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A septic B-spline collocation method for solving nonlinear singular boundary value problems arising in physiological models

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Abstract. In this paper, we present a numerical collocation method for nonlinear singular two-point boundary value problems of second order based on septic B-spline function. This method depends on different physiological processes such as steady-state oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis-Menten and heat sources distribution in human head. The proposed method has uniform convergence for the exact solution. We will provide some physiological models proving that our method is very effective with acceptable maximum absolute errors and absolute residual errors.

1. Introduction

Consider the nonlinear singular boundary value problems of the following form:

$$z''(t) + \frac{k}{t} z'(t) = g(t, z(t)), \quad t \in [0,1], \tag{1}$$

under the boundary conditions:

$$z'(0) = 0, \tag{2}$$

and:

$$\alpha\, z(1) + \beta\, z'(1) = \gamma, \tag{3}$$

where $\alpha > 0, \beta \geq 0, k \geq 1, \gamma \in R$ and $g(t, z(t))$ is a nonlinear continuous function with the partial derivative of z; also, $\frac{\partial g}{\partial z} \geq 0$, and it is continuous in the domain of $[0,1]$. In this paper, we introduce three models arising in physiology. The solutions to Eqs. (1)–(3) are unique ones in [1–3]. For $k = 1$ and $g(t, z) = l\, e^{z(t)}$, where l is a physical parameter, the first model arises in electro-hydrodynamics [4] following Eq. (1) under boundary conditions in Eqs. (2) and (3) with $\alpha = 1$, $\beta = 0$, and $\gamma = 0$, in thermal explosions [5–8]. For different values of $k = 0, 1, 2, 3$ and $g(t, z) = \frac{m\, z(t)}{z(t)+c}$, where $m > 0$, and $c > 0$ in Eq. (1), the second model describes the steady-state oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis-Menten [9–12]. The existence and uniqueness of the second model has been proved in [13]. Many authors studied the oxygen diffusion problem with:

$$m = 0.76129, \quad \text{and} \quad c = 0.03119$$

subject to the boundary conditions in Eqs. (2) and (3) with $\alpha = \gamma = 5$, and $\beta = 1$ [4–7]. The last model studies heat sources distribution in human head [14–16] at $k = 2$ and $g(t, z) = -d\, e^{-p\, z(t)}$, where

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$d > 0$, and $p > 0$ in Eq. (1), under boundary conditions in Eqs. (2) and (3) with $\alpha = \gamma = 0$, while β takes three different values between [4–6] and [8,17]. Many authors have introduced methods for solving a class of linear and nonlinear singular boundary value problems arising in applied science and engineering [18–25]. Also, many applications of ordinary and partial differential equations arising in mechanical engineering, computer science, and electrical engineering have been investigated in [26–33].

Eq. (1) has a singular point at $t = 0$. Thus, we first use L'Hôpital's rule to modify Eq. (1) at $t = 0$ and then, convert the boundary value problems (1)–(3) to:

$$z''(t) + r(t)\, z'(t) = q(t, z(t)),$$

$$z'(0) = 0, \quad \alpha\, z(1) + \beta\, z'(1) = \gamma, \tag{4}$$

where:

$$r(t) = \begin{cases} 0, & t = 0, \\ \frac{k}{t}, & t \neq 0, \end{cases}$$

$$q(t, z(t)) = \begin{cases} \frac{g(t, z(t))}{k+1}, & t = 0, \\ g(t, z(t)), & t \neq 0. \end{cases}$$

This paper is organized as follows. In Section 2, we analyze the proposed collocation method. In Section 3, the uniform convergence of the septic B-spline collocation method is derived. Section 4 enters into a discussion of some numerical models. Finally, in Section 5, conclusions are drawn.

2. Septic B-spline collocation method

Let [0,1] be the domain of the proposed problem, which is divided into n subintervals $\lfloor t_j, t_{j+1} \rfloor$, $j = 0, 1, \ldots, n-1$ with equal step size of $h = \frac{1}{n}$ by the knots $t_j = jh$, $\{j = 0, 1, \ldots, n\}$, where $0 = t_0 < t_1 < \ldots < t_n = 1$. We find additional knots $t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}$, and t_{n+3} outside the domain. Then, the septic B-spline function, $B_j(t)$, is introduced to find a numerical solution to the nonlinear singular boundary value problem (4); Eq. (5) is shown in Box I. Suppose $s(t)$ is the septic B-spline approximate solution to the exact solution $z(t)$ to problem (4), given by:

$$s(t) = \sum_{j=-3}^{N+3} \delta_j B_j(t), \tag{6}$$

where δ_j's are constants found by the collocation points t_j, $j = 0, 1, ..., n$ and the boundary conditions. The septic B-spline function, $B_j(t)$, and its six derivatives $B_j'(t)$, $B_j''(t)$, $B_j'''(t)$, $B_j^{(4)}(t)$, $B_j^{(5)}(t)$, and $B_j^{(6)}(t)$ at the knots are summarized in Table 1.

Based on Eqs. (5) and (6), the values of $s(t_j)$ at the nodal points and their six derivatives are:

$$s(t_j) = \delta_{j-3} + 120\,\delta_{j-2} + 1191\,\delta_{j-1} + 2416\,\delta_j + 1191\,\delta_{j+1} + 120\,\delta_{j+2} + \delta_{j+3}$$

$$s'(t_j) = \frac{1}{h}\Big(-7\,\delta_{j-3} - 392\,\delta_{j-2} - 1715\,\delta_{j-1} + 1715\,\delta_{j+1} + 392\,\delta_{j+2} + 7\,\delta_{j+3}\Big)$$

$$s''(t_j) = \frac{1}{h^2}\Big(42\,\delta_{j-3} + 1008\,\delta_{j-2} + 630\,\delta_{j-1} - 3360\,\delta_j + 630\,\delta_{j+1} + 1008\,\delta_{j+2} + 42\,\delta_{j+3}\Big)$$

$$s'''(t_j) = \frac{1}{h^3}\Big(-210\,\delta_{j-3} - 1680\,\delta_{j-2} + 3990\,\delta_{j-1}$$

$$B_j(t) = \frac{1}{h^7}\begin{cases} (t - t_{j-4})^7 & t \in [t_{j-4}, t_{j-3}], \\ (t - t_{j-4})^7 - 8(t - t_{j-3})^7 & t \in [t_{j-3}, t_{j-2}], \\ (t - t_{j-4})^7 - 8(t - t_{j-3})^7 + 28(t - t_{j-2})^7 & t \in [t_{j-2}, t_{j-1}], \\ (t - t_{j-4})^7 - 8(t - t_{j-3})^7 + 28(t - t_{j-2})^7 - 56(t - t_{j-1})^7 & t \in [t_{j-1}, t_j], \\ (t_{j+4} - t)^7 - 8(t_{j+3} - t)^7 + 28(t_{j+2} - t)^7 - 56(t_{j+1} - t)^7 & t \in [t_j, t_{j+1}], \\ (t_{j+4} - t)^7 - 8(t_{j+3} - t)^7 + 28(t_{j+2} - t)^7 & t \in [t_{j+1}, t_{j+2}], \\ (t_{j+4} - t)^7 - 8(t_{j+3} - t)^7 & t \in [t_{j+2}, t_{j+3}], \\ (t_{j+4} - t)^7 & t \in [t_{j+3}, t_{j+4}], \\ 0, & \text{otherwise.} \end{cases}$$

$$(j = -3, -2, -1, 0, \cdots, N+2, N+3). \tag{5}$$

Box I

Table 1. Nodal values of $B_j'(t)$, $B_j''(t)$, $B_j'''(t)$, $B_j^{(4)}(t)$, $B_j^{(5)}(t)$, and $B_j^{(6)}(t)$.

	t_{j-4}	t_{j-3}	t_{j-2}	t_{j-1}	t_j	t_{j+1}	t_{j+2}	t_{j+3}	t_{j+4}
$B_j(t)$	0	1	120	1191	2416	1191	120	1	0
$B_j'(t)$	0	$\frac{-7}{h}$	$\frac{-392}{h}$	$\frac{-1715}{h}$	0	$\frac{1715}{h}$	$\frac{392}{h}$	$\frac{7}{h}$	0
$B_j''(t)$	0	$\frac{42}{h^2}$	$\frac{1008}{h^2}$	$\frac{630}{h^2}$	$\frac{-3360}{h^2}$	$\frac{630}{h^2}$	$\frac{1008}{h^2}$	$\frac{42}{h^2}$	0
$B_j'''(t)$	0	$\frac{-210}{h^3}$	$\frac{-1680}{h^3}$	$\frac{3990}{h^3}$	0	$\frac{-3990}{h^3}$	$\frac{1680}{h^3}$	$\frac{210}{h^3}$	0
$B_j^{(4)}(t)$	0	$\frac{840}{h^4}$	0	$\frac{-7560}{h^4}$	$\frac{13440}{h^4}$	$\frac{-7560}{h^4}$	0	$\frac{840}{h^4}$	0
$B_j^{(5)}(t)$	0	$\frac{-2520}{h^5}$	$\frac{10080}{h^5}$	$\frac{-12600}{h^5}$	0	$\frac{12600}{h^5}$	$\frac{-10080}{h^5}$	$\frac{2520}{h^5}$	0
$B_j^{(6)}(t)$	0	$\frac{5040}{h^6}$	$\frac{-30240}{h^6}$	$\frac{75600}{h^6}$	$\frac{-100800}{h^6}$	$\frac{75600}{h^6}$	$\frac{-30240}{h^6}$	$\frac{5040}{h^6}$	0

$$-3990\,\delta_{j+1}+1680\,\delta_{j+2}+210\,\delta_{j+3}\Big)$$

$$s^{(4)}(t_j)=\frac{1}{h^4}\Big(840\,\delta_{j-3}-7560\,\delta_{j-1}+13440\,\delta_j-7560\,\delta_{j+1}+840\,\delta_{j+3}\Big)$$

$$s^{(5)}(t_j)=\frac{1}{h^5}\Big(-2520\,\delta_{j-3}+10080\,\delta_{j-2}-12600\,\delta_{j-1}+12600\,\delta_{j+1}-10080\,\delta_{j+2}+2520\,\delta_{j+3}\Big)$$

$$s^{(6)}(t_j)=\frac{1}{h^6}\Big(5040\,\delta_{j-3}-30240\,\delta_{j-2}+75600\,\delta_{j-1}-100800\,\delta_j+75600\,\delta_{j+1}-30240\,\delta_{j+2}+5040\,\delta_{j+3}\Big)$$

$$j=0,1,...,n. \tag{7}$$

Replacing Eq. (7) in Eq. (4), we have:

$$\frac{1}{h^2}\Big(42\,\delta_{j-3}+1008\,\delta_{j-2}+630\,\delta_{j-1}-3360\,\delta_j+630\,\delta_{j+1}+1008\,\delta_{j+2}+42\,\delta_{j+3}\Big)=\frac{g(t_j,\,z(t_j))}{k+1},\qquad j=0, \tag{8a}$$

$$\frac{1}{h^2}\Big(42\,\delta_{j-3}+1008\,\delta_{j-2}+630\,\delta_{j-1}-3360\,\delta_j+630\,\delta_{j+1}+1008\,\delta_{j+2}+42\,\delta_{j+3}\Big)+\frac{k}{ht_j}\Big(-7\,\delta_{j-3}-392\,\delta_{j-2}-1715\,\delta_{j-1}+1715\,\delta_{j+1}+392\,\delta_{j+2}+7\,\delta_{j+3}\Big)=g(t_j,\,z(t_j)),$$

$$j=1,\,2,\,...,\,n. \tag{8b}$$

After simplifying the above two equations, we get:

$$42\,\delta_{-3}+1008\,\delta_{-2}+630\,\delta_{-1}-3360\,\delta_0+630\,\delta_1+1008\,\delta_2+42\,\delta_3=h^2\,g_0, \tag{9a}$$

$$\xi_1(t_j)\,\delta_{j-3}+\xi_2(t_j)\,\delta_{j-2}+\xi_3(t_j)\,\delta_{j-1}+\xi_4(t_j)\,\delta_j+\xi_5(t_j)\,\delta_{j+1}+\xi_6(t_j)\,\delta_{j+2}+\xi_7(t_j)\,\delta_{j+3}=h^2\,t_j\,g_j,$$

$$j=1,\,2,\,...,\,n, \tag{9b}$$

where:

$$g_0=\frac{g(t_0,\,z(t_0))}{k+1},$$

$$\xi_1(t_j)=42\,t_j-7\,h,\qquad \xi_2(t_j)=1008\,t_j-392\,h\,k,$$

$$\xi_3(t_j)=630\,t_j-1715\,h,\qquad \xi_4(t_j)=-3360\,t_j,$$

$$\xi_5(t_j)=630\,t_j+1715\,h,\qquad \xi_6(t_j)=1008\,t_j+392\,h\,k,$$

$$\xi_7(t_j)=42\,t_j+7\,h\,k,\qquad g_j=g(t_j,\,z(t_j)),$$

$$j=1,\,2,\,...,\,n,$$

and the boundary conditions are:

$$-7\,\delta_{-3}-392\,\delta_{-2}-1715\,\delta_{-1}+1715\,\delta_1+392\,\delta_2+7\,\delta_3=0, \tag{10}$$

and:

$$\lambda_1(t_n)\,\delta_{n-3}+\lambda_2(t_n)\,\delta_{n-2}+\lambda_3(t_n)\,\delta_{n-1}+\lambda_4(t_n)\,\delta_n+\lambda_5(t_n)\,\delta_{n+1}+\lambda_6(t_n)\,\delta_{n+2}+\lambda_7(t_n)\,\delta_{n+3}=h\,\gamma, \tag{11}$$

where:

$$\lambda_1(t_n) = h\,\alpha - 7\,\beta, \qquad \lambda_2(t_n) = 120\,h\,\alpha - 392\,\beta,$$

$$\lambda_3(t_n) = 1191\,h\,\alpha - 1715\,\beta, \qquad \lambda_4(t_n) = 2416\,h\,\alpha,$$

$$\lambda_5(t_n) = 1191\,h\,\alpha + 1715\,\beta,$$

$$\lambda_6(t_n) = 120\,h\,\alpha + 392\,\beta, \qquad \lambda_7(t_n) = h\,\alpha + 7\,\beta.$$

However, still four equations are required. By differentiating Eq. (4) in t fifth times, we get:

$$z^{(5)}(t) + r(t)\; z^{(4)}(t) + 3r'(t)\; z'''(t) + 3r''(t)\; z''(t)$$
$$+r'''(t)\; z'(t) = q_1(t,\, z(t)), \tag{12}$$

where:

$$r'(t) = \begin{cases} 0, & t = 0, \\ \frac{-k}{t^2}, & t \neq 0, \end{cases} \qquad r''(t) = \begin{cases} 0, & t = 0, \\ \frac{2\,k}{t^3}, & t \neq 0, \end{cases}$$

$$r'''(t) = \begin{cases} 0, & t = 0, \\ \frac{-6\,k}{t^4}, & t \neq 0, \end{cases}$$

and:

$$q_1(t, z(t)) = \begin{cases} 0, & t = 0, \\ \frac{dg(t,\, z(t))}{dz}\, z'''(t) \\ \quad +3\frac{d^2 g(t,\, z(t))}{dz^2},\, z'(t)\; z''(t) \\ \quad +\frac{d^3 g(t,\, z(t))}{dz^3}(z'(t))^3, & t \neq 0. \end{cases}$$

Putting $t = t_0$ in Eq. (12) and using Eq. (7), we have:

$$-2520\;\delta_{-3} + 10080\;\delta_{-2} - 12600\;\delta_{-1} + 12600\;\delta_1$$
$$-10080\;\delta_2 + 2520\;\delta_3 = 0, \tag{13}$$

similarly, when $t = t_n$, we get:

$$\mu_1(t_n)\;\delta_{n-3} + \mu_2(t_n)\;\delta_{n-2} + \mu_3(t_n)\;\delta_{n-1}$$
$$+\mu_4(t_n)\;\delta_n + \mu_5(t_n)\;\delta_{n+1} + \mu_6(t_n)\;\delta_{n+2}$$
$$+\mu_7(t_n)\;\delta_{n+3} = h^5\,\psi_n, \tag{14}$$

where:

$$\mu_1(t_n) = -2520 + 840\,h\,k + 630\,h^2\,k + 252\,h^3\,k$$
$$+42\,h^4\,k,$$

$$\mu_2(t_n) = 10080 + 5040\,h^2\,k + 6048\,h^3\,k + 2352\,h^4\,k,$$

$$\mu_3(t_n) = -12600 - 7560\,h\,k - 11970\,h^2\,k$$
$$+3780\,h^3\,k + 10290\,h^4\,k,$$

$$\mu_4(t_n) = 13440\,h\,k - 20160\,h^3\,k,$$

$$\mu_5(t_n) = 12600 - 7560\,h\,k + 11970\,h^2\,k$$
$$+3780h^3\,k - 10290h^4\,k,$$

$$\mu_6(t_n) = -10080 - 5040\,h^2 k + 6048\,h^3 k$$
$$-2352\,h^4 k,$$

$$\mu_7(t_n) = 2520 + 840\,h\,k - 630\,h^2\,k + 252\,h^3\,k$$
$$-42\,h^4\,k,$$

$$\psi_n = \frac{dg(t,\, z(t))}{dz}\, z'''(t) + 3\,\frac{d^2 g(t,\, z(t))}{dz^2},$$
$$z'(t)\; z''(t) + \frac{d^3 g(t,\, z(t))}{dz^3}\,(z'(t))^3\bigg|_{t=t_n}.$$

Differentiating Eq. (12) again with t, we get:

$$z^{(6)}(t) + r(t)\; z^{(5)}(t) + 2\,r'(t)\; z^{(4)}(t) + 6\,r''(t)$$
$$z'''(t) + 4\,r'''(t)\; z''(t) + r^{(4)}(t)\; z'(t)$$
$$= q_2(t,\, z(t)), \tag{15}$$

where:

$$r^{(4)}(t) = \begin{cases} 0, & t = 0, \\ \frac{24\,k}{t^5}, & t \neq 0, \end{cases}$$

Also, we obtain $q_2(t, z(t))$ by the expression shown in Box II, where:

$$q_2(t, z(t)) = \begin{cases} \dfrac{\frac{dg(t,\, z(t))}{dz}\, z^{(4)}(t) + 3\,\frac{d^2 g(t,\, z(t))}{dz^2}\,\left(z''(t)\right)^2}{1+\frac{k}{5}} & t = 0 \\ \frac{dg(t,\, z(t))}{dz}\; z^{(4)}(t) + \frac{d^2 g(t,\, z(t))}{dz^2}\;\left(4\,z'(t)\;\; z'''(t) + 3\;(z''(t))^2\right) + 6\frac{d^3 g(t,\, z(t))}{dz^3}(z'(t))^2 z''(t) \\ \qquad + \frac{d^4 g(t,\, z(t))}{dz^4}\,(z'(t))^4 & t \neq 0 \end{cases}$$

Box II

$$z''(t)\big|_{t=0} = \frac{g(t,\, z(t))\big|_{t=t_0}}{1+k} \quad \text{and :}$$

$$z^{(4)}(t)\Big|_{t=0} = \frac{\frac{dg(t,\, z(t))}{dz}\Big|_{t=t_0} z''(t)\big|_{t=0}}{1+\frac{k}{3}}.$$

Putting $t = t_0$ in Eq. (15) and using Eq. (7), we get:

$$5040\,\delta_{-3} - 30240\,\delta_{-2} + 75600\,\delta_{-1} - 100800\,\delta_0 + 75600\,\delta_1 + -30240\,\delta_2 + 5040\,\delta_3 = h^6\,\sigma_0, \quad (16)$$

where:

$$\sigma_0 = \left.\frac{\frac{dg(t,\, z(t))}{dz}\, z^{(4)}(t) + 3\,\frac{d^2 g(t,\, z(t))}{dz^2}\,(z''(t))^2}{1+\frac{k}{5}}\right|_{t=t_0}.$$

Similarly, when $t = t_n$, we have:

$$\eta_1(t_n)\,\delta_{n-3} + \eta_2(t_n)\,\delta_{n-2} + \eta_3(t_n)\,\delta_{n-1} + \eta_4(t_n)\,\delta_n + \eta_5(t_n)\,\delta_{n+1} + \eta_6(t_n)\,\delta_{n+2} + \eta_7(t_n)\,\delta_{n+3} = h^6\,\sigma_n, \quad (17)$$

where:

$$\eta_1(t_n) = 5040 - 2520\,h\,k - 3360\,h^2\,k - 2520\,h^3\,k - 1008\,h^4\,k - 168\,h^5\,k,$$

$$\eta_2(t_n) = -30240 + 10080\,h\,k - 20160\,h^3\,k - 24192\,h^4\,k - 9408\,h^5\,k,$$

$$\eta_3(t_n) = 75600 - 12600\,h\,k + 30240\,h^2\,k + 47880\,h^3\,k - 15120\,h^4\,k - 41160\,h^5\,k,$$

$$\eta_4(t_n) = -100800 - 53760\,h^2\,k + 80640\,h^4\,k,$$

$$\eta_5(t_n) = 75600 + 12600\,h\,k + 30240\,h^2\,k - 47880\,h^3\,k - 15120\,h^4\,k + 41160\,h^5\,k,$$

$$\eta_6(t_n) = -30240 - 10080\,h\,k + 20160\,h^3\,k - 24192\,h^4\,k + 9408\,h^5\,k,$$

$$\eta_7(t_n) = 5040 + 2520\,h\,k - 3360\,h^2\,k + 2520\,h^3\,k - 1008\,h^4\,k + 168\,h^5\,k,$$

and σ_n is obtained by the expression shown in Box III. Then, from Eqs. (9)–(11), (13), (14), (16), and (17), we get a matrix in the following form:

$$AT = B, \quad (18)$$

where A is non-singular square matrix $(n+7)\times(n+7)$ (as shown in Box IV), T is dimensional vector $(n+7)$ with components δ_j, and B on the right-hand side is an $(n+7)$-dimensional vector as:

$$T = \begin{pmatrix} \delta_{-3} & \delta_{-2} & \delta_{-1} & \delta_0 & \delta_1 & \dots & \delta_{n-1} & \delta_n & \delta_{n+1} & \delta_{n+2} & \delta_{n+3} \end{pmatrix}^T,$$

and:

$$\sigma_n = \left.\left(\begin{array}{l} \frac{dg(t,\, z(t))}{dz}\, z^{(4)}(t) + \frac{d^2 g(t,\, z(t))}{dz^2}\left(4\,z'(t)\;z'''(t) + 3\,(z''(t))^2\right) + 6\,\frac{d^3 g(t,\, z(t))}{dz^3}\,(z'(t))^2\, z''(t) \\ +\frac{d^4 g(t,\, z(t))}{dz^4}\,(z'(t))^4 \end{array} \right)\right|_{t=t_n}.$$

Box III

$$A = \begin{pmatrix}
-7 & -392 & -1715 & 0 & 1715 & 392 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2520 & 10080 & -12600 & 0 & 12600 & -10080 & 2520 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5040 & -30240 & 75600 & -100800 & 75600 & -30240 & 5040 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & 1008 & 630 & -3360 & 630 & 1008 & 42 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi_1(t_1) & \xi_2(t_1) & \xi_3(t_1) & \xi_4(t_1) & \xi_5(t_1) & \xi_6(t_1) & \xi_7(t_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_1(t_2) & \xi_2(t_2) & \xi_3(t_2) & \xi_4(t_2) & \xi_5(t_2) & \xi_6(t_2) & \xi_7(t_2) & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \xi_1(t_{n-2}) & \xi_2(t_{n-2}) & \xi_3(t_{n-2}) & \xi_4(t_{n-2}) & \xi_5(t_{n-2}) & \xi_6(t_{n-2}) & \xi_7(t_{n-2}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi_1(t_{n-1}) & \xi_2(t_{n-1}) & \xi_3(t_{n-1}) & \xi_4(t_{n-1}) & \xi_5(t_{n-1}) & \xi_6(t_{n-1}) & \xi_7(t_{n-1}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_1(t_n) & \xi_2(t_n) & \xi_3(t_n) & \xi_4(t_n) & \xi_5(t_n) & \xi_6(t_n) & \xi_7(t_n) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_1(t_n) & \eta_2(t_n) & \eta_3(t_n) & \eta_4(t_n) & \eta_5(t_n) & \eta_6(t_n) & \eta_{71}(t_n) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1(t_n) & \mu_2(t_n) & \mu_3(t_n) & \mu_4(t_n) & \mu_5(t_n) & \mu_6(t_n) & \mu_{71}(t_n) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1(t_n) & \lambda_2(t_n) & \lambda_3(t_n) & \lambda_4(t_n) & \lambda_5(t_n) & \lambda_6(t_n) & \lambda_7(t_n)
\end{pmatrix}.$$

Box IV

$$B=\begin{pmatrix}0 & 0 & \sigma_0 & g_0 & g_1 & \dots & g_{n-1} & g_n & \sigma_n & \psi_n & \gamma\end{pmatrix}^T.$$

3. Uniform convergence

We estimate the truncation error for the septic B-spline collocation method in the interval $[a,b]$. Let $z(t)$ have continuous derivatives for all $t\in[a,b]$. Using Eq. (7), we get:

$$s'(t_{j-3})+120\,s'(t_{j-2})+1191\,s'(t_{j-1})+2416\,s'(t_j)+1191\,s'(t_{j+1})+120\,s'(t_{j+2})+s'(t_{j+3})=\frac{1}{h}\Big(-7\,z(t_{j-3})-392\,z(t_{j-2})-1715\,z(t_{j-1})+1715\,z(t_{j+1})+392\,z(t_{j+2})+7\,s'(t_{j+3})\Big),\tag{19}$$

$$s''(t_{j-3})+120\,s''(t_{j-2})+1191\,s''(t_{j-1})+2416\,s''(t_j)+1191\,s''(t_{j+1})+120\,s''(t_{j+2})+s''(t_{j+3})=\frac{1}{h^2}\Big(42\,z(t_{j-3})+1008\,z(t_{j-2})+630\,z(t_{j-1})-3360\,z(t_j)+630\,z(t_{j+1})+1008\,z(t_{j+2})+42z(t_{j+3})\Big),\tag{20}$$

$$s'''(t_{j-3})+120\,s'''(t_{j-2})+1191\,s'''(t_{j-1})+2416\,s'''(t_j)+1191\,s'''(t_{j+1})+120\,s'''(t_{j+2})+s'''(t_{j+3})=\frac{1}{h^3}\Big(-210\,z(t_{j-3})-1680\,z(t_{j-2})+3990\,z(t_{j-1})-3990\,z(t_{j+1})+1680\,z(t_{j+2})+210\,z(t_{j+3})\Big),\tag{21}$$

$$s^{(4)}(t_{j-3})+120\,s^{(4)}(t_{j-2})+1191\,s^{(4)}(t_{j-1})+2416\,s^{(4)}(t_j)+1191\,s^{(4)}(t_{j+1})+120\,s^{(4)}(t_{j+2})+s^{(4)}(t_{j+3})=\frac{1}{h^4}\Big(840\,z(t_{j-3})-7560\,z(t_{j-1})+13440\,z(t_j)-7560\,z(t_{j+1})+840\,z(t_{j+3})\Big),\tag{22}$$

$$s^{(5)}(t_{j-3})+120\,s^{(5)}(t_{j-2})+1191\,s^{(5)}(t_{j-1})+2416\,s^{(5)}(t_j)+1191\,s^{(5)}(t_{j+1})+120\,s^{(5)}(t_{j+2})+s^{(5)}(t_{j+3})=\frac{1}{h^5}\Big(-2520\,z(t_{j-3})+10080\,z(t_{j-2})-12600\,z(t_{j-1})+12600\,z(t_{j+1})-10080\,z(t_{j+2})+2520\,z(t_{j+3})\Big),\tag{23}$$

$$s^{(6)}(t_{j-3})+120\,s^{(6)}(t_{j-2})+1191\,s^{(6)}(t_{j-1})+2416\,s^{(6)}(t_j)+1191\,s^{(6)}(t_{j+1})+120\,s^{(6)}(t_{j+2})+s^{(6)}(t_{j+3})=\frac{1}{h^6}\Big(5040\,z(t_{j-3})-30240\,z(t_{j-2})+75600\,z(t_{j-1})-100800z(t_j)+75600\,z(t_{j+1})-30240\,z(t_{j+2})+5040\,z(t_{j+3})\Big).\tag{24}$$

Using the operator notation $E(z(t_j))=z(t_{j+1})$ [34], Eqs. (19)–(24) are expressed by Eqs. (25)–(30) as shown in Box V. In Eqs. (25)-(30), putting the operator notation $E=e^{hD}$ as the expansion to the power of hD, we have:

$$s'(t_j)=z'(t_j)-\frac{h^8}{151200}z^{(9)}(t_j)+\frac{h^{10}}{399168}z^{(11)}(t_j)+O(h^{11}),\tag{31}$$

$$s''(t_j)=z''(t_j)-\frac{h^6}{30240}z^{(8)}(t_j)+\frac{h^8}{60480}z^{(10)}(t_j)-\frac{h^{10}}{532224}z^{(12)}(t_j)+O(h^{11}),\tag{32}$$

$$s'''(t_j)=z'''(t_j)+\frac{h^6}{6048}z^{(9)}(t_j)-\frac{h^8}{33600}z^{(11)}(t_j)+\frac{h^{10}}{295680}z^{(13)}(t_j)+O(h^{11}),\tag{33}$$

$$s^{(4)}(t_j)=z^{(4)}(t_j)+\frac{h^4}{720}z^{(8)}(t_j)-\frac{h^6}{3024}z^{(10)}(t_j)+\frac{17\,h^8}{604800}z^{(12)}(t_j)-\frac{h^{10}}{997920}z^{(14)}(t_j)+O(h^{11}),\tag{34}$$

$$s'(t_j) = \frac{1}{h}\left(\frac{-7E^{-3} - 392E^{-2} - 1715E^{-1} + 1715E + 392E^2 + 7E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j), \tag{25}$$

$$s''(t_j) = \frac{1}{h^2}\left(\frac{42E^{-3} + 1008E^{-2} + 630E^{-1} - 3360 + 630E + 1008E^2 + 42E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j), \tag{26}$$

$$s'''(t_j) = \frac{1}{h^3}\left(\frac{-210E^{-3} - 1680E^{-2} + 3990E^{-1} - 3990E + 1680E^2 + 210E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j), \tag{27}$$

$$s^{(4)}(t_j) = \frac{1}{h^4}\left(\frac{840E^{-3} - 7560E^{-1} + 13440 - 7560E + 840E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j), \tag{28}$$

$$s^{(5)}(t_j) = \frac{1}{h^5}\left(\frac{-2520E^{-3} + 10080E^{-2} - 12600E^{-1} + 12600E - 10080E^2 + 2520E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j), \tag{29}$$

$$s^{(6)}(t_j) = \frac{1}{h^6}\left(\frac{5040E^{-3} - 30240E^{-2} + 75600E^{-1} - 100800 + 75600E - 30240E^2 + 5040E^3}{E^{-3} + 120E^{-2} + 1191E^{-1} + 2461 + 1191E + 120E^2 + E^3}\right) z(t_j). \tag{30}$$

Box V

$$s^{(5)}(t_j) = z^{(5)}(t_j) - \frac{h^4}{240} z^{(9)}(t_j) + \frac{h^6}{3024} z^{(11)}(t_j) - \frac{11\,h^8}{604800} z^{(13)}(t_j) + \frac{13h^{10}}{665280} z^{(15)}(t_j) + O\left(h^{11}\right), \tag{35}$$

$$s^{(6)}(t_j) = z^{(6)}(t_j) - \frac{h^2}{12} z^{(8)}(t_j) + \frac{h^4}{240} z^{(10)}(t_j) - \frac{h^6}{6048} z^{(12)}(t_j) + \frac{h^8}{210600} z^{(14)}(t_j) + \frac{101\,h^{10}}{159667200} z^{(16)}(t_j) + O\left(h^{11}\right). \tag{36}$$

By defining the truncation error $e(t) = z(t) - s(t)$ and replacing Eqs. (31)–(36) in the Taylor series expansion of $e(t_j + \omega h)$, we find:

$$e(t_j + \omega h) = \frac{\omega^2\left(2 - 7\omega^2 + 14\omega^4\right)}{120960} h^8 z^{(8)}(t_j) + \frac{\omega\left(12 - 50\omega^2 + 63\omega^4\right)}{1814400} h^9 z^{(9)}(t_j) - \frac{\omega^2\left(30 - 50\omega^2 + 21\omega^4\right)}{3628800} h^{10} z^{(10)}(t_j) + O\left(h^{11}\right), \tag{37}$$

where $0 \le \omega \le 1$.

Theorem 1. Let the nonlinear singular two-point boundary value problems of the second order have the form of Eqs. (1)–(3) with the exact solution, $z(t)$, and approximate solution, $s(t)$. Then, the septic B-spline collocation method has a truncation error of $O(h^8)$ and the convergence of this method is $O(h^6)$ for sufficiently small.

4. Numerical problems and discussion

We introduce some physiological applications related to nonlinear singular two-point boundary value problems of the second order. The results are generated with Mathematica using FindRoot function. The absolute value of the difference between the exact solution and the numerical solution is calculated and the absolute residual errors for the problems with no exact solutions are found.

Problem 1. Consider the following nonlinear singular boundary value problem [5]:

$$z''(t) + \frac{1}{t} z'(t) = -e^{z(t)},$$
$$z'(0) = 0, \quad z(1) = 0, \tag{38}$$

where the exact solution to Model (38) takes the form $z(t) = 2\log\left(\frac{A+1}{At^2+1}\right)$, with $A = 3 - 2\sqrt{2}$. Table 2 shows

Table 2. Values of the maximum absolute errors for Problem 1.

n	Method in [6]	n	Method in [5]	n	Method in [7]	n	Method in [8]	n	Present method
20	3.16×10^{-5}	6	4.00×10^{-4}	20	5.03×10^{-10}	16	3.11×10^{-6}	10	1.08×10^{-10}
90	1.55×10^{-6}	8	2.53×10^{-5}	40	4.11×10^{-12}	32	2.35×10^{-7}	20	7.10×10^{-13}
161	4.90×10^{-7}	10	2.10×10^{-6}	90	2.54×10^{-14}	64	1.50×10^{-8}	40	6.05×10^{-15}

Table 3. Numerical results for Problem 2.

t	Method in [6] $n = 60$	Method in [7] $n = 60$, $j_u = 5$	Present method $n = 20$	Present method $n = 60$
0.0	0.82848327295802	—	0.828483290367166	0.8284832903611578
0.1	0.82970607521884	0.829706075229	0.829706092441288	0.8297060924352763
0.2	0.83337471691089	0.833374716949	0.8333757335984896	0.8333747335924774
0.3	0.83948989814383	0.839489898181	0.8394899139611924	0.8394899139551798
0.4	0.84805277036165	0.848052770408	0.8480527850035551	0.8480527849975414
0.5	0.85906491397434	0.859064914012	0.8590649271767254	0.8590649271707098
0.6	0.87252830841853	0.872528308451	0.8725283199657823	0.872528319959764
0.7	0.88844529589927	0.888445295928	0.8884453056306942	0.8884453056246733
0.8	0.90681854026297	0.906818540286	0.9068185480743092	0.9068185480682849
0.9	0.92765098252660	0.927650982542	0.9276509883730854	0.9276509883970594
1.0	0.95094579461056	0.950945794637	0.9509457985037988	0.9509457984979333

that the maximum absolute errors of the septic B-spline collocation method are acceptable in comparison with other methods [5–8].

Problem 2. Consider another nonlinear singular boundary value problem [5] as follows:

$$z''(t) + \frac{2}{t} z'(t) = \frac{0.76129\, z(t)}{z(t) + 0.03119},$$

$$z'(0) = 0, \qquad 5\, z(1) + z'(1) = 5, \tag{39}$$

Table 3 and Figure 1 show the numerical solutions of the present method in comparison with those of other methods [6,7]. Also, Figure 2 illustrates the absolute residual errors $R_n(t) = \left| z''(t) + \frac{2}{t} z'(t) - \frac{0.76129\, z(t)}{z(t)+0.03119} \right|$, $0 < t \leq 1$, at $n = 60$ for Model (39), which does not have any exact solution.

Problem 3. Consider the following nonlinear singular boundary value problem [5]:

$$z''(t) + \frac{2}{t} z'(t) = -e^{-z(t)}, \tag{40}$$

subject to boundary conditions in three cases:

(i) $z'(0) = 0, \qquad 0.1\, z(1) + z'(1) = 0,$

(ii) $z'(0) = 0, \qquad z(1) + z'(1) = 0,$

(iii) $z'(0) = 0, \qquad 2\, z(1) + z'(1) = 0.$

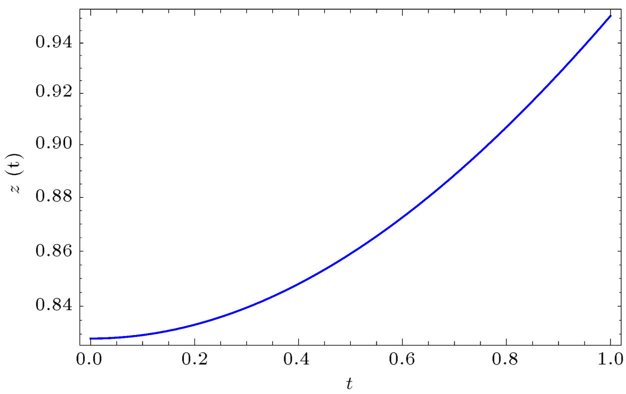

Figure 1. Numerical solutions to Problem 2 by the present method for $n = 40$.

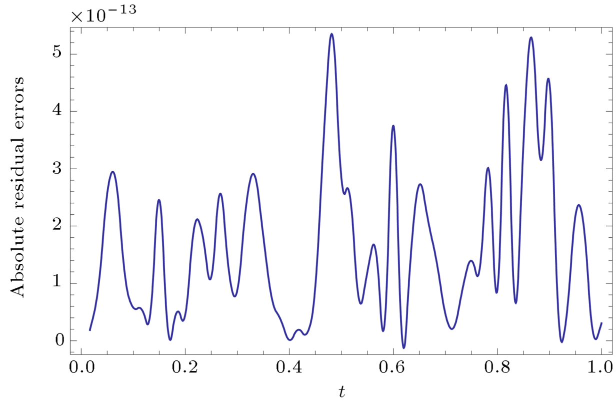

Figure 2. Absolute residual errors for Problem 2 at $n = 60$.

Table 4. Numerical results for Problem 3.

t	Method in [6] $\alpha = 0.1$, $\beta = 1$, $\gamma = 0$, $n = 60$	Present method $\alpha = 0.1$, $\beta = 1$, $\gamma = 0$, $n = 40$	Method in [17] $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $n = 40$	Present method $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $n = 40$
0.0	1.14703993670271	1.147039019329926	0.3675181074	0.3675168151351779
0.1	1.14651055946170	1.1465096424107886	0.3663637561	0.36636232924657036
0.2	1.14492141825538	1.1449205020922848	0.3628959378	0.362894066125307
0.3	1.14226947822689	1.1422685635712044	0.3570991429	0.3570975457273243
0.4	1.13854966085306	1.1385487483652494	0.3489499903	0.3489484206280108
0.5	1.13375481292594	1.1337539033259227	0.3384136581	0.3384121487566203
0.6	1.12787566262296	1.1278747567071357	0.3254450019	0.3254435224405896
0.7	1.12090076206338	1.120899860725792	0.3099878567	0.309986040238538
0.8	1.11281641561478	1.112815519868545	0.2919789654	0.29197110306288504
0.9	1.10360659299888	1.103605704000184	0.2713185637	0.2713170101649496
1.0	1.09325282603337	1.0932519451088027	0.2479292837	0.24792772332382068

Table 5. Absolute residual errors and numerical results for Problem 3.

t	Method in [5] $\alpha = 2$, $\beta = 1$, $\gamma = 0$, $n = 10$	Method in [8] $\alpha = 2$, $\beta = 1$, $\gamma = 0$, $n = 10$	Present method $\alpha = 2$, $\beta = 1$, $\gamma = 0$, $n = 10$	Present method $\alpha = 2$, $\beta = 1$, $\gamma = 0$, $n = 40$
0.0	—	—	—	0.27002964789651424
0.1	8.2261E–04	1.17552E–05	1.4627×10^{-12}	0.26875690062949353
0.2	7.7789E–04	5.59474E–06	1.37901×10^{-12}	0.2649328175381816
0.3	7.2546E–04	1.77891E–06	1.26776×10^{-12}	0.258539789381345
0.4	6.5808E–04	7.15070E–07	1.11477×10^{-12}	0.2495481802537009
0.5	5.8074E–04	4.01100E–07	9.49685×10^{-13}	0.23791588758927304
0.6	4.9869E–04	1.37600E–06	7.78599×10^{-13}	0.22358770718136314
0.7	4.1676E–04	2.70372E–06	6.17506×10^{-13}	0.20649448302373294
0.8	3.3896E–04	6.88321E–06	4.69846×10^{-13}	0.18655201416659845
0.9	2.6816E–04	1.83404E–05	3.44058×10^{-13}	0.16365968158038072
1.0	2.0607E–04	2.53189E–05	2.41918×10^{-13}	0.137698746613583

Table 4 and Figure 3 show the numerical solutions of the present method with different values of α in comparison with other methods [6,17]. Furthermore, Table 5 shows the absolute residual errors $R_n(t) = \left|z''(t) + \frac{2}{t}z'(t) + e^{-z(t)}\right|$, $0 < t \leq 1$, at $\alpha = 2$, $n = 10$ and indicates that numerical solutions of Model (40) at $\alpha = 2$, $n = 40$ do not have any exact solution.

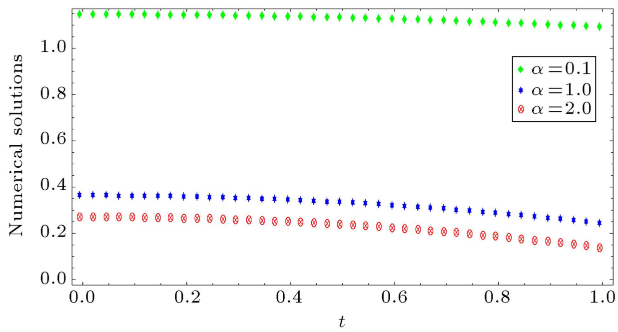

Figure 3. Numerical solutions to Problem 3 at different values of α.

5. Conclusion

We presented a septic B-spline collocation method for finding numerical solutions to the nonlinear singular two-point boundary value problems of the second order. At different values of n, the numerical results showed that the proposed method had efficient solutions for the studied models. Three applications were investigated, which played a vital role in physiological models. We observed that numerical accuracy of the proposed method for the thermal explosions problem was better than the results obtained by [5–8] and for the problems of oxygen diffusion in a spherical cell with the kinetics uptake of Michaelis-Menten and heat sources distribution in human head, numerical solutions were similar to the results obtained by [5–8] and [17]. Nevertheless, it should be noted that these problems do not have exact solutions. Accordingly, we calculated the absolute residual errors and presented the results.

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