0, \\ \frac{2k}{t^3}, & t \neq 0, \end{cases} \quad r'''(t) = \begin{cases} 0, & t = 0, \\ -\frac{6k}{t^4}, & t \neq 0, \end{cases} \]

and

\[ q_1(t, z(t)) = \frac{dg(t, z(t))}{dz}z'''(t) + 3\frac{d^2g(t, z(t))}{dz^2}z''(t)z''(t) + \frac{d^3g(t, z(t))}{dz^3}(z'(t))^3, \quad t \neq 0. \]

Putting \( t = t_0 \) in Eq. (12) and using Eq. (7), we have

\[ -2520\delta_{-3} + 10080\delta_{-2} - 12600\delta_{-1} + 12600\delta_{1} - 10080\delta_{2} + 2520\delta_{3} = 0, \]  

(13)

Similarly, when \( t = t_n \), we get

\[ \mu_{1}(t_n)\delta_{n+3} + \mu_{2}(t_n)\delta_{n+2} + \mu_{3}(t_n)\delta_{n+1} + \mu_{4}(t_n)\delta_{n} + \mu_{5}(t_n)\delta_{n+1} + \mu_{6}(t_n)\delta_{n+2} + \mu_{7}(t_n)\delta_{n+3} = h^5\psi_n, \]  

(14)
\[
\mu_1(t_n) = -2520 + 840hk + 630h^2k + 252h^3k + 42h^4k, \\
\mu_2(t_n) = 10080 + 5040h^2k + 6048h^3k + 2352h^4k, \\
\mu_3(t_n) = -12600 - 7560hk - 11970h^2k + 3780h^3k + 10290h^4k, \\
\mu_4(t_n) = 13440hk - 20160h^3k, \\
\mu_5(t_n) = 12600 - 7560hk + 11970h^2k + 3780h^3k - 10290h^4k, \\
\mu_6(t_n) = -10080 - 5040h^2k + 6048h^3k - 2352h^4k, \\
\mu_7(t_n) = 2520 + 840hk - 630h^2k + 252h^3k - 42h^4k, \\
\psi_n = \frac{dg(t, z(t))}{dz}z^n(t) + 3 \frac{d^2g(t, z(t))}{dz^2}z'(t)z^n(t) + \frac{d^3g(t, z(t))}{dz^3}(z'(t))^3 \bigg|_{z=z_0}.
\]

Differentiating Eq. (12) with respect to \( t \) again, we get
\[
z^{(6)}(t) + r(t)z^{(5)}(t) + 2r'(t)z^{(4)}(t) + 6r''(t)z^{(3)}(t) + 4r'''(t)z^{(2)}(t) + r^{(4)}(t)z'(t) = q_z(t, z(t)),
\]
where
\[
r^{(4)}(t) = \begin{cases} 
0, & t = 0, \\
\frac{24k}{t^5}, & t \neq 0,
\end{cases}
\]
and
\[
q_z(t, z(t)) = \begin{cases} 
\frac{dg(t, z(t))z^{(4)}(t)}{dz} + 3 \frac{d^2g(t, z(t))}{dz^2}(z'(t))^2, & t = 0, \\
\frac{1+k}{5} \left( \frac{dg(t, z(t))}{dz}z^{(4)}(t) + \frac{d^2g(t, z(t))}{dz^2}(4z'(t)z^{(3)}(t) + 3(z'(t))^2) + 6 \frac{d^3g(t, z(t))}{dz^3}(z'(t))^3 \right), & t \neq 0,
\end{cases}
\]
where
\[
z^{(4)}(t)_{t=0} = \frac{g(t, z(t))}{1+k} \Bigg|_{z=z_0} \text{ and } z^{(4)}(t)_{t=0} = \frac{dg(t, z(t))}{dz}z^n(t)_{t=0}.
\]
Putting \( t = t_0 \) in Eq. (15) and using Eq. (7), we get

\[
5040\delta_{-3} - 30240\delta_{-2} + 75600\delta_{-1} - 100800\delta_0 + 75600\delta_1 + 30240\delta_2 + 5040\delta_3 = 5\sigma_0,
\]

where

\[
\sigma_0 = \left. \frac{d^2 g(t, z(t))}{dz^2} \right|_{z=z_0} (z'(t))^2 + \left. \frac{d^2 g(t, z(t))}{dz^2} \right|_{z=z_0} (z''(t))^2 + \left. \frac{d^4 g(t, z(t))}{dz^4} \right|_{z=z_0} (z''(t))^4.
\]

Similarly, when \( t = t_n \) we have

\[
\eta_1(t_n)\delta_{n-3} + \eta_2(t_n)\delta_{n-2} + \eta_3(t_n)\delta_{n-1} + \eta_4(t_n)\delta_n + \eta_5(t_n)\delta_{n+1} + \eta_6(t_n)\delta_{n+2} + \eta_7(t_n)\delta_{n+3} = 5\sigma_n,
\]

where

\[
\eta_1(t_n) = 5040 - 2520h_k - 53760h^2k - 2520h^3k - 1008h^4k - 168h^5k,
\]

\[
\eta_2(t_n) = -30240 + 10080h_k - 20160h^2k - 24192h^3k - 9408h^4k,
\]

\[
\eta_3(t_n) = 75600 - 12600h_k + 30240h^2k + 47880h^3k - 15120h^4k - 41160h^5k,
\]

\[
\eta_4(t_n) = -100800 + 35760h^2k + 80640h^4k,
\]

\[
\eta_5(t_n) = 75600 + 12600h_k + 30240h^2k - 47880h^3k - 15120h^4k + 41160h^5k,
\]

\[
\eta_6(t_n) = -30240 - 10080h_k + 20160h^2k + 24192h^3k + 9408h^5k,
\]

\[
\eta_7(t_n) = 5040 + 2520h_k - 35760h^2k + 2520h^3k - 1008h^4k + 168h^5k,
\]

Then from Eqs. (9a), (9b), (10), (11), (13), (14), (16) and (17), we get a matrix in the form;

\[
AT = B,
\]

where \( A \) is non-singular square matrix \((n+7) \times (n+7)\), \( T \) is dimensional vector \((n+7)\) with components \( \delta_j \) and the right hand side \( B \) is an \((n+7)\) dimensional vector as shown:
\[
A = \begin{pmatrix}
-7 & -392 & -1715 & 0 & 1715 & 392 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2520 & 10080 & -12600 & 0 & 12600 & -10080 & 2520 & 0 & 0 & 0 & 0 & 0 & 0 \\
5040 & -30240 & 75600 & -100800 & 75600 & -30240 & 5040 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & 1008 & 630 & -3360 & 630 & 1008 & 42 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & \xi(t_1) & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & \xi(t_{n-1}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) & \eta(t_{n-1}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu(t_{n-1}) & \mu(t_{n-1}) & \mu(t_{n-1}) & \mu(t_{n-1}) & \mu(t_{n-1}) & \mu(t_{n-1}) & \mu(t_{n-1}) \end{pmatrix}
\]

\[
T = \begin{pmatrix}
\delta_{-3} & \delta_{-2} & \delta_{-1} & \delta_0 & \delta_1 & \ldots & \delta_{n-1} & \delta_n & \delta_{n+1} & \delta_{n+2} & \delta_{n+3}
\end{pmatrix}^T,
\]

and

\[
B = \begin{pmatrix}
0 & 0 & \sigma_0 & g_0 & g_1 & \ldots & g_{n-1} & g_n & \sigma_n & \psi_n & \gamma
\end{pmatrix}^T,
\]

3. Uniform convergence

We estimate the truncation error for the septic B-spline collocation method in the interval \([a,b]\).

Let \(z(t)\) has continuous derivatives for all \(t \in [a,b]\). Using Eq. (7), we get

\[
s'(t_{j-3}) + 120 s'(t_{j-2}) + 1191 s'(t_{j-1}) + 2416 s'(t_j) + 1191 s'(t_{j+1}) + 120 s'(t_{j+2}) + s'(t_{j+3})
= \frac{1}{h} \left( -7 z(t_{j-3}) - 392 z(t_{j-2}) - 1715 z(t_{j-1}) + 1715 z(t_{j+1}) + 392 z(t_{j+2}) + 7 s'(t_{j+3}) \right),
\]

(19)

\[
s''(t_{j-3}) + 120 s''(t_{j-2}) + 1191 s''(t_{j-1}) + 2416 s''(t_j) + 1191 s''(t_{j+1}) + 120 s''(t_{j+2}) + s''(t_{j+3})
= \frac{1}{h^2} \left( 42 z(t_{j-3}) + 1008 z(t_{j-2}) + 630 z(t_{j-1}) - 3360 z(t_j) + 630 z(t_{j+1}) \right),
\]

(20)

\[
s'''(t_{j-3}) + 120 s'''(t_{j-2}) + 1191 s'''(t_{j-1}) + 2416 s'''(t_j) + 1191 s'''(t_{j+1}) + 120 s'''(t_{j+2}) + s'''(t_{j+3})
= \frac{1}{h^3} \left( -210 z(t_{j-3}) - 1680 z(t_{j-2}) + 3990 z(t_{j-1}) - 3990 z(t_j) \right),
\]

(21)

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\[
s^{(4)}(t_{j-3}) + 120 s^{(4)}(t_{j-2}) + 1191 s^{(4)}(t_{j-1}) + 2416 s^{(4)}(t_{j}) + 1191 s^{(4)}(t_{j+1}) + 120 s^{(4)}(t_{j+2}) + s^{(4)}(t_{j+3})
= \frac{1}{\hbar^4} \left( 840 z(t_{j-3}) - 7560 z(t_{j-1}) + 13440 z(t_{j}) - 7560 z(t_{j+1}) + 840 z(t_{j+3}) \right),
\]

(22)

\[
s^{(5)}(t_{j-3}) + 120 s^{(5)}(t_{j-2}) + 1191 s^{(5)}(t_{j-1}) + 2416 s^{(5)}(t_{j}) + 1191 s^{(5)}(t_{j+1}) + 120 s^{(5)}(t_{j+2}) + s^{(5)}(t_{j+3})
= \frac{1}{\hbar^5} \left( -2520 z(t_{j-3}) + 10080 z(t_{j-2}) - 12600 z(t_{j-1}) + 12600 z(t_{j+1}) \right),
\]

(23)

\[
s^{(6)}(t_{j-3}) + 120 s^{(6)}(t_{j-2}) + 1191 s^{(6)}(t_{j-1}) + 2416 s^{(6)}(t_{j}) + 1191 s^{(6)}(t_{j+1}) + 120 s^{(6)}(t_{j+2}) + s^{(6)}(t_{j+3})
= \frac{1}{\hbar^6} \left( 5040 z(t_{j-3}) - 30240 z(t_{j-2}) + 75600 z(t_{j-1}) - 100800 z(t_{j}) + 75600 z(t_{j+1}) \right),
\]

(24)

Using the operator notation \(E(z(t_j)) = z(t_{j+1})\) [34], then Eqs. (19) - (24) become as follows:

\[
s'(t_j) = \frac{1}{\hbar} \left( -7 E^{-3} - 392 E^{-2} - 1715 E^{-1} + 1715 E + 392 E^2 + 7 E^3 \right) z(t_j),
\]

(25)

\[
s''(t_j) = \frac{1}{\hbar^2} \left( 42 E^{-3} + 1008 E^{-2} + 630 E^{-1} - 3360 + 630 E + 1008 E^2 + 42 E^3 \right) z(t_j),
\]

(26)

\[
s'''(t_j) = \frac{1}{\hbar^3} \left( -210 E^{-3} - 1680 E^{-2} + 3990 E^{-1} - 3990 E + 1680 E^2 + 210 E^3 \right) z(t_j),
\]

(27)

\[
s^{(4)}(t_j) = \frac{1}{\hbar^4} \left( 840 E^{-3} - 7560 E^{-1} + 13440 - 7560 E + 840 E^3 \right) z(t_j),
\]

(28)

\[
s^{(5)}(t_j) = \frac{1}{\hbar^5} \left( -2520 E^{-3} + 10080 E^{-2} - 12600 E^{-1} + 12600 E - 10080 E^2 + 2520 E^3 \right) z(t_j),
\]

(29)

\[
s^{(6)}(t_j) = \frac{1}{\hbar^6} \left( 5040 E^{-3} - 30240 E^{-2} + 75600 E^{-1} - 100800 + 75600 E - 30240 E^2 + 5040 E^3 \right) z(t_j)
\]

(30)

In Eqs. (25)- (30), we put the operator notation \(E = e^{\hbar D}\) as expansion in powers of \(\hbar D\), we have
\[ s'(t_j) = z'(t_j) - \frac{h^8}{151200} z^{(8)}(t_j) + \frac{h^{10}}{399168} z^{(10)}(t_j) + O(h^{11}) \]  
\[ s''(t_j) = z''(t_j) - \frac{h^6}{30240} z^{(6)}(t_j) + \frac{h^8}{60480} z^{(8)}(t_j) - \frac{h^{10}}{532224} z^{(10)}(t_j) + O(h^{11}) \]  
\[ s'''(t_j) = z'''(t_j) + \frac{h^6}{6048} z^{(9)}(t_j) - \frac{h^8}{33600} z^{(10)}(t_j) + \frac{h^{10}}{295680} z^{(13)}(t_j) + O(h^{11}) \]  
\[ s^{(4)}(t_j) = z^{(4)}(t_j) + \frac{h^6}{720} z^{(8)}(t_j) - \frac{h^6}{3024} z^{(10)}(t_j) + \frac{17h^8}{604800} z^{(12)}(t_j) \]  
\[ - \frac{h^{10}}{997920} z^{(14)}(t_j) + O(h^{11}) \]  
\[ s^{(5)}(t_j) = z^{(5)}(t_j) - \frac{h^4}{240} z^{(9)}(t_j) + \frac{h^6}{3024} z^{(10)}(t_j) - \frac{11h^8}{604800} z^{(13)}(t_j) \]  
\[ + \frac{13h^{10}}{665280} z^{(15)}(t_j) + O(h^{11}) \]  
\[ s^{(6)}(t_j) = z^{(6)}(t_j) - \frac{h^2}{12} z^{(8)}(t_j) + \frac{h^4}{240} z^{(10)}(t_j) - \frac{h^6}{6048} z^{(12)}(t_j) + \frac{h^8}{210600} z^{(14)}(t_j) \]  
\[ + \frac{101h^{10}}{159667200} z^{(16)}(t_j) + O(h^{11}) \]  

We define the truncation error \( e(t) = z(t) - s(t) \) and replacing Eqs. (31)-(36) in the Taylor series expansion of \( e(t_j + \omega h) \), we find

\[ e(t_j + \omega h) = \frac{\omega^2 (2 - 7\omega^2 + 14\omega^4)}{120960} h^8 z^{(8)}(t_j) + \frac{\omega (12 - 50\omega^2 + 63\omega^4)}{1814400} h^9 z^{(9)}(t_j) \]  
\[ - \frac{\omega^2 (30 - 50\omega^2 + 21\omega^4)}{3628800} h^{10} z^{(10)}(t_j) + O(h^{11}) \]  

where \( 0 \leq \omega \leq 1 \).

**Theorem 7.1.** Let nonlinear singular two-point boundary value problems of second-order have the form (1)-(3) with the exact solution \( z(t) \) and the approximate solution \( s(t) \), then the septic
B-spline collocation method has a truncation error of $O(h^8)$ and the convergence of this method is $O(h^6)$ for sufficiently small $h$.

4. Numerical Problems and discussions

We introduce some physiological applications related to nonlinear singular two-point boundary value problems of second order. We generate the results with Mathematica using FindRoot function. We calculate the absolute value of the difference between the exact solution and the numerical solution and find absolute residual errors of the problems that have not the exact solution.

**Problem 1.** We consider nonlinear singular boundary value problem [5]

\[
\begin{align*}
z''(t) + \frac{1}{t} z'(t) &= -e^{z(t)}, \\
z'(0) &= 0, \quad z(1) = 0,
\end{align*}
\]

where the exact solution of the model (38) take the form $z(t) = 2\log\left(\frac{A+1}{At^2 + 1}\right)$, where $A = 3 - 2\sqrt{2}$. Table 2 show that the septic B-spline collocation method has the maximum absolute errors are acceptable comparing with other methods [5-8].

**Problem 2.** We present another nonlinear singular boundary value problem [5]

\[
\begin{align*}
\frac{2}{t} z'(t) &= \frac{0.76129 z(t)}{z(t) + 0.03119}, \\
z'(0) &= 0, \quad 5z(1) + z'(1) = 5,
\end{align*}
\]

Table 3 and Fig. 1 show the numerical solutions of the present method and it is comparison with other methods [6, 7], while Fig. 2 appears the absolute residual errors $R_n(t) = \left|z''(t) + \frac{2}{t} z'(t) - \frac{0.76129 z(t)}{z(t) + 0.03119}\right|, 0 < t \leq 1$, at $n = 60$ of the model (39) doesn’t have any exact solution.
**Problem 3.** We investigate nonlinear singular boundary value problem [5]

\[ z'(t) + \frac{2}{t} z(t) = -e^{-e^{z(t)}}, \]  

subject to boundary conditions in three cases:

(i) \( z'(0) = 0, \quad 0.1z(1) + z'(1) = 0, \)

(ii) \( z'(0) = 0, \quad z(1) + z'(1) = 0, \)

(iii) \( z'(0) = 0, \quad 2z(1) + z'(1) = 0, \)

Table 4 and Fig. 3 show the numerical solutions of the present method at different cases of \( \alpha \) and it is comparison with other methods [6, 17], while Table [5] shows the absolute residual errors \( R_n(t) = \left| z'(t) + \frac{2}{t} z(t) + e^{-e^{z(t)}} \right|, 0 < t \leq 1, \) at \( \alpha = 2, n = 10, \) and the numerical solutions at \( \alpha = 2, n = 40, \) of the model (40) doesn’t have any exact solution.

### 5. Conclusion

In this paper, we presented septic B-spline collocation method to find numerical solutions for the nonlinear singular two-point boundary value problems of second-order. At different values of \( n, \) the numerical results are shown that the proposed method has efficient solutions of the studied models. We investigated three applications which play a vital role in physiological models. We can see that the numerical accuracy in thermal explosions problem is better than results obtained by references [5-8] as in Table 2, while the problems of oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis–Menten and heat sources distribution in the human head give us numerical solutions are similar to results obtained by references [5-8] and [17] as in Figs. 1 and 3 and Tables 3 and 4. Meanwhile, these problems have not exact solutions, so we calculated the absolute residual errors as in Fig. 2 and Table 5.

### Acknowledgments

The authors are grateful and express sincere thanks to the reviewers for their detailed reading, precious proposals and comments.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.
References


**Table Captions**

**Table 1**: The nodal values of $B_j'(t), B_j''(t), B_j'''(t), B_j^{(4)}(t), B_j^{(5)}(t)$ and $B_j^{(6)}(t)$.

**Table 2**: Values of the maximum absolute errors of problem 1.
Table 3: Numerical results of problem 2.

Table 4: Numerical results of problem 3.

Table 5: Absolute residual errors and numerical results of problem 3.

Figure Captions

Fig. 1: Numerical solutions of problem 2 by the present method for \( n = 40 \).

Fig. 2: Absolute residual errors of problem 2 at \( n = 60 \).

Fig. 3: Numerical solutions of problem 3 at different values of \( \alpha \).

Table 1

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Fig. 1
Biographies

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