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Optimal design of fractional-order digital integrators: An evolutionary approach

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Abstract. This paper presents an optimal approach to design Fractional-Order Digital Integrators (FODIs) using a metaheuristic technique, called Hybrid Flower Pollination Algorithm (HFPA). HFPA is a hybrid approach which combines the exploitation and exploration capabilities of two different evolutionary optimization algorithms, namely, Particle Swarm Optimization (PSO) and Flower Pollination Algorithm (FPA). The proposed HFPA based designs are compared with the designs based on Real Coded Genetic Algorithm (RGA), PSO, Differential Evolution (DE), and FPA. Simulation results demonstrate that HFPA based FODIs of all the different orders consistently achieve the best magnitude responses. The proposed technique yields FODIs which surpass all the designs based on both classical and evolutionary optimization approaches reported in recent literature.

1. Introduction

Fractional Calculus (FC) is regarded as a generalization of the integer-order calculus [1]. Since the fractional-order models illustrate the dynamical behavior of physical systems more accurately [2], the applications of FC are explored in various disciplines such as power electronics and control systems [3-4].

A Fractional-Order Integrator (FOI) of order r is a system of infinite dimension characterized by the frequency response as given in Eq. (1):

$$H_{FOI}(j\omega) = \frac{1}{(j\omega)^r}, \tag{1}$$

where, $r \in (0,1)$, and the angular frequency is ω.

The magnitude and the phase responses of the FOI are given by Eqs. (2) and (3), respectively:

$$|H_{FOI}(j\omega)| = \frac{1}{\omega^r}, \tag{2}$$

$$\angle H_{FOI}(j\omega) = -90^o \times r. \tag{3}$$

For practical realization, the FODIs should be finite-dimensional and closely approximate the ideal FOI. In the literature, the direct discretization based design procedure applies the series expansion along with a suitable generating function. The series expansion techniques include power series expansion [5], Continued Fraction Expansion (CFE) [6-7], Taylor series expansion [8], and MacLaurin series expansion [9] with suitable generating functions such as the Tustin operator [10-11], the Simpson operator [12], the Al-Alaoui operator [13], and the mixed Tustin-Simpson operator [14]. Barbosa et al. [15] reported the design of FODIs based on the least squares method. Thiele's continued fractions as rational approximations to design differintegrators are also proposed by Maione [16]. Romero et al. [17] used Chebyshev polynomials to design FODIs. In the design method based on indirect discretization [18-20], the frequency domain mapping

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is performed in the analog domain and the resultant transfer function is discretized. In the recent literature, the design of FODIs has been reported by Gupta and Yadav [21] by using the discretization of PSO optimized integer-order digital integrators. PSO algorithm based optimized FODIs of various orders with much improved magnitude response have also been reported by Yadav and Gupta [22].

While evolutionary algorithms can converge on a near global optimal solution for objective functions of discontinuous, non-linear, and non-differentiable nature [23], an individual algorithm such as Genetic Algorithm (GA) shows two major drawbacks: (a) early convergence and (b) lack of good exploitation characteristics [24-25]. Similarly, PSO and DE suffer from (a) stagnation of agents in the search space and (b) premature convergence [26-28]. The stagnation of agents in the search space leads to poor exploration. Hence, the agents are stuck at the local optima and are unable to reach the global optimal position, thereby producing solutions of poor quality.

In this paper, design of FODIs using a recently proposed algorithm by Abdel-Raouf et al. [29], called HFPA, is proposed. Since HFPA combines both FPA [30] and PSO algorithms [31], HFPA achieves a much better exploration and exploitation of the error landscape and can attain superior solution quality to the capability of the individual algorithms.

1.1. Differences between HFPA and the algorithm of Abdel-Raouf et al. [29]

The major differences between the algorithm used in this paper and the one reported in [29] are described below:

(a) In the algorithm reported in [29], the FPA phase begins only after the PSO phase completely terminates, i.e., reaches the stopping criteria. However, in the HFPA algorithm used in this paper, all the agents go through the PSO phase and then, proceed to the FPA phase in each and every iteration;

(b) The switching factor (p) is obtained by employing a chaotic map in [29], whereas this work uses a fixed value of p at run-time. The main reason for avoiding the selection of p based on chaotic maps is to reduce the execution time (t_x) of the algorithm.

1.2. Contributions of this work

(a) It shows the applicability of HFPA to the design of stable and accurate Infinite Impulse Response (IIR) FODIs of different orders. FODIs for integrators of any arbitrary fractional order have also been designed. This proves that the proposed FODI design procedure is a generic one;

(b) It justifies the consistently improved performance of HFPA based FODIs in comparison with the FODIs designed using RGA [32], PSO [31], DE [33], and FPA [30], by using different hypothesis tests;

(c) It demonstrates the superiority of the proposed FODIs over all state-of-the-art designs with respect to magnitude response performance.

The rest of the paper is organized as follows. In Section 2, the problem of designing FODIs is formulated. In Section 3, the HFPA is presented. Section 4 shows the simulation results. Conclusions of this work are discussed in Section 5.

2. Problem formulation

The Nth order IIR FODI is represented by Eq. (4):

$$H_{\mathrm{FODI}}(z) = \frac{a_1 + a_2 z^{-1} + a_3 z^{-2} + \ldots + a_{N+1} z^{-N}}{b_1 + b_2 z^{-1} + b_3 z^{-2} + \ldots + b_{N+1} z^{-N}}, \tag{4}$$

where, the numerator and denominator coefficients of $H_{\mathrm{FODI}}(z)$ are denoted by a_k and b_k, respectively, $k = 1, 2, \ldots, N + 1$.

The cost function RMSME (Root Mean Square Magnitude Error) given by Eq. (5) is minimized by obtaining the optimal coefficients of $H_{\mathrm{FODI}}(z)$ using HFPA and the other algorithms:

$$\mathrm{RMSME} = \sqrt{\frac{\sum \left| \left| H(\omega)_{\mathrm{proposed}} \right| - \left| H(\omega)_{\mathrm{ideal}} \right| \right|^2}{n}}, \tag{5}$$

where, n is the number of sampled frequency points. In this work, $n = 512$, $N = 2, 3, \ldots, 6$ and $0.02\pi \leq \omega \leq \pi$.

Group delay response of $H_{\mathrm{FODI}}(z)$ is defined by Eq. (6):

$$\tau = -\frac{d\theta(\omega)}{d\omega}, \tag{6}$$

where $\theta(\omega)$ is the phase of the FODI.

3. Hybrid Flower Pollination Algorithm (HFPA)

Evolutionary optimization algorithms have efficiently solved various optimization design problems [34-46]. HFPA combines the search capabilities of FPA and PSO to attain the global optimal solution. In HFPA, the optimal solution achieved by the PSO phase ($g$best) is considered as the best starting solution B for the FPA phase, i.e., $B = g$best, in HFPA. Instead of initially starting with the best solution (B) as done in the basic FPA, in which B is determined as the agent with the lowest value of error fitness that is obtained from random distribution of flowers in the search space, considering $B = g$best in HFPA helps in starting with an improved value of B, since it is obtained at the end of the PSO phase of HFPA. This improves both the diversification and the intensification efficiency of HFPA. In this work, the steps used to design HFPA based FODIs are:

Step 1: Initialize the control parameters and n_p (=50) number of real coded particle vectors consisting of the coefficients of $H_{\text{FODI}}(z)$. The total number of coefficients/decision variables is $D = (N+1) \times 2$;

Step 2: Randomly generate n_p number of particles/solution vectors in the search space. Also, determine the initial fitness for all the agents.

PSO Phase:

Step 3: Compute the initial personal best solution vectors, $p$best, and best solution vector of the group, $g$best;

Step 4: Modify the velocity, V_i, and the position S_i of each particle as per Eqs. (7) and (8), respectively:

$$V_i^{k+1} = w \times V_i^{k} + C_1 \times \text{rand}_1 \times (p\text{best}_i^k - S_i^{k}) + C_2 \times \text{rand}_2 \times (g\text{best}_i^k - S_i^{k}), \tag{7}$$

where rand_1 and rand_2 are two random vectors and rand_1, $\text{rand}_2 \in [0\ 1]$; C_1 and C_2 are the learning parameters:

$$S_i^{k+1} = S_i^{k} + V_i^{k+1}. \tag{8}$$

Step 5: Compute the fitness of all agents for their new positions and update the $g$best and the $p$best vectors:

FPA Phase:

Step 6: Consider $B = g$best;

Step 7: Generate a random number rand ($\in$ [0 1]) for each particle/pollen. If rand $<$ switch probability (p), execute global pollination or else, local pollination according to Eqs. (9) and (10), respectively:

$$S_i^{k+1} = S_i^{k} + \gamma L(\lambda)(B - S_i^{k}), \tag{9}$$

where, γ is a scaling factor, which controls the step size, and $L(\lambda)$ determines the strength of the pollination.

$$S_i^{k+1} = S_i^{k} + U(S_j^{k} - S_i^{k}), \tag{10}$$

where $U \in [0, 1]$.

Step 8: Evaluate the fitness of all new solution vectors. Update $p$best and $g$best if the new solution vectors are fitter;

Step 9: Repeat from Step 4 on until the algorithm terminates;

Step 10: Proclaim B as the final optimal solution.

Figure 1 shows the flowchart of the HFPA.

4. Results and discussions

The control parameters of all the algorithms are selected after conducting exhaustive trial runs. The best values of the parameters and the justification of their settings for this optimization problem are shown in Table 1. The programming environment used is CPU: i3 processor (1.70 GHz), RAM: 2 GB, Operating System: Windows 7, and Programming Language: MATLAB (software version: MATLAB 7.5).

4.1. Design of FODIs for $r = 1/2$

The optimal coefficients attained for the designed FODIs with $r = 1/2$ are shown in Table 2.

4.1.1. Frequency response performance comparison

Table 3 shows the comparison of RMSMEs, Maximum Absolute Relative Magnitude Errors (MARMEs), and maximum sample deviations achieved from constant group delay (τ_m) for all the designed FODIs. RMSME achieved by HFPA is lower than those by the competing metaheuristics based FODIs of orders 2 to 6 with the values of −21.9 dB, −30.4 dB, −32.1 dB, −47.8 dB, and −30.5 dB, respectively. The HFPA based FODIs of orders 2 to 6 also outperform the other designs by achieving the lowest MARMEs of −18.2 dB, −22.7 dB, −29.9 dB, −31.9 dB, and −22.6 dB, respectively. The HFPA based designs of orders 2 to 6 achieve τ_ms of 5.14, 4.82, 5.48, 1.21, and 7.23 samples, respectively. Figures 2(a)-(e) show the Absolute Magnitude Error (AME) response comparison plots for the designed FODIs and demonstrate the superior solution quality of HFPA. Figure 2(f) shows the magnitude (dB) response plot for the proposed HFPA based FODIs for the design of FOI with $r = 1/2$.

The improved performance of HFPA in comparison with the other algorithms for the design of FODIs is due to the following reasons:

1. The hybridization technique involving PSO (known for its efficient exploitation capability) and FPA (recognized for its efficient search space exploration ability) improves both the global and local search competencies of HFPA as compared with individual algorithms, such as RGA, which exhibits poor exploitation characteristics, and PSO, which has early convergence and stagnation issues. The FPA phase, which follows the PSO phase in HFPA, helps

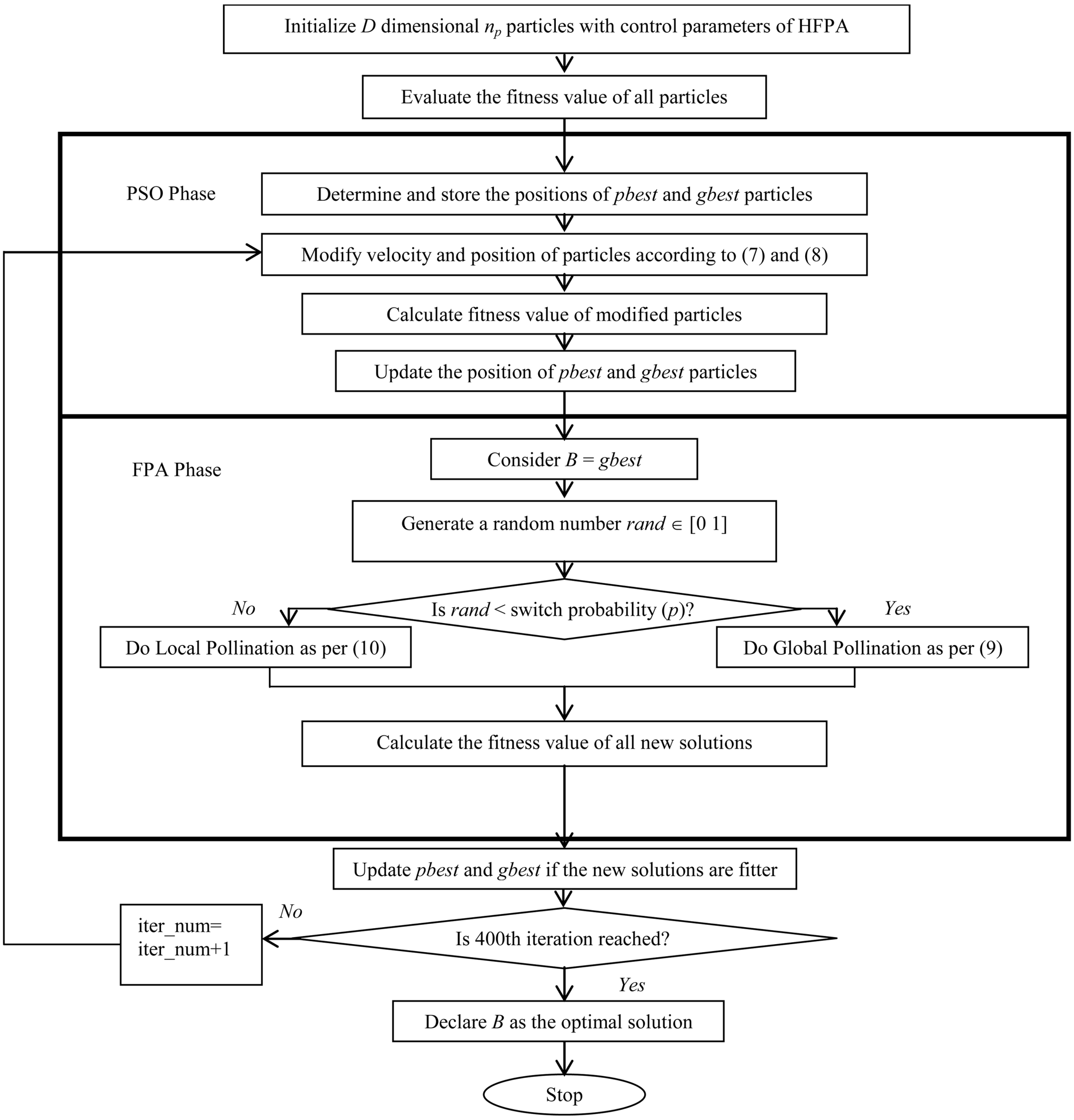

Figure 1. Flowchart of the HFPA.

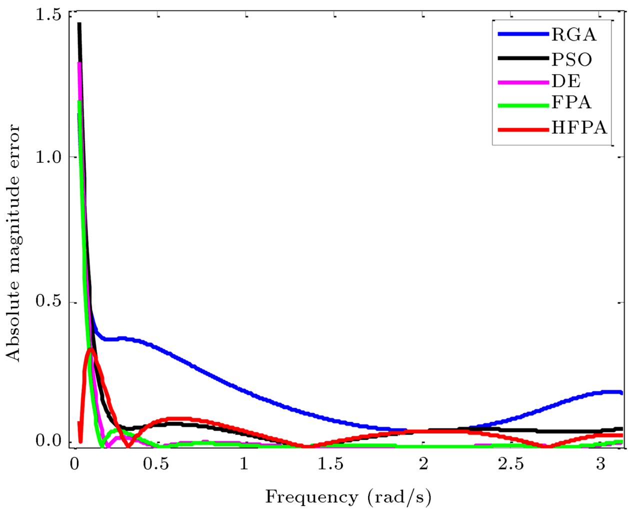

Figure 2(a). AME plots of the designed FODIs ($N = 2$, $r = 1/2$).

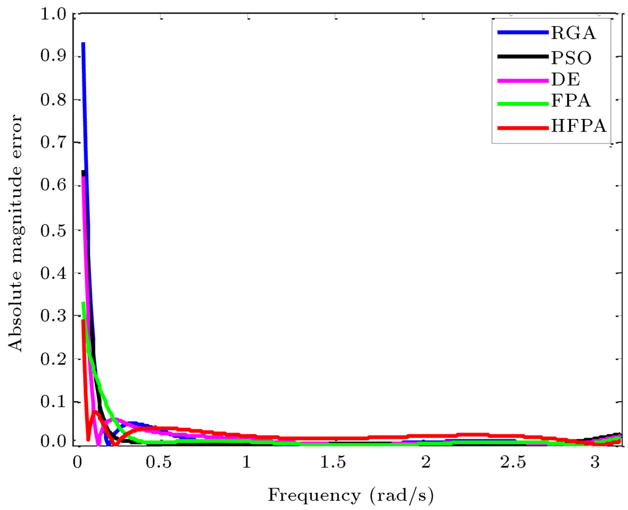

Figure 2(b). AME plots of the designed FODIs ($N = 3$, $r = 1/2$).

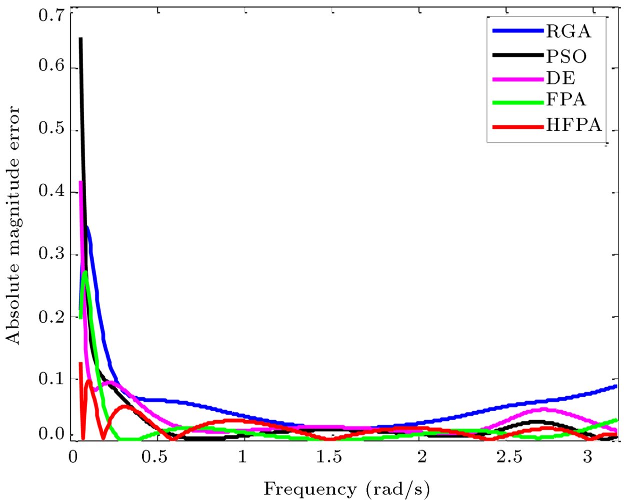

Figure 2(c). AME plots of the designed FODIs ($N = 4$, $r = 1/2$).

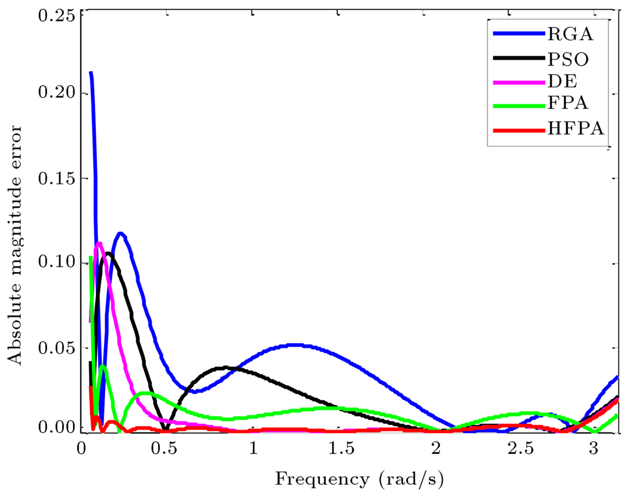

Figure 2(d). AME plots of the designed FODIs ($N = 5$, $r = 1/2$).

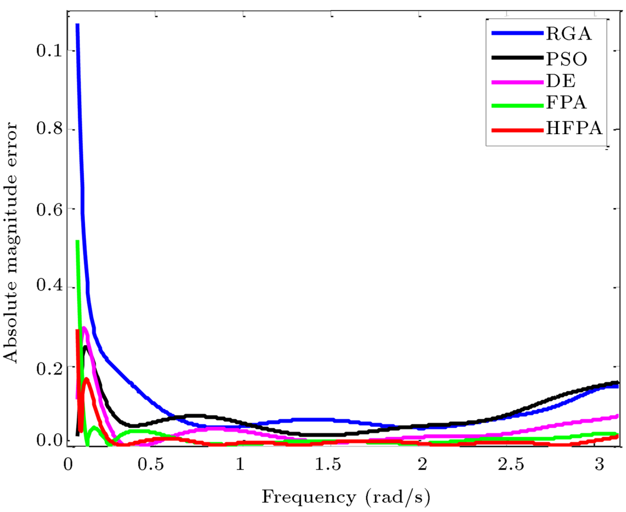

Figure 2(e). AME plots of the designed FODIs ($N = 6$, $r = 1/2$).

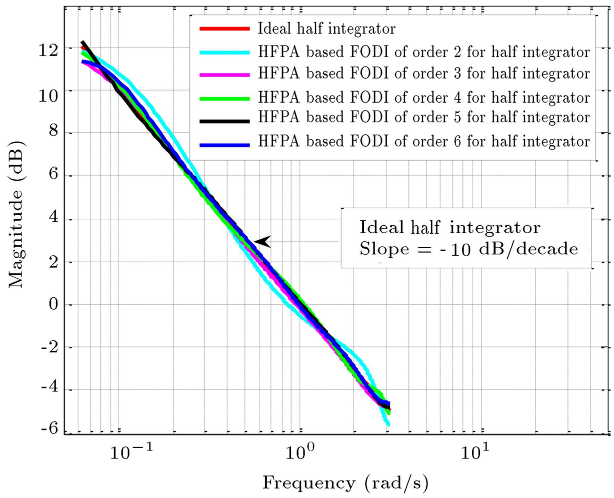

Figure 2(f). Magnitude (dB) versus frequency (rad/s) of HFPA based FODIs ($r = 1/2$).

in preventing the agents from getting stuck in sub-optimal solution, whereas the use of PSO phase in HFPA helps in achieving a better convergence on the global optima;

2. The FPA phase of HFPA uses global random walks instead of simple isotropic randomization process of the classical algorithms such as RGA, PSO, and DE for the implementation of the diversification phase of the algorithm. The random walks are modelled by using the Lévy flight distribution. The measure of the variance of the Lévy flight is given by Eq. (11):

$$\sigma^2(t) \sim t^{3-\upsilon}, \tag{11}$$

where, $1 \leq \upsilon \leq 2$, such a variance characteristic results in a very fast increase in the change of position of agents in comparison with the simple random walks of classical optimizers, which only provide a linear change of variance, i.e., $\sigma^2(t) \sim t$. Thus, the possibility of getting stuck in the local optima is greatly reduced and a more diversified exploration of the multimodal error landscape by the agents of HFPA is possible.

4.1.2. Comparison using hypothesis tests

The parametric and non-parametric hypothesis tests [47] are used to determine whether a significant difference occurs in the performance of two algorithms, namely, the HFPA and the other algorithms. It is a recommended practice in the field of evolutionary computation to examine the competency of an algorithm in solving a particular optimization problem. If the solution quality evaluated in terms of RMSME based on the proposed algorithm (HFPA) is not significantly different from the solution based on the classical algorithms, then the new methodology does not consistently provide an accurate quality of

Table 1. Selection and justification of control parameters.

Control parameter	RGA	PSO	DE	FPA	HFPA	Justification of the parameter settings
Population size	50	50	50	50	50	Smaller size of population (comprising of 20 search agents) creates less diversity of solutions, leading to poor exploration of the search space, whereas a higher population size (consisting of 100 search agents) leads to increased optimization time without any significant increase in the quality of solutions. Population size of 50 agents demonstrates the best balance between the computational time and the solution quality.
Algorithm termination condition	400 iterations	400 iterations	400 iterations	400 iterations	400 iterations	All the algorithms achieved convergence within 400 iterations. Using more than 400 iterations leads to increased computational time.
Crossover rate, crossover	1, Two point	–	–	–	–	Selecting the rate of crossover equal to 1 alleviates the problems of early convergence, stagnation of agents (chromosomes), and yielding the same solution repeatedly.
Mutation rate, mutation	0.01, Gaussian	–	–	–	–	Rate of mutation < 0.01 leads to poor diversity in solutions, whereas mutation rate > 0.01 causes deviations from global optimal solution.
Selection	Roulette wheel	–	–	–	–	The smallest value of RMSME is produced among all the other selection types.
C_1, C_2	–	2.05, 2.05	–	–	2.05, 2.05	$C_1, C_2 < 2.05$ leads to a reduction in the learning capability of agents, thereby, leading to sub-optimal solution. $C_1, C_2 > 2.05$ leads to movement of agents farther away from the global optimal, since the dependence on memory componence is reduced.
$V_i^{\min}, V_i^{\max}$	–	0.01, 1.0	–	–	0.01, 1.0	$\nu_i^{\max} = 1.0$ helps to stabilize the PSO by avoiding overshooting.
$w_{\max}, w_{\max}$	–	1.0, 0.4	–	–	1.0, 0.4	Linear decrease in inertia factor (w) from $w_{\max} = 1.0$ to $w_{\min} = 0.4$ provides the particles with an improved transition from the exploration to the exploitation phases, i.e., a high starting value of w provides a good global search in the initial stages, while the linear reduction helps the particles in acquiring both social and cognitive skills.
C_r	–	–	0.3	–	–	$C_r < 0.3$ leads to a reduced crossover of agents, resulting in poor exploration characteristic.
F	–	–	0.5	–	–	$F < 0.5$ causes a poor diversity in the solutions, leading to an inadequate exploration, whereas, $F > 0.5$ produces significant oscillations, leading to a poor convergence.
p	–	–	–	0.8	0.8	$p < 0.8$ leads to a poor global search, which causes reduced diversity of solutions, whereas $p > 0.80$ causes reduced exploitation of the search space.
γ	–	–	–	0.1	0.1	$\gamma < 0.1$ leads to a reduced diversity of solutions, resulting in a poor exploration, while $\gamma > 0.1$ produces a high divergence.
λ	–	–	–	1.5	1.5	$\lambda > 1.5$ causes a wide divergence of flowers in the search space, whereas $\lambda < 1.5$ results in a meagre global search.

Table 2. Coefficients of the designed FODIs ($r = 1/2$).

N	Algorithm	$[a_1 a_2 \ldots a_{N+1}]$	$[b_1 b_2 \ldots b_{N+1}]$
2	RGA	[0.797202 0.069410 –0.275015]	[1.056014 –0.505205 –0.360048]
	PSO	[0.971504 0.021414 –0.184209]	[1.048003 –0.453106 –0.284112]
	DE	[1.110406 –0.244508 –0.108813]	[1.224003 –0.930304 –0.017511]
	FPA	[1.084920 –0.278339 –0.103257]	[1.188590 –0.954331 0.008985]
	HFPA	**[0.989904 0.109907 –0.284911]**	**[1.070019 –0.472705 –0.407932]**
3	RGA	[0.910302 –0.937406 0.085916 0.052813]	[0.995505 –1.584204 0.628810 –0.006527]
	PSO	[0.936607 –0.944110 0.069909 0.057725]	[1.000711 –1.597018 0.608410 0.004139]
	DE	[0.931902 –0.945506 0.069117 0.058441]	[1.008007 –1.610115 0.609916 0.005214]
	FPA	[0.928130 –0.951829 0.048214 0.068076]	[1.013606 –1.612829 0.594631 0.022813]
	HFPA	**[0.888006 –0.879507 0.111238 0.040162]**	**[0.984904 –1.541101 0.635134 –0.041127]**
4	RGA	[1.104301 0.397313 –0.489927 –0.352431 0.031343]	[1.130015 –0.214821 –0.831140 –0.177531 0.229901]
	PSO	[1.040011 0.393204 –0.469616 –0.361621 0.031134]	[1.154901 –0.217108 –0.806617 –0.198532 0.234541]
	DE	[1.003906 0.401707 –0.465611 –0.349229 0.029630]	[1.126003 –0.205611 –0.806821 –0.189535 0.223941]
	FPA	[1.096868 0.358728 –0.474306 –0.353985 0.029785]	[1.169454 –0.288467 –0.793210 –0.190883 0.228520]
	HFPA	**[0.967203 0.426609 –0.431416 –0.202825 –0.039133]**	**[1.047007 –0.138913 –0.781516 –0.027728 0.058549]**
5	RGA	[0.958101 0.079210 –0.656223 –0.081207 0.123233 –0.039642]	[1.060015 –0.505609 –0.861912 0.237324 0.225126 –0.090104]
	PSO	[0.935201 –1.619005 0.720613 0.019424 –0.046011 0.008717]	[1.001009 –2.317005 1.745018 –0.428315 –0.011025 0.012631]
	DE	[0.952800 –1.622004 0.718017 0.000519 –0.039047 0.009331]	[1.025006 –2.344005 1.738012 –0.432434 0.016227 0.000238]
	FPA	[0.968901 –1.620078 0.695911 0.027516 –0.042447 0.001424]	[1.041006 –2.357013 1.741032 –0.431717 0.016323 –0.005342]
	HFPA	**[0.952003 –1.618104 0.717038 –0.000583 –0.041006 0.010526]**	**[1.024910 –2.340520 1.734621 –0.433959 0.015812 0.001419]**
6	RGA	[0.996702 0.699406 –0.184510 –0.251616 –0.089337 –0.033541 –0.028033]	[1.207001 0.109200 –0.684904 –0.276207 0.013419 0.004216 –0.024435]
	PSO	[1.100101 0.667603 –0.192901 –0.271216 –0.093710 –0.032721 –0.026934]	[1.098700 0.107601 –0.670507 –0.285515 0.009916 0.012216 –0.021025]
	DE	[1.043901 0.670804 –0.188901 –0.262711 –0.093816 –0.016224 –0.022310]	[1.088002 0.101103 –0.671700 –0.276604 0.000711 0.020542 –0.029058]
	FPA	[1.065510 0.666067 –0.238241 –0.302687 –0.066192 0.024803 –0.015537]	[1.179800 0.054048 –0.720700 –0.295348 0.057573 0.056835 –0.035469]
	HFPA	**[0.948301 –1.627003 0.711701 0.008611 –0.038915 –0.005317 0.015229]**	**[1.027003 –2.345001 1.727007 –0.423501 0.015028 –0.000531 0.003801]**

Table 3. RMSME (dB)/MARME (dB)/τ_m (samples)/t_x (seconds) comparison for the designed FODIs ($r = 1/2$).

N	RGA	PSO	DE	FPA	HFPA
2	–12.9/–9.5/ 5.59/80.236	–16.1/–8.7/ 3.03/67.350	–18.2/–9.5/ 3.05/69.772	–19.5/–10.5/ 3.27/68.991	–21.9/–18.2/ 5.14/79.169
3	–21.7/–12.6/ 3.84/84.824	–24.0/–15.9/ 4.59/69.560	–25.5/–16.1/ 3.85/78.067	–27.5/–21.6/ 3.79/75.115	–30.4/–22.7/ 4.82/83.324
4	–22.2/–16.1/ 5.36/89.469	–24.2/–15.7/ 4.39/72.054	–26.5/–19.5/ 4.87/80.116	–27.6/–21.8/ 5.27/78.169	–32.1/–29.9/ 5.48/88.581
5	–26.8/–24.6/ 5.09/93.724	–30.0/–26.8/ 1.83/74.384	–33.3/–28.7/ 2.86/81.684	–37.2/–30.6/ 2.64/80.034	–47.8/–31.9/ 1.21/92.867
6	–17.0/–11.4/ 4.25/97.104	–21.2/–10.8/ 5.19/77.689	–24.6/–17.5/ 5.28/85.428	–28.1/–17.7/ 4.58/82.973	–30.5/–22.6/ 7.23/95.831

solution as compared with the solutions produced by the other algorithms.

(a) ***t*-Test Results**: The results of t-test for the different algorithm pairs for the design of FODIs with $r = 1/2$ are shown in Table 4. The RMSME sample length taken for each algorithm is 60. Since HFPA rejects the null hypothesis ('no significant difference between the algorithms') with a high level of confidence (CL), the HFPA based designs consistently outperform the other algorithm based designs;

(b) ***Wilcoxon Rank-Sum Test Results***: Table 5 summarizes the Wilcoxon rank-sum test results for the algorithm pairs with different sample sizes (n_1, n_2) for the competing algorithm and HFPA used in the design of FODIs with $r = 1/2$ to confirm the solution consistency. Small values of W_1 and a high value of W_2 for almost all cases confirm that HFPA based FODIs consistently provide the lowest RMSME.

Table 4. Results of the t-test.

N	Algorithm pair	t_{cal}	Hypothesis decision
2	HFPA-RGA	2.237	Reject with 97.5% CL
	HFPA-PSO	1.126	Reject with 75% CL
	HFPA-DE	0.753	Reject with 75% CL
	HFPA-FPA	0.328	Reject with 60% CL
3	HFPA-RGA	1.035	Reject with 75% CL
	HFPA-PSO	0.589	Reject with 60% CL
	HFPA-DE	0.467	Reject with 60% CL
	HFPA-FPA	0.422	Reject with 60% CL
4	HFPA-RGA	1.619	Reject with 90% CL
	HFPA-PSO	0.952	Reject with 75% CL
	HFPA-DE	0.613	Reject with 60% CL
	HFPA-FPA	0.524	Reject with 60% CL
5	HFPA-RGA	1.497	Reject with 90% CL
	HFPA-PSO	0.814	Reject with 75% CL
	HFPA-DE	0.324	Reject with 60% CL
	HFPA-FPA	0.279	Reject with 60% CL
6	HFPA-RGA	1.512	Reject with 90% CL
	HFPA-PSO	0.715	Reject with 75% CL
	HFPA-DE	0.316	Reject with 60% CL
	HFPA-FPA	0.293	Reject with 60% CL

4.1.3. Comparison with the literature

Table 6 confirms that the proposed HFPA based designs outperform all the reported FODI designs and exhibit the most accurate magnitude response. Figures 3(a)-3(e) present the AME response compar-

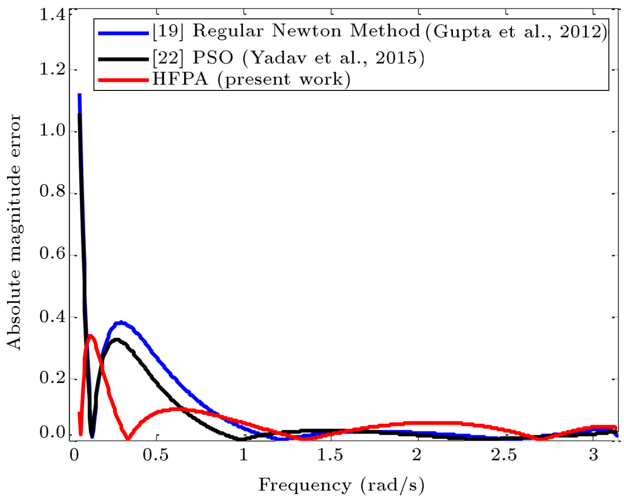

Figure 3(a). AME plots for comparison with the reported FODIs for $N = 2$ and $r = 1/2$.

Table 5. Results of the Wilcoxon rank-sum test.

N	Sample length n_1	Sample length n_2	U_C $\alpha = 0.05$	U_C $\alpha = 0.01$	RGA W_1	RGA W_2	PSO W_1	PSO W_2	DE W_1	DE W_2	FPA W_1	FPA W_2
2	6	9	31	26	28	92	43	77	43	77	45	75
	8	12	58	51	55	155	73	137	74	136	76	134
3	6	9	31	26	41	79	49	71	49	71	49	71
	8	12	58	51	72	138	76	134	79	131	80	130
4	6	9	31	26	33	87	42	78	51	69	53	67
	8	12	58	51	64	146	72	138	79	131	79	131
5	6	9	31	26	34	86	41	79	48	72	47	73
	8	12	58	51	60	150	73	137	79	131	81	129
6	6	9	31	26	33	87	43	77	51	69	52	68
	8	12	58	51	62	148	77	133	77	133	77	133

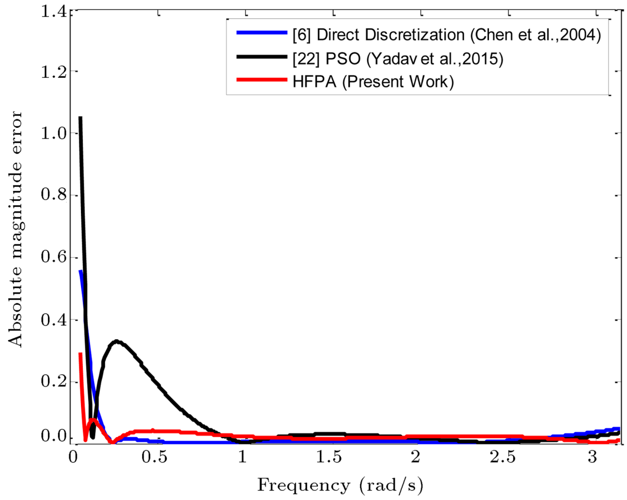

Figure 3(b). AME plots for comparison with the reported FODIs for $N = 3$ and $r = 1/2$.

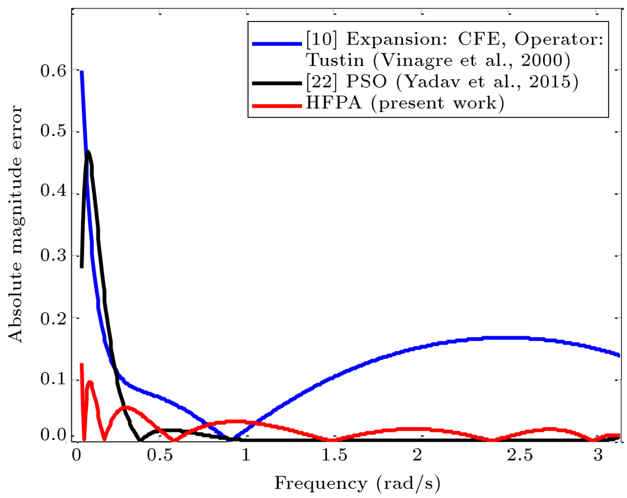

Figure 3(c). AME plots for comparison with the reported FODIs for $N = 4$ and $r = 1/2$.

Table 6. Comparison of HFPA based FODIs ($r = 1/2$) with the literature.

N	Reference	Approach	RMSME (dB)	MARME (dB)	τ_m (samples)
2	[19]	Regular Newton	–16.1	–10.9	4.74
	[22]	PSO	–17.7	–11.4	2.93
	Present work	**HFPA**	**–21.9**	**–18.2**	**5.14**
3	[6]	Direct discretization	–23.7	–16.9	5.39
	[22]	PSO	–17.7	–11.5	2.89
	Present work	**HFPA**	**–30.4**	**–22.7**	**4.82**
4	[10]	Expansion method: CFE operator: Tustin	–17.0	–11.3	3.85
	[22]	PSO	–21.5	–16.3	5.81
	Present work	**HFPA**	**–32.1**	**–29.9**	**5.48**
5	[16]	Thiele's continued Fractions discretized–1	–30.1	–3.3	16.32
	[16]	Thiele's continued Fractions discretized–2	–27.7	–3.0	10.01
	[17]	Approximation: Chebyshev–Padé Operator: Tustin	–23.2	–9.3	151.7
	[17]	Approximation: Rational Chebyshev Operator: Tustin	–23.0	–9.3	145.8
	[17]	Approximation: Least squares Operator: Tustin	–19.7	–7.7	26.73
	[17]	Approximation: Chebyshev– Padé Operator: Al–Alaoui	–10.6	–1.7	3.71
	[17]	Approximation: Rational Chebyshev Operator: Al–Alaoui	–14.8	–2.5	149.7
	[18]	Indirect discretization Operator: Al–Alaoui	–12.6	–7.1	14.54
	[20]	Indirect discretization	–33.2	–25.1	2.40
	[22]	PSO	–24.9	–17.0	5.11
	Present work	**HFPA**	**–47.8**	**–31.9**	**1.21**
6	[22]	PSO	–21.3	–16.2	5.78
	Present work	**HFPA**	**–30.5**	**–22.6**	**7.23**

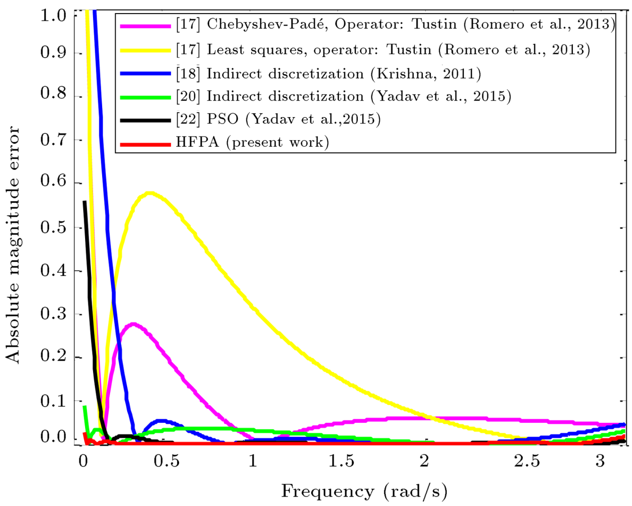

Figure 3(d). AME plots for comparison with the reported FODIs for $N = 5$ and $r = 1/2$.

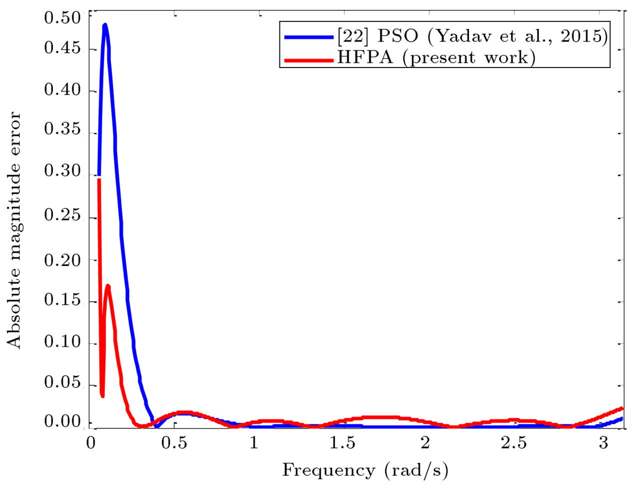

Figure 3(e). AME plots for comparison with the reported FODIs for $N = 6$ and $r = 1/2$.

ison plots for the proposed FODIs for $r = 1/2$ with the published designs. The reasons for the superior performance of the proposed HFPA based designs over the reported ones are the following:

(a) The designs reported in [6,10,16-20] are based on non-optimization procedures. However, it is well known that the error landscape of a multimodal and non-linear optimization problem, such as the FODI design, can only be solved accurately using an optimization process;

(b) The designs reported in [22] employ the PSO algorithm. The main reasons for generation of sub-optimal solution by PSO are discussed in Section 1 of this paper.

4.2. Design of FODIs for $r = 1/3$

The optimal coefficients attained for the FODIs with $r = 1/3$ are shown in Table 7.

4.2.1. Frequency response performance comparison

Table 8 shows the statistical analysis for RMSME, MARME, and τ_m for the FODIs with $r = 1/3$. RMSMEs achieved by the proposed HFPA based designs, as compared with those by the competing benchmark algorithm based FODIs, of orders 2 to 6 are the lowest with values of -25.8 dB, -20.9 dB, -21.1 dB, -32.2 dB, and -42.5 dB, respectively. The HFPA based FODIs of orders 2 to 6 also achieve the lowest MARMEs of -22.0 dB, -9.8 dB, -13.8 dB, -17.9 dB, and -27.1 dB, respectively. The proposed FODIs of orders 2 to 6 also achieve the maximum deviations of 5.98, 1.31, 6.25, 2.72, and 3.42 samples, respectively, from the constant group delay. Figures 4(a)-4(e) show the AME response comparison plots for the designed FODIs. The magnitude response plots of the HFPA based designs are shown in Figure 4(f).

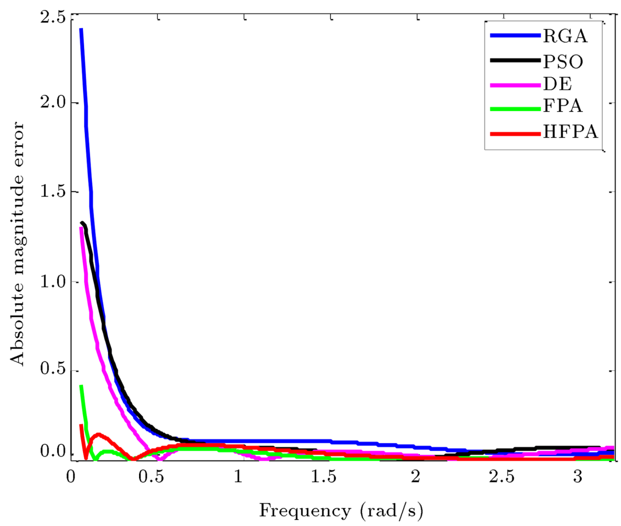

Figure 4(a). AME plots of the designed FODIs ($N = 2$, $r = 1/3$).

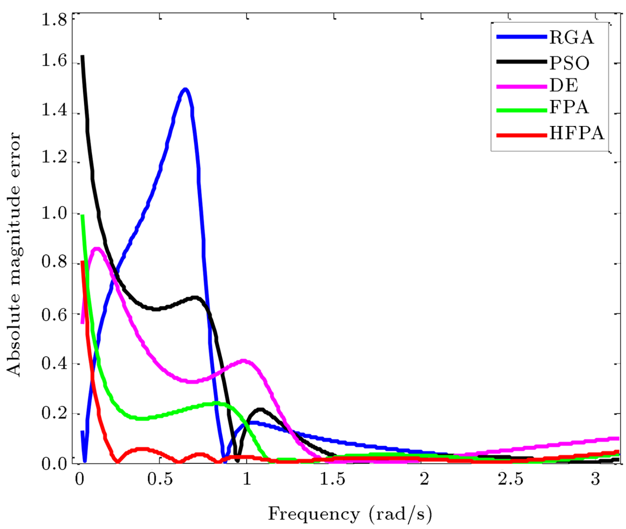

Figure 4(b). AME plots of the designed FODIs ($N = 3$, $r = 1/3$).

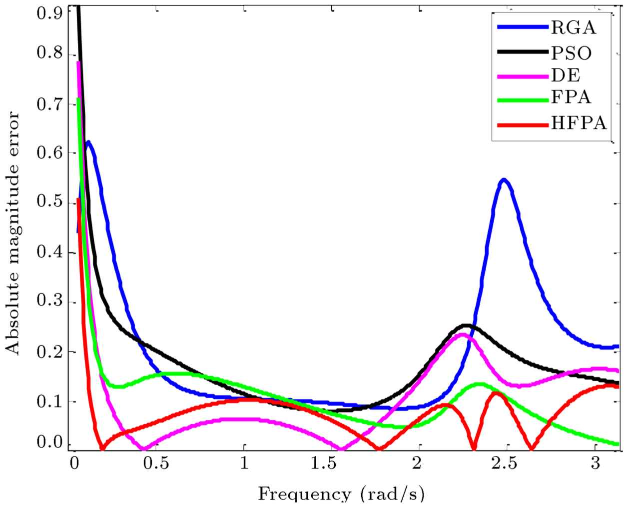

Figure 4(c). AME plots of the designed FODIs ($N = 4$, $r = 1/3$).

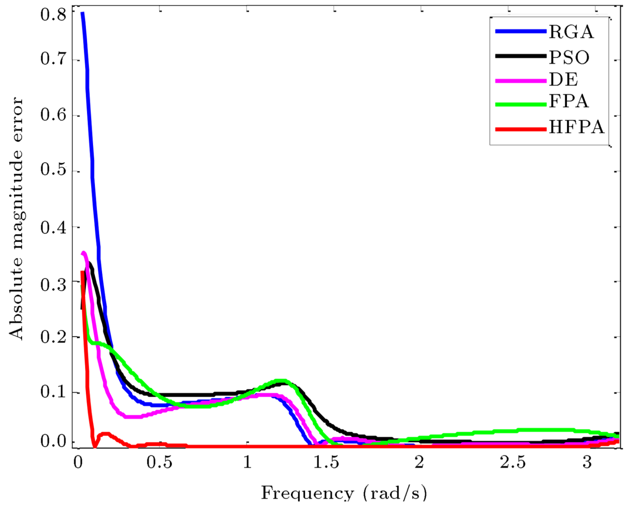

Figure 4(d). AME plots of the designed FODIs ($N = 5$, $r = 1/3$).

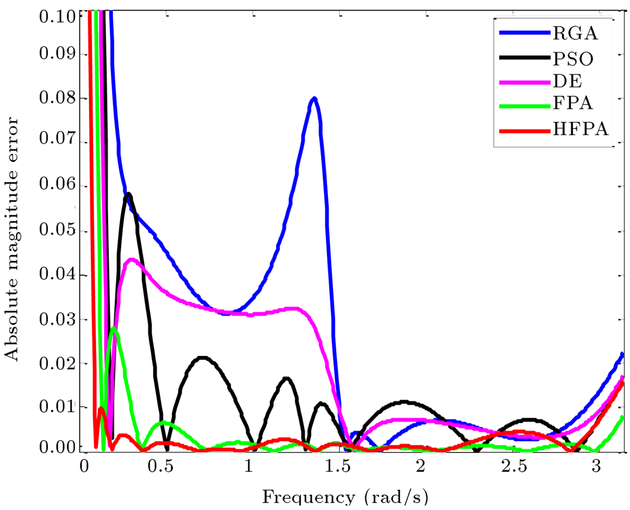

Figure 4(e). AME plots of the designed FODIs ($N = 6$, $r = 1/3$).

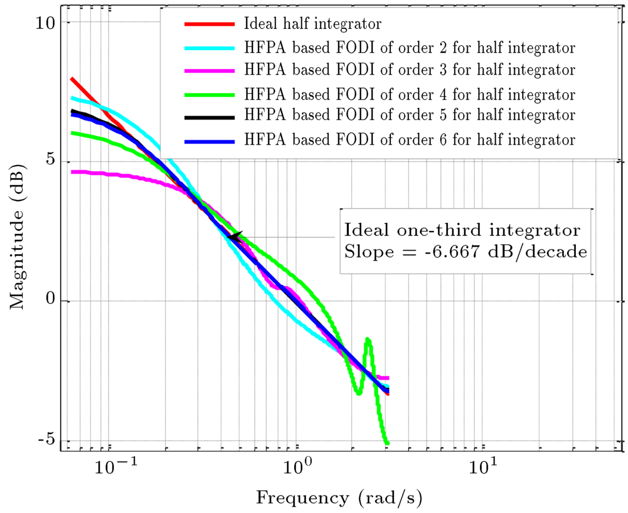

Figure 4(f). Magnitude (dB) versus frequency (rad/s) of HFPA based FODIs ($r = 1/3$).

4.2.2. Comparison using hypothesis tests

(a) ***t-Test Results.*** Table 9 shows the t-test results in terms of RMSME metric for HFPA-RGA, HFPA-PSO, HFPA-DE, and HFPA-FPA pair FODIs for the one-third-order integrator. Since HFPA rejects the null hypothesis with significantly high CL, it can be concluded that the HFPA based designs outperform the competing algorithm based designs in achieving the lowest RMSME consistently;

(b) ***Wilcoxon Rank-Sum Test Results:*** Table 10 shows the Wilcoxon Rank-Sum test results for FODIs for the one-third-order integrator to substantiate the consistently superior performance of HFPA based FODIs. Results confirm that HFPA based designs consistently attain the lowest RMSMEs.

4.2.3. Comparison with the literature

Table 11 reveals that the proposed HFPA based FODI of order 2 yields the lowest values of RMSME and MARME metrics, and hence achieves better magnitude response than the FODI introduced in [19]. Figure 5 shows the AME response comparison plots with the literature.

4.3. Design of FODIs for $r = 1/4$

The optimal coefficients attained for the designed FODIs with $r = 1/4$ are shown in Table 12.

4.3.1. Frequency response performance comparison

Table 13 shows that the HFPA based FODIs of orders 2 to 6 achieve the lowest RMSME values of −26.6 dB, −24.9 dB, −41.8 dB, −41.9 dB, and −40.7 dB, respectively. The proposed HFPA based FODIs of orders 2 to 6 also outperform the other algorithms by achieving the lowest MARME values of −15.1

Table 7. Coefficients of the designed FODIs ($r = 1/3$).

N	Algorithm	$[a_1 a_2 ... a_{N+1}]$	$[b_1 b_2 ... b_{N+1}]$
2	RGA	[3.849702 –1.656906 –0.540519]	[–3.582701 3.294709 0.029722]
	PSO	[3.559601 –1.167211 –0.572626]	[–3.580500 3.125610 0.026844]
	DE	[0.457204 –1.181013 1.981105]	[2.180009 –1.886018 0.744228]
	FPA	[0.923444 1.813501 –1.208672]	[2.467210 –1.177242 –0.577933]
	HFPA	**[–1.790011 2.887020 0.834334]**	**[3.608007 –2.330032 –0.474819]**
3	RGA	[1.209901 –1.580061 1.108011 –0.167809]	[1.175003 –2.209109 1.768214 –0.495523]
	PSO	[1.094004 –1.578013 1.114007 –0.167231]	[1.411701 –2.085005 1.687015 –0.495827]
	DE	[1.436001 –1.474011 1.134020 –0.314546]	[1.310010 –2.017031 1.600127 –0.646113]
	FPA	[1.451834 –1.734620 1.385282 –0.311056]	[1.663138 –2.408248 1.992977 –0.732253]
	HFPA	**[1.105006 –1.585014 1.105048 –0.172639]**	**[1.181006 –2.099010 1.680136 –0.498668]**
4	RGA	[–0.257101 –1.419022 1.132006 2.420152 1.670195]	[2.283807 1.108016 –0.866811 –1.402064 0.0184
	PSO	[–0.570907 –1.456029 1.699074 2.362017 1.883098]	[3.819704 0.988415 –0.498414 –1.722094 –0.216716]
	DE	[–0.649701 –1.243012 1.363024 2.253019 1.651085]	[2.818900 1.013001 –0.547105 –1.346001 –0.015947]
	FPA	[–0.016409 –1.537480 1.431536 2.198137 1.919138]	[3.580835 0.690101 –0.543795 –1.595067 0.036167]
	HFPA	**[–0.814501 –1.365010 1.437030 2.465046 1.688062]**	**[2.644014 1.293032 –0.628511–1.555037 –0.097804]**
5	RGA	[1.768002 –1.596008 1.099032 –0.261913 –0.400014 0.048226]	[1.787300 –2.410012 1.691001 –0.719211 –0.348036 0.163517]
	PSO	[1.878103 –1.569005 1.090016 –0.298718 –0.372041 0.038862]	[1.880001 –2.379003 1.670008 –0.767612 –0.298827 0.148095]
	DE	[1.749001 –1.584003 1.034210 –0.349029 –0.373724 0.074148]	[1.784303 –2.373017 1.624006 –0.771131 –0.273321 0.172831]
	FPA	[1.649850 –1.567750 1.016260 –0.314635 –0.358904 0.045763]	[1.845160 –2.392870 1.625080 –0.793486 –0.256408 0.158132]
	HFPA	**[1.577002 –1.648020 1.228016 –0.143326 –0.397535 0.054146]**	**[1.670016 –2.368003 1.847014 –0.624874 –0.385361 0.153058]**
6	RGA	[1.863005 –1.460021 1.012007 –0.555811 –0.480542 0.024239 0.065921]	[1.896301 –2.291015 1.520004 –1.013046 –0.316713 0.191852 0.100079]
	PSO	[1.766005 –1.636008 1.107001 –0.225811 –0.397419 0.0339347 0.086958]	[1.879002 –2.419003 1.705021 –0.676714 –0.343926 0.151835 0.088375]
	DE	[1.715005 –1.666503 0.894618 –0.423832 –0.351538 0.148742 0.015122]	[1.808019 –2.444702 1.449114 –0.801124 –0.205929 0.273433 –0.003315]
	FPA	[1.756016 –1.302020 1.141044 –0.369500 –0.340301 0.105635 –0.022971]	[1.861001 –2.073029 1.609010 –0.865711 –0.250845 0.187438 –0.038168]
	HFPA	**[1.743210 –1.708200 0.958822 –0.417338 –0.233577 0.031543 0.029168]**	**[1.840270 –2.502540 1.574310 –0.814188 –0.093902 0.107073 0.039445]**

Table 8. RMSME (dB)/MARME (dB)/τ_m (samples)/t_x (seconds) comparison for the designed FODIs ($r = 1/3$).

N	RGA	PSO	DE	FPA	HFPA
2	–9.1/–0.3/ 5.94/103.336	–11.4/–4.8/ 5.37/82.679	–14.0/–5.7/ 3.11/91.212	–22.2/–15.4/ 2.43/89.116	–25.8/–22.0/ 5.98/102.039
3	–6.0/2.2/ 5.51/107.351	–8.0/–3.7/ 4.76/84.180	–10.3/–6.5/ 2.92/93.168	–15.6/–8.0/ 1.33/91.397	–20.9/–9.8/ 1.31/105.915
4	–11.9/–2.6/ 7.08/110.920	–14.0/–8.9/ 5.11/88.236	–16.9/–10.1/ 5.16/96.664	–17.9/–10.9/ 5.45/94.912	–21.1/–13.8/ 6.25/108.297
5	–18.2/–10.0/ 4.61/112.700	–20.8/–16.0/ 3.78/89.560	–22.9/–16.5/ 4.15/97.960	–24.0/–18.2/ 3.54/96.005	–32.2/–17.9/ 2.72/110.949
6	–22.0/–10.5/ 2.92/116.152	–24.4/–12.1/ 1.84/94.584	–26.0/–14.2/ 3.51/102.044	–31.2/–17.1/ 2.61/99.257	–42.5/–27.1/ 3.42/114.613

Table 9. Results of the t-test.

N	Algorithm pair	t_{cal}	Hypothesis decision
2	HFPA-RGA	3.125	Reject with 99.75% CL
	HFPA-PSO	1.954	Reject with 95% CL
	HFPA-DE	0.913	Reject with 75% CL
	HFPA-FPA	0.742	Reject with 75% CL
3	HFPA-RGA	3.997	Reject with 99.95% CL
	HFPA-PSO	2.647	Reject with 99% CL
	HFPA-DE	1.315	Reject with 90% CL
	HFPA-FPA	0.851	Reject with 75% CL
4	HFPA-RGA	2.259	Reject with 99% CL
	HFPA-PSO	1.429	Reject with 90% CL
	HFPA-DE	0.814	Reject with 75% CL
	HFPA-FPA	0.685	Reject with 75% CL
5	HFPA-RGA	1.513	Reject with 90% CL
	HFPA-PSO	0.716	Reject with 75% CL
	HFPA-DE	0.421	Reject with 60% CL
	HFPA-FPA	0.274	Reject with 60% CL
6	HFPA-RGA	0.729	Reject with 75% CL
	HFPA-PSO	0.427	Reject with 60% CL
	HFPA-DE	0.365	Reject with 60% CL
	HFPA-FPA	0.286	Reject with 60% CL

dB, −12.2 dB, −31.5 dB, −33.4 dB, and −29.4 dB, respectively. The proposed designs of orders 2 to 6 also achieve a shorter group delay by yielding the maximum deviations of only 0.99, 0.96, 2.85, 3.45, and 1.17 samples, respectively, from the constant group delay. Figures 6(a)-6(e) show the AME response comparison

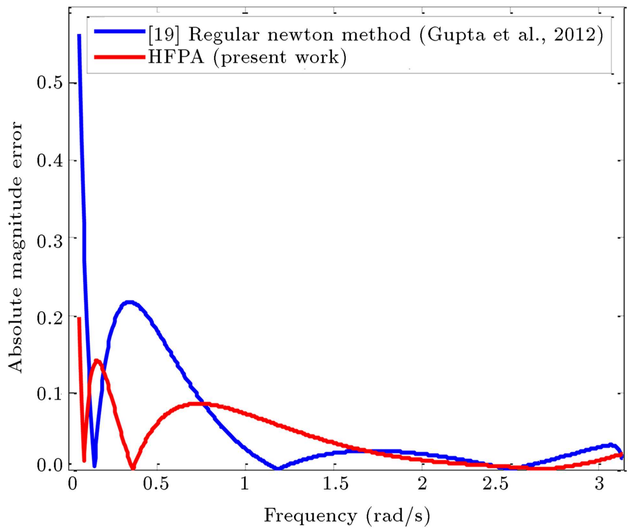

Figure 5. AME plots for comparison with the reported FODIs for $N = 2$ and $r = 1/3$.

plots and Figure 6(f) shows the magnitude responses of the proposed FODIs.

4.3.2. Comparison using hypothesis tests

(a) ***t*-Test Results:** Table 14 shows the t-test results in terms of the RMSME metric for different algorithm pairs for FODIs with $r = 1/4$. The proposed designs consistently outperform the benchmark algorithm based designs by yielding the smallest value of RMSME.

(b) ***Wilcoxon Rank-Sum Test Results:*** Table 15 shows the Wilcoxon rank-sum test results for HFPA-RGA, HFPA-PSO, HFPA-DE, and HFPA-FPA pairs for the one-fourth-order integrator to confirm the consistently improved magnitude response characteristics of the HFPA based designs. Test results confirm that the proposed FODIs consistently attain the lowest RMSME.

Table 10. Results of the Wilcoxon rank-sum test.

N	Sample length		U_C		Competing algorithm							
					RGA		PSO		DE		FPA	
	n_1	n_2	$\alpha = 0.05$	$\alpha = 0.01$	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
2	6	9	31	26	21	99	29	91	45	75	46	74
	8	12	58	51	36	174	56	154	76	134	75	135
3	6	9	31	26	21	99	25	95	34	86	36	84
	8	12	58	51	36	174	48	162	63	147	66	144
4	6	9	31	26	25	95	35	85	51	69	52	68
	8	12	58	51	49	161	69	141	79	131	80	130
5	6	9	31	26	34	86	51	69	51	69	53	67
	8	12	58	51	61	149	74	136	79	131	79	131
6	6	9	31	26	42	78	53	67	51	6 9	52	68
	8	12	58	51	71	139	81	129	79	131	75	135

Table 11. Comparison of HFPA based FODIs ($r = 1/3$) with the literature.

N	Reference	Approach	RMSME (dB)	MARME (dB)	τ_m (samples)
2	[19]	Regular Newton	−20.8	−12.9	1.49
	Present work	**HFPA**	**−25.8**	**−22.0**	**5.98**

4.3.3. Comparison with the literature

Table 16 confirms that the proposed HFPA based FODIs achieve the best magnitude response performances in comparison with the design approaches reported in recent literature. Figures 7(a)-7(c) show the AME response comparison plots for the proposed designs and the literature.

4.4. FODIs for integrators of any arbitrary fractional order

Although FODIs for $r = 1/2$, 1/3, and 1/4 are proposed in this paper, FODIs for integrators of any arbitrary fractional order can be implemented by the presented approach. This section demonstrates the implementation of FODIs for two arbitrarily chosen orders, namely, $r = 8/9$ and $r = 0.9876$. It justifies that the proposed FODI design technique can cope with any value of fractional order.

The optimal coefficients obtained for the proposed HFPA based FODIs for the 8/9 and 0.9876 FOIs by minimizing the objective function in Eq. (5) are shown in Table 17.

The magnitude and AME response plots for the HFPA based FODIs for $r = 8/9$ and $r = 0.9876$ are presented in Figure 8(a) to 8(d). Table 18 shows that the HFPA based FODIs of orders 2 to 6 achieve the RMSME values of −15.0 dB, −20.9 dB, −31.2 dB, −33.3 dB, and −40.8 dB, respectively, for the 8/9-order integrator, and −25.1 dB, −25.6 dB, −34.0 dB, −36.4 dB, and −49.2 dB, respectively, for the 0.9876-order integrator. The proposed HFPA based FODIs of orders 2 to 6 achieve the MARME values of −13.2 dB, −20.1 dB, −20.8 dB, −31.0 dB, and −27.7 dB, respectively, for the 8/9-order integrator, and −18.4 dB, −20.2 dB, −17.9 dB, −23.8 dB, and −25.7 dB, respectively, for the 0.9876-order integrator. The proposed designs also achieve a smaller group delay.

4.5. Sensitivity analysis

The performances of the designed FODIs are investigated by varying the control parameter values of HFPA. After conducting 30 runs for each set of parameter values, the sensitivity test results are presented in Tables 19-21 for the HFPA based FODIs for $r = 1/2$, 1/3, and 1/4, respectively. Results reveal that the most accurate magnitude responses of FODIs are obtained by choosing the parameter values of HFPA as shown in Table 1. It is worthwhile to note that the choice of the orders of the FODIs for presenting the results is completely arbitrary, and similar results are also obtained for the other FODI orders.

5. Conclusions

In this paper, HFPA has efficiently been used to design accurate IIR Fractional Order Digital Integrators. HFPA is a hybrid meta-heuristic optimization algorithm that combines the exploration and exploitation capabilities of both PSO and FPA to achieve global optimality. The major contributions underlined in this

Table 12. Coefficients of the designed FODIs ($r = 1/4$).

N	Algorithm	$[a_1 a_2 ... a_{N+1}]$	$[b_1 b_2 ... b_{N+1}]$
2	RGA	[2.948901 0.705415 –0.428325]	[2.163103 0.074809 –0.755734]
	PSO	[2.834402 0.257109 –0.447134]	[2.947401 0.760905 –0.794253]
	DE	[2.162600 1.192006 –0.196624]	[2.971103 0.674713 –0.535231]
	FPA	[2.589320 0.558791 –0.758813]	[2.820110 –0.166985 –1.132730]
	HFPA	**[1.268007 0.028401 –0.235016]**	**[1.327004 –0.324501 –0.325227]**
3	RGA	[1.290004 –1.544606 1.296017 –0.259645]	[1.721402 –1.934001 1.695088 –0.537459]
	PSO	[1.123005 –1.666014 1.213116 –0.217809]	[1.302000 –2.059032 1.649109 –0.472256]
	DE	[1.103901 –1.603022 1.182048 –0.227539]	[1.090008 –1.962018 1.597124 –0.466941]
	FPA	[1.109804 –1.607013 1.202517 –0.211928]	[1.168701 –1.987060 1.649099 –0.471545]
	HFPA	**[1.120976 –1.625051 1.262763 –0.242217]**	**[1.173807 –2.012434 1.695455 –0.516428]**
4	RGA	[0.977301 –1.804806 0.963112 –0.054734 –0.047662]	[1.010003 –2.177011 1.478022 –0.278957 –0.028674]
	PSO	[0.952101 –1.809016 0.960114 –0.047728 –0.035835]	[1.001912 –2.179034 1.467022 –0.265119 –0.020872]
	DE	[0.974802 –1.790014 0.964217 –0.060732 –0.063145]	[1.019308 –2.164029 1.487011 –0.293714 –0.031869]
	FPA	[0.961005 –1.803014 0.965920 –0.068415 –0.025623]	[1.001010 –2.165003 1.471058 –0.269719 –0.021437]
	HFPA	**[0.973507 –1.781163 0.963079 –0.062528 –0.055459]**	**[1.012722 –2.150031 1.482519 –0.301120 –0.027812]**
5	RGA	[0.971601 –1.778803 0.943814 –0.059927 –0.050838 0.007264]	[1.011001 –2.150006 1.450108 –0.277741 –0.029362 0.006998]
	PSO	[0.971103 –1.786007 0.942419 –0.061234 –0.050047 0.007304]	[1.011102 –2.155001 1.449006 –0.278317 –0.029321 0.007523]
	DE	[0.971200 –1.788001 0.943318 –0.060724 –0.049529 0.007047]	[1.011006 –2.154009 1.450025 –0.277833 –0.028324 0.006716]
	FPA	[0.974559 –1.790670 0.945260 –0.064520 –0.047037 0.006702]	[1.011100 –2.151510 1.449660 –0.279880 –0.026036 0.006440]
	HFPA	**[0.957501 –1.802007 0.945010 –0.053515 –0.039428 0.000731]**	**[0.987403 –2.161004 1.471029 –0.271831 –0.034915 0.010245]**
6	RGA	[0.982501 –1.829001 0.958906 –0.059510 –0.030004 0.003418 –0.008361]	[1.023004 –2.191002 1.482014 –0.286721 –0.001130 –0.010524 –0.003142]
	PSO	[0.988403 –1.820001 0.958712 –0.060224 –0.029716 0.003617 –0.009149]	[1.024001 –2.192003 1.481009 –0.286914 –0.001719 –0.010125 –0.003434]
	DE	[0.982504 –1.826002 0.958504 –0.059408 –0.030041 0.002937 –0.008204]	[1.023001 –2.192009 1.482011 –0.287702 –0.001124 –0.011319 –0.002617]
	FPA	[0.979704 –1.829480 0.957939 –0.065007 –0.026939 0.002873 –0.006679]	[1.022040 –2.199320 1.482410 –0.288917 0.002291 –0.012114 –0.000234]
	HFPA	**[0.969801 –1.814006 0.952511 –0.051324 –0.051019 0.003341 0.003400]**	**[1.005000 –2.177001 1.475014 –0.280817 –0.024224 0.002636 0.002605]**

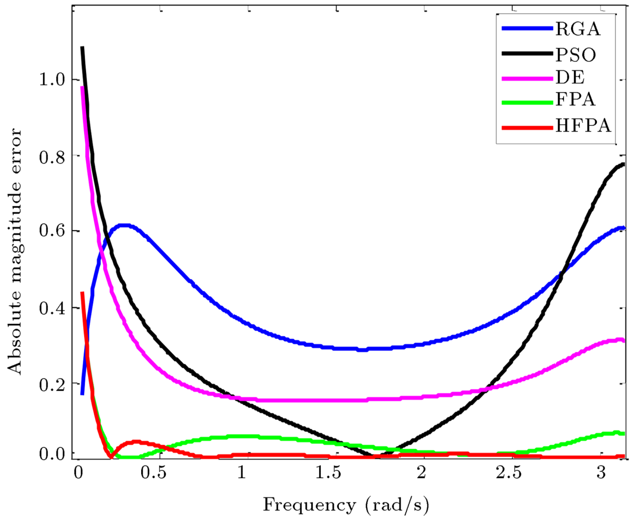

Figure 6(a). AME plots of the designed FODIs ($N = 2$, $r = 1/4$).

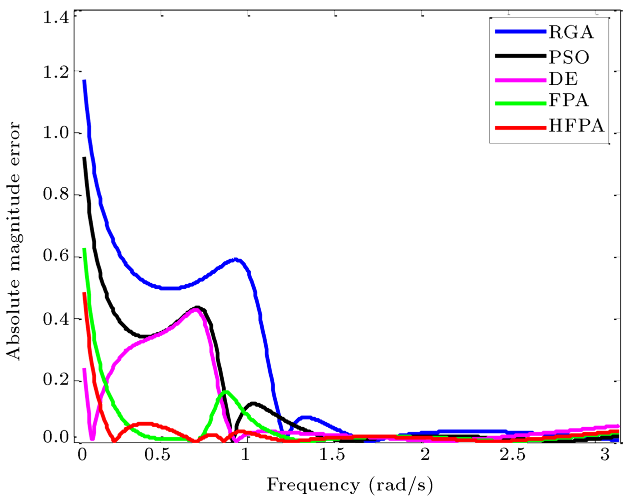

Figure 6(b). AME plots of the designed FODIs ($N = 3$, $r = 1/4$).

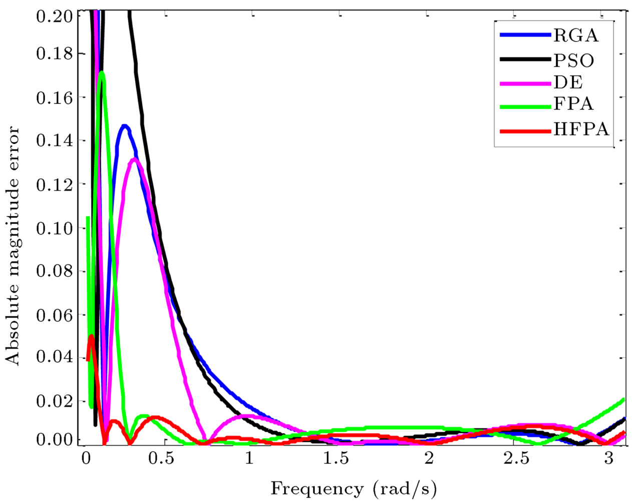

Figure 6(c). AME plots of the designed FODIs ($N = 4$, $r = 1/4$).

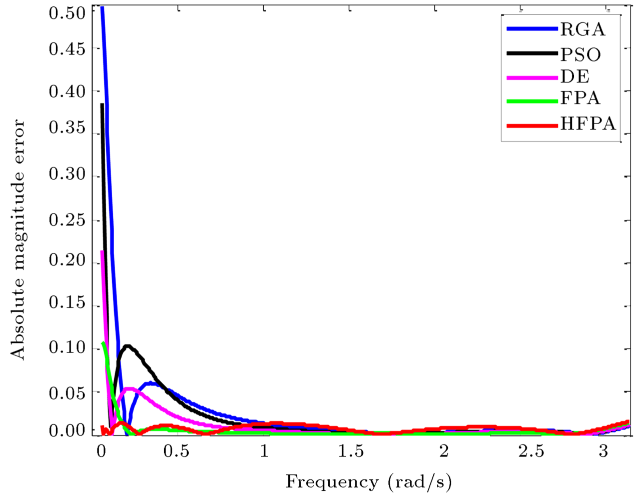

Figure 6(d). AME plots of the designed FODIs ($N = 5$, $r = 1/4$).

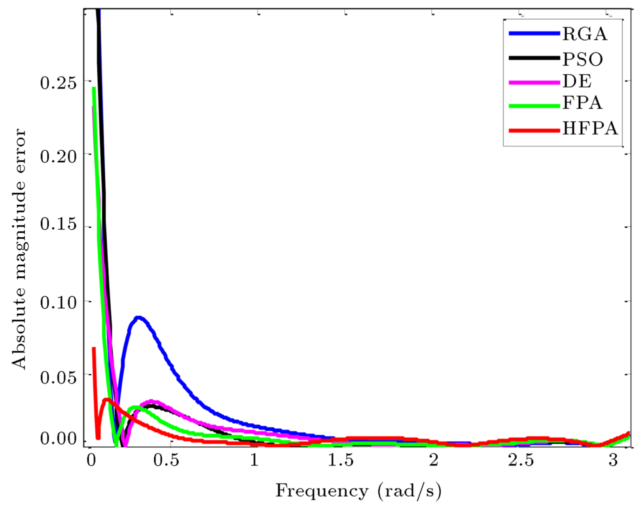

Figure 6(e). AME plots of the designed FODIs ($N = 6$, $r = 1/4$).

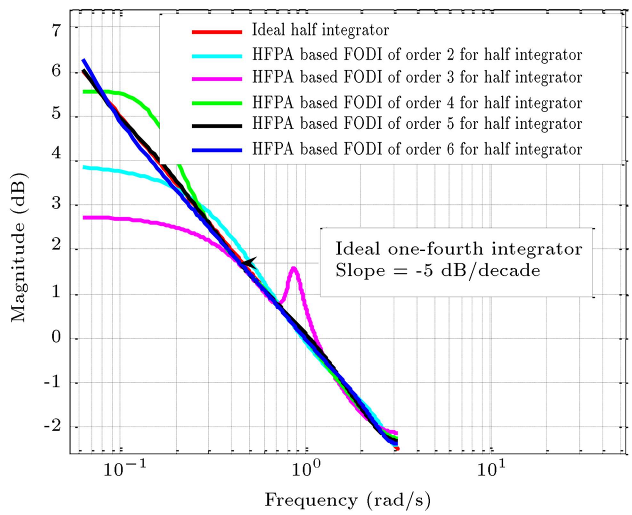

Figure 6(f). Magnitude (dB) versus frequency (rad/s) of HFPA based FODIs ($r = 1/4$).

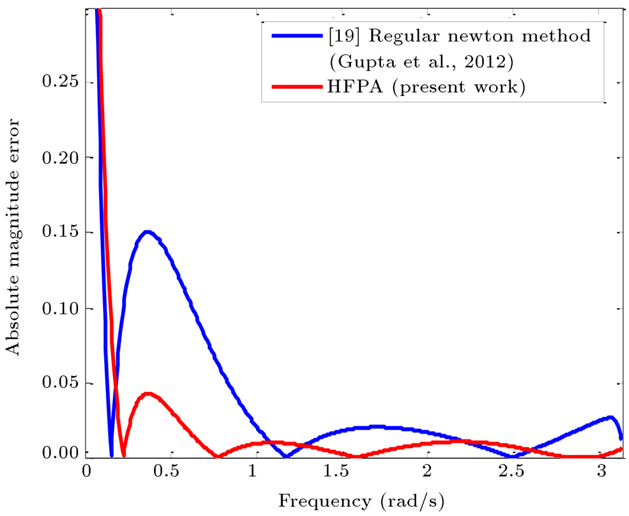

Figure 7(a). AME plots for comparison with the reported FODIs for $N = 2$ and $r = 1/4$.

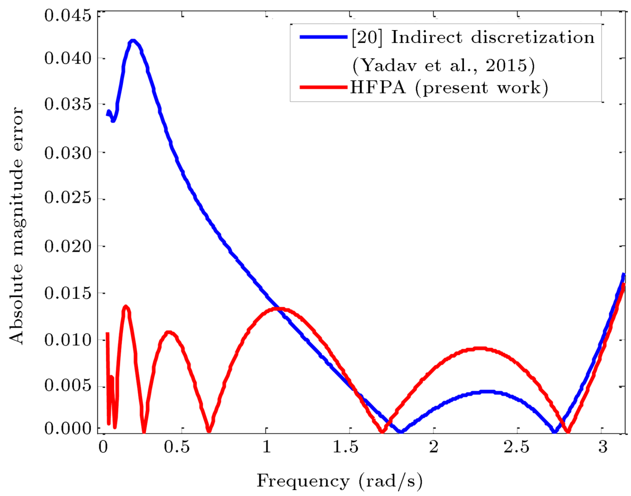

Figure 7(b). AME plots for comparison with the reported FODIs for $N = 5$ and $r = 1/4$.

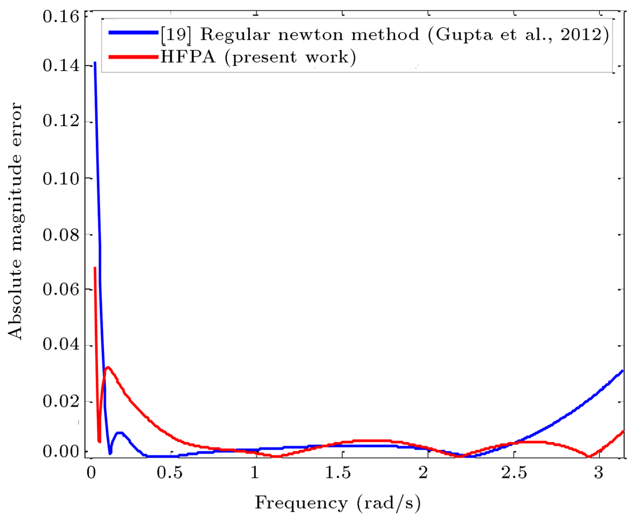

Figure 7(c). AME plots for comparison with the reported FODIs for $N = 6$ and $r = 1/4$.

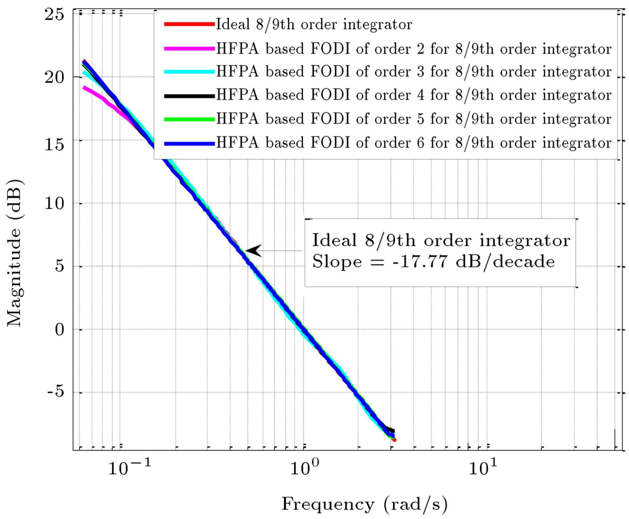

Figure 8(a). Magnitude (dB) versus frequency (rad/s) of HFPA based FODIs ($r = 8/9$).

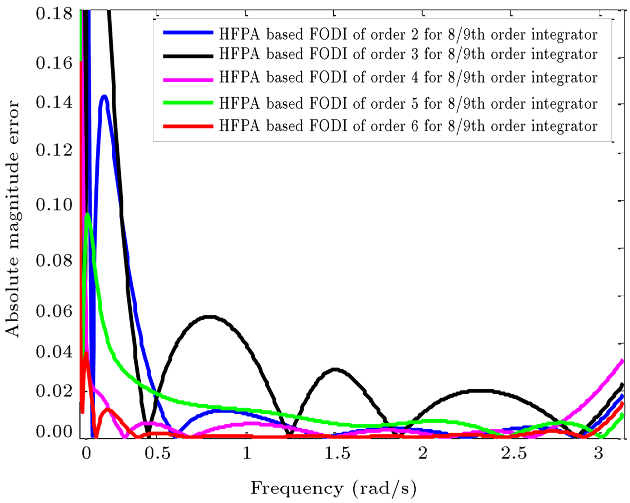

Figure 8(b). AME plots of HFPA based FODIs ($r = 8/9$).

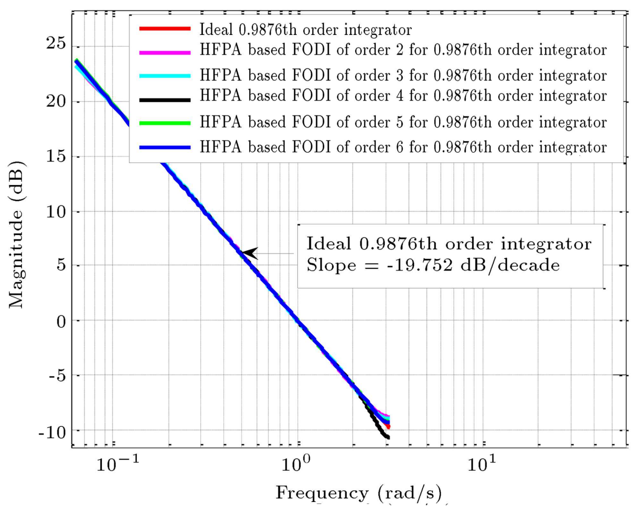

Figure 8(c). Magnitude (dB) versus frequency (rad/s) of HFPA based FODIs ($r = 0.9876$).

Table 13. RMSME (dB)/MARME (dB)/τ_m (samples)/t_x (seconds) comparison for the designed FODIs ($r = 1/4$).

N	RGA	PSO	DE	FPA	HFPA
2	–7.4/–1.8/ 3.68/82.952	–9.0/0.3/ 1.34/76.096	–11.7/–6.1/ 1.07/78.993	–24.5/–12.1/ 1.26/78.269	–26.6/–15.1/ 0.99/82.242
3	–9.0/–4.6/ 4.17/107.368	–12.9/–6.6/ 3.22/84.363	–15.8/–8.1/ 2.11/91.432	–20.9/–10.0/ 1.61/88.016	–24.9/–12.2/ 0.96/105.614
4	–19.6/–4.5/ 2.57/114.952	–22.2/–10.6/ 2.73/92.348	–24.8/–12.6/ 1.24/100.540	–30.5/–19.6/ 2.59/97.382	–41.8/–31.5/ 2.85/112.349
5	–24.4/–11.9/ 4.16/115.876	–28.0/–14.2/ 1.49/92.524	–33.1/–19.4/ 2.21/100.704	–37.5/–25.3/ 2.71/98.156	–41.9/–33.4/ 3.45/113.913
6	–25.1/–11.9/ 1.31/116.988	–27.2/–14.3/ 3.49/93.265	–30.9/–18.5/ 1.53/103.316	–32.0/–19.5/ 1.09/99.671	–40.7/–29.4/ 1.17/115.785

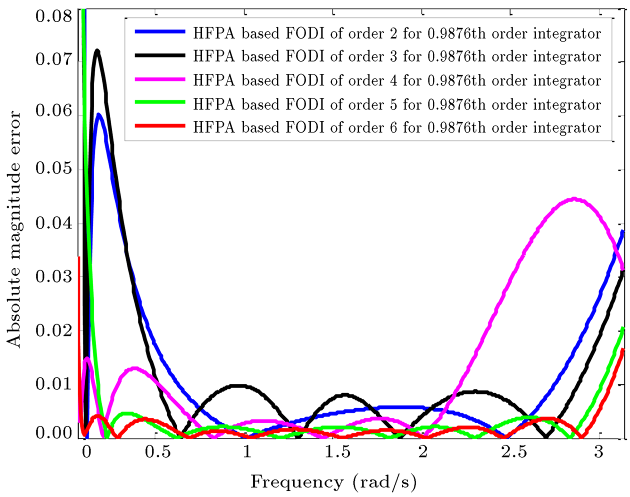

Figure 8(d). AME plots of HFPA based FODIs ($r = 0.9876$).

paper are:

(a) The proposed designs obtain superior magnitude responses over those of the designs based on 4 other evolutionary optimization algorithms;

(b) Results of the hypothesis tests show that HFPA based FODIs consistently achieve the most accurate designs;

(c) It is demonstrated that the proposed approach is a generic one, which can provide an optimal solution for the digital realization of FOIs for any arbitrary value of the fractional order;

(d) The proposed designs are suitable for real-time embedded digital signal processing and control applications due to the lower orders of the filters;

(e) The proposed designs also outperform the designs recently published in the literature.

Table 14. Results of the t-test.

N	Algorithm pair	t_{cal}	Accept/reject null hypothesis
2	HFPA-RGA	3.593	Reject with 99.95% CL
	HFPA-PSO	2.426	Reject with 99% CL
	HFPA-DE	1.161	Reject with 75% CL
	HFPA-FPA	0.982	Reject with 75% CL
3	HFPA-RGA	3.269	Reject with 99.9% CL
	HFPA-PSO	1.541	Reject with 90% CL
	HFPA-DE	0.724	Reject with 75% CL
	HFPA-FPA	0.548	Reject with 60% CL
4	HFPA-RGA	1.372	Reject with 90% CL
	HFPA-PSO	0.764	Reject with 75% CL
	HFPA-DE	0.394	Reject with 60% CL
	HFPA-FPA	0.371	Reject with 60% CL
5	HFPA-RGA	2.637	Reject with 99% CL
	HFPA-PSO	1.724	Reject with 95% CL
	HFPA-DE	0.719	Reject with 75% CL
	HFPA-FPA	0.595	Reject with 60% CL
6	HFPA-RGA	1.419	Reject with 90% CL
	HFPA-PSO	0.783	Reject with 75% CL
	HFPA-DE	0.419	Reject with 60% CL
	HFPA-FPA	0.386	Reject with 60% CL

Thus, HFPA exhibits a promising performance in solving this multimodal optimization problem in the domain of signal processing. Since only one-dimensional integrators of fractional order are proposed here, in future, this research work can be ex-

Table 15. Results of the Wilcoxon rank-sum test.

N	Sample length		U_C		Competing algorithm							
					RGA		PSO		DE		FPA	
	n_1	n_2	$\alpha = 0.05$	$\alpha = 0.01$	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
2	6	9	31	26	21	99	25	95	48	72	48	72
	8	12	58	51	36	174	49	161	77	133	79	131
3	6	9	31	26	21	99	34	86	49	71	50	70
	8	12	58	51	36	174	60	150	77	133	77	133
4	6	9	31	26	33	87	43	77	58	62	56	64
	8	12	58	51	62	148	74	136	82	128	84	126
5	6	9	31	26	24	96	28	92	48	72	51	69
	8	12	58	51	48	162	56	154	75	135	76	134
6	6	9	31	26	36	84	47	73	58	62	57	63
	8	12	58	51	61	149	76	134	85	125	86	124

Table 16. Comparison of HFPA based FODIs ($r = 1/4$) with the literature.

N	Reference	Approach	RMSME (dB)	MARME (dB)	τ_m (samples)
2	[19]	Regular Newton	–23.9	–14.9	1.07
	Present work	**HFPA**	**–26.6**	**–15.1**	**0.99**
5	[20]	Indirect discretization	–35.3	–30.7	1.62
	Present work	**HFPA**	**–41.9**	**–33.4**	**3.45**
6	[19]	Regular Newton	–36.6	–22.9	2.30
	Present work	**HFPA**	**–40.7**	**–29.4**	**1.17**

Table 17. Coefficients for the HFPA based FODIs ($r = 8/9$ and $r = 0.9876$).

r	N	$[a_1 a_2 \ldots a_{N+1}]$	$[b_1 b_2 \ldots b_{N+1}]$
8/9	2	[0.036001 –0.208811 –0.822509]	[0.953205 –0.717602 –0.150321]
	3	[0.146101 1.045007 –0.202914 0.374126]	[1.254004 –1.407009 0.710116 –0.467320]
	4	[0.982800 –0.323804 –0.149301 –0.099015 –0.009626]	[1.124001 –1.512006 0.345904 –0.058922 0.117907]
	5	[0.978901 –0.304705 –0.173211 –0.103227 –0.029836 0.012224]	[1.117002 –1.503009 0.339116–0.072512 0.127609 0.007630]
	6	[0.975905 –0.315001 –0.154521 –0.089618 –0.019310 0.011409 –0.003134]	[1.119001 –1.512004 0.355403 –0.064114 0.117319 0.000807 –0.001116]
0.9876	2	[0.398401 1.499006 0.038712]	[1.724000 –1.440012 –0.229106]
	3	[0.224100 1.087001 –0.123518 0.382124]	[1.310009 –1.460002 0.642316 –0.447924]
	4	[0.979805 –0.222609 –0.272414 –0.046291 –0.025127]	[1.162003 –1.678001 0.382912 0.048422 0.084712]
	5	[0.870502 –0.286301 –0.146911 –0.124619 0.004327 0.011742]	[1.004000 –1.493003 0.437401 –0.106509 0.198821 –0.039322]
	6	[1.044007 –0.207103 –0.132110 –0.037121 0.052753 –0.024621 –0.011746]	[1.208002 –1.634010 0.397306 –0.028294 0.147518 –0.132822 0.048972]

Table 18. RMSME (dB)/MARME (dB)/τ_m (samples) comparison for the designed FODIs ($r = 8/9$, $r = 0.9876$).

r	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
8/9	–15.0/–13.2/7.65	–20.9/–20.1/8.24	–31.2/–20.8/6.27	–33.3/–31.0/5.38	–40.8/–27.7/5.21
0.9876	–25.1/–18.4/5.88	–25.6/–20.2/5.94	–34.0/–17.9/0.78	–36.4/–23.8/1. 22	–49.2/–25.7/2.51

Table 19. Sensitivity test results for RMSME (dB)/MARME (dB) for HFPA based FODI ($N = 2$, $r = 1/2$).

Control parameter						Best	Worst	Mean	Standard deviation
n	C_1, C_2	$w_{\max}, w_{\min}$	p	γ	λ				
20						−20.0/−16.8	−8.6/−5.1	−14.9/−12.3	2.59/2.53
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
100						−22.0/−18.1	−9.8/−7.6	−15.9/−13.2	2.61/2.67
	1.50, 1.50					−19.2/−16.0	−9.3/−6.2	−15.8/−12.6	2.71/2.69
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
	2.50, 2.50					−21.1/−17.7	−9.7/−7.0	−15.4/−12.0	2.76/2.63
		0.8, 0.4				−19.7/−16.4	−8.7/−5.5	−14.9/−11.6	2.62/2.55
50	2.05, 2.05	0.9, 0.4	0.8	0.1	1.5	−20.3/−17.5	−9.9/−7.1	−15.7/−12.4	2.59/2.64
		1.0, 0.4				−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
			0.5			−18.1/−15.4	−7.9/−6.0	−14.5/−11.7	2.63/2.75
50	2.05, 2.05	1.0, 0.4	0.7	0.1	1.5	−19.3/−16.7	−8.9/−6.7	−14.9/−12.9	2.58/2.61
			0.8			−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
			0.9			−21.0/−17.4	−9.7/−7.1	−15.8/−12.8	2.61/2.68
				0.05		−19.0/16.8	−9.1/−6.2	−15.3/−11.2	2.77/2.74
50	2.05, 2.05	1.0, 0.4	0.8	0.10	1.5	−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
				0.15		−19.6/−17.5	−9.3/−6.5	−14.9/−11.6	2.62/2.70
					1.2	−20.2/−16.0	−9.1/−7.0	−14.4/−12.7	2.66/2.59
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−21.9/−18.2	−10.4/−7.6	−16.2/−13.1	2.56/2.52
					1.8	−19.7/−16.3	−9.4/−6.1	−15.1/−11.9	2.57/2.61

Table 20. Sensitivity test results for RMSME (dB)/MARME (dB) for HFPA based FODI ($N = 2$, $r = 1/3$).

Control parameter						Best	Worst	Mean	Standard deviation
n	C_1, C_2	$w_{\max}, w_{\min}$	p	γ	λ				
20						−22.3/−19.3	−14.0/−11.6	−19.2/−15.7	2.68/2.25
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
100						−26.0/−20.1	−15.2/−12.7	−20.6/−17.9	2.77/2.34
	1.50, 1.50					−23.1/−19.4	−14.4/−11.2	−18.1/−14.3	2.81/2.38
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
	2.50, 2.50					−25.0/−20.6	−15.0/−12.8	−19.3/−15.8	2.75/2.28
		0.8, 0.4				−23.9/−20.7	−13.6/−11.5	−19.0/−14.7	2.66/2.51
50	2.05, 2.05	0.9, 0.4	0.8	0.1	1.5	−24.8/−21.5	−14.5/−13.2	−20.1/−16.9	2.79/2.40
		1.0, 0.4				−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
			0.5			−20.8/−17.8	−12.2/−9.7	−17.5/−15.1	2.74/2.35
50	2.05, 2.05	1.0, 0.4	0.7	0.1	1.5	−24.0/−21.4	−14.8/−11.9	−19.8/−16.0	2.88/2.26
			0.8			−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
			0.9			−24.9/−21.1	−14.9/−12.7	−20.1/−15.7	2.62/2.24
				0.05		−22.5/−19.4	−13.2/−10.9	−18.0/−15.2	2.89/2.50
50	2.05, 2.05	1.0, 0.4	0.8	0.10	1.5	−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
				0.15		−23.7/−20.6	−12.9/−12.5	−18.6/−15.9	2.76/2.29
					1.2	−22.9/−19.5	−13.7/−11.1	−19.1/−14.8	2.66/2.36
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	−25.8/−22.0	−15.9/−13.4	−20.7/−17.4	2.60/2.19
					1.8	−23.7/−18.9	−13.0/−11.6	−18.3/−16.0	2.78/2.44

Table 21. Sensitivity test results for RMSME (dB)/MARME (dB) for HFPA based FODI ($N = 5$, $r = 1/4$).

Control parameter						Best	Worst	Mean	Standard deviation
n	C_1, C_2	$w_{\max}, w_{\min}$	p	γ	λ				
20						–38.7/–30.9	–30.2/–20.1	–35.2/–25.6	2.31/2.50
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
100						–41.7/–33.5	–32.9/–23.0	–38.0/–27.5	2.44/2.65
	1.50, 1.50					–39.4/–32.0	–31.8/–22.1	–36.1/–25.3	2.51/2.70
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
	2.50, 2.50					–40.2/–31.4	–32.0/–21.8	–35.4/–25.9	2.40/2.61
		0.8, 0.4				–40.0/–31.9	–31.4/–20.9	–35.2/–24.8	2.55/2.72
50	2.05, 2.05	0.9, 0.4	0.8	0.1	1.5	–41.2/–32.3	–32.3/–22.4	–36.9/–27.0	2.67/2.58
		1.0, 0.4				–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
			0.5			–38.4/–29.7	–29.1/–19.6	–34.5/–24.6	2.37/2.65
50	2.05, 2.05	1.0, 0.4	0.7	0.1	1.5	–39.2/–30.4	–30.0/–20.1	–36.1/–25.1	2.48/2.81
			0.8			–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
			0.9			–40.7/–31.9	–31.8/–22.0	–37.2/–26.9	2.57/2.74
				0.05		–39.5/–32.8	–31.1/–20.2	–35.4/–25.6	2.36/2.63
50	2.05, 2.05	1.0, 0.4	0.8	0.10	1.5	–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
				0.15		–40.1/–32.3	–31.9/–21.3	–36.3/–26.2	2.41/2.68
					1.2	–39.7/–30.9	–30.1/–19.9	–33.7/–24.6	2.63/2.82
50	2.05, 2.05	1.0, 0.4	0.8	0.1	1.5	–41.9/–33.4	–32.7/–22.9	–37.8/–27.7	2.33/2.59
					1.8	–38.4/–31.6	–31.0/–21.2	–35.2/–26.3	2.46/2.67

tended towards using HFPA to realize two-dimensional fractional-order integrators.

Nomenclature

r	Order of fractional-order integrator
ω	Angular frequency in radians/second
N	Order of fractional-order digital integrator
n	Total number of frequency sample points
a_i	Numerator coefficients of the fractional-order digital integrator
b_i	Denominator coefficients of the fractional-order digital integrator
τ	Group delay
τ_m	Maximum sample deviation from the flat group delay response
$\theta(\omega)$	Phase angle in degrees of the fractional-order digital integrator
C_r	Rate of crossover
F	Differential weight
$g$best	Global best solution in the particle swarm optimization phase
$p$best	Personal best solution in the particle swarm optimization phase
B	Best solution in the flower pollination algorithm phase
n_p	Total number of search agents
D	Total number of decision variables
p	Switch probability
γ	Scaling factor
$L(\lambda)$	Lévy flight parameter
$\text{rand}_1, \text{rand}_2$	Random number $\in [0, 1]$
w	Inertia factor
$w_{\max}$	Maximum value of the inertia factor
$w_{\min}$	Minimum value of the inertia factor
C_1, C_2	Learning parameters
V_i	Velocity of the ith search agent
S_i	Position of the ith search agent
t_x	Computational time in seconds
t_{cal}	t-test value
n_1	Sample length for the competing algorithm
n_2	Sample length for HFPA
α	Level of significance

W_1	Sum of the ranks in the smaller sample
W_2	Sum of the ranks in the other sample
U_C	Critical value for hypothesis testing in Wilcoxon rank-sum test

Abbreviations

IIR	Infinite Impulse Response
FODI	Fractional-Order Digital Integrator
HFPA	Hybrid Flower Pollination Algorithm
PSO	Particle Swarm Optimization
FPA	Flower Pollination Algorithm
RGA	Real coded Genetic Algorithm
DE	Differential Evolution
FOI	Fractional-Order Integrator
CFE	Continued Fractions Expansion
RMSME	Root Mean Square Magnitude Error
ARME	Absolute Relative Magnitude Error
MARME	Maximum Absolute Relative Magnitude Error
AME	Absolute Magnitude Error
CL	Confidence Level

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