Scientia Iranica D (2019) 26(6), 3510–3514

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
http://scientiairanica.sharif.edu

Two sufficient conditions for the existence of path factors in graphs

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Received 9 September 2017; received in revised form 20 June 2018; accepted 15 September 2018

KEYWORDS
Network;
$P_{\geq n}$-factor;
$(P_{\geq n}, m)$-factor deleted graph;
$(P_{\geq n}, k)$-factor critical graph;
Toughness;
Connectivity.

Abstract. A graph G is called a $(P_{\geq n}, k)$-factor critical graph if $G-U$ has a $P_{\geq n}$-factor for any $U \subseteq V(G)$ with $|U| = k$. A graph G is called a $(P_{\geq n}, m)$-factor deleted graph if $G-E'$ contains a $P_{\geq n}$-factor for any $E' \subseteq E(G)$ with $|E'| = m$. In this paper, we obtain two results for graphs to be $(P_{\geq n}, k)$-factor critical graphs or $(P_{\geq n}, m)$-factor deleted graphs. The two results are best possible in some conditions.

1. Introduction

Many real-world networks can conveniently be modelled by graphs or networks. Examples include a railroad network with nodes presenting railroad stations, and links corresponding to railways between two stations, a communication network with nodes and links modelling cities and communication channels, respectively, or the World Wide Web with nodes presenting Web pages, and links corresponding to hyperlinks between them. In our daily life, many problems in network design and optimization, e.g., building blocks, the file transfer problems in computer networks, coding design, scheduling problems and so on, are related to the factors and factorizations of graphs [1,2]. For example, file transfer problem in computer networks can be converted into factorizations of graphs. The problem on telephone network design can be converted into P_2-factors of graphs. Many other applications in this field can be found in a current survey [1]. It is well known that a graph can represent a network. Vertices and edges of the graph model nodes and links between the nodes in the network. Henceforth, we use the term *graph* instead of *network*.

In this paper, we deal with only finite and undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. Given $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and set $G - S = G[V(G) \setminus S]$. Especially, we write:

$$G - x = G - \{x\} \text{ if } S = \{x\}.$$

Given a vertex x of G, let $N_G(x)$ denote the set of vertices adjacent to x in G and $d_G(x) = |N_G(x)|$ denote the degree of x in G. Given $E' \subseteq E(G)$, let $G - E'$ denote the subgraph obtained from G by deleting E'. In particular, we write $G - e = G - \{e\}$ if $E' = \{e\}$. Let $\kappa(G)$ denote the connectivity of G and $\lambda(G)$ denote the edge-connectivity of G. Chvátal [3] presented the parameter of toughness of a graph G, denoted by $t(G)$,

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doi: 10.24200/sci.2018.5151.1122

defined as follows:

$$t(G) = \min\{\frac{|X|}{\omega(G-X)} : X \subseteq V(G),$$
$$\omega(G-X) \geq 2\},$$

if G is not a complete graph, where $\omega(G-X)$ denotes the number of connected components in $G-X$. Otherwise, $t(K_n) = +\infty$, where K_n denotes a complete graph of order n. We refer the reader to [4] for the notation and terminology which are used but not described here.

A path factor of a graph G is a spanning subgraph of G with each component being a path. We denote by P_n the path on n vertices. A P_n-factor of G is defined as a spanning subgraph of G with each component isomorphic to P_n. A path factor with each component having at least n vertices is called a $P_{\geq n}$-factor. It is obvious that a path factor is a generalization of a perfect matching, which is a P_2-factor. Path factors in graphs have attracted a great deal of attention. Wang [5] characterized the bipartite graphs with $P_{\geq 3}$-factors. Kaneko [6] generalized this theorem to general graphs. For other results on graph factors, see [7-21].

A graph G is called a $(P_{\geq n}, k)$-factor critical graph if $G-U$ has a $P_{\geq n}$-factor for any $U \subseteq V(G)$ with $|U| = k$. It is obvious that a $(P_{\geq n}, 0)$-factor critical graph admits a $P_{\geq n}$-factor. A graph G is called a $(P_{\geq n}, m)$-factor deleted graph if $G-E'$ contains a $P_{\geq n}$-factor for any $E' \subseteq E(G)$ with $|E'| = m$. Clearly, a $(P_{\geq n}, 0)$-factor deleted graph has a $P_{\geq n}$-factor. A $(P_{\geq n}, m)$-factor deleted graph is simply called a $P_{\geq n}$-factor deleted graph if $m = 1$.

A graph R is called a factor-critical graph if $R-x$ has a perfect matching for every vertex x in R. A graph H is called a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph, R, having at least 3 vertices, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge, vw, for every $v \in V(R)$. A big sun is a sun having at least 6 vertices. A component of G is called a sun component if it is isomorphic to a sun. The number of sun components of G is denoted by $sun(G)$.

The following result for $P_{\geq 3}$-factor was first obtained by Kaneko [6]. Kano et al. [22] gave a new proof of this theorem.

Theorem 1 [6]. A graph G contains a $P_{\geq 3}$-factor if and only if $sun(G-S) \leq 2|S|$ for every $S \subseteq V(G)$.

A claw is a graph isomorphic to $K_{1,3}$. A graph is said to be a claw-free graph if it does not contain induced claw. For a 2-connected claw-free graph G, Kelmans [23] showed a sufficient condition for the existence of P_3-factors in $G-\{x\}$ for any $x \in V(G)$, and presented a sufficient condition for the existence of P_3-factors in $G-\{e\}$ for any $e \in E(G)$. Naturally, motivated by the two results, we consider two more general problems:

1. Does $G-U$ have a $P_{\geq 3}$-factor for any $U \subseteq V(G)$ with $|U| = k$, or is a graph G a $(P_{\geq 3}, k)$-factor critical graph?
2. Does $G-E'$ have a $P_{\geq 3}$-factor for any $E' \subseteq E(G)$ with $|E'| = m$, or is a graph G a $(P_{\geq 3}, m)$-factor deleted graph?

In this paper, we investigate the relationship between toughness and $(P_{\geq 3}, m)$-factor deleted graphs and $(P_{\geq 3}, k)$-factor critical graphs, and obtain two toughness conditions for the existence of $(P_{\geq 3}, m)$-factor deleted graphs and $(P_{\geq 3}, k)$-factor critical graphs. Our main results are the following theorems, which imply that the two problems above are true.

Theorem 2. Let m be an integer with $m \geq 1$, and let G be a graph with $\kappa(G) \geq \frac{3m+1}{2}$. If $t(G) > \frac{3}{4} - \frac{1}{2(m+1)}$, then G is a $(P_{\geq 3}, m)$-factor deleted graph.

Remark 1. The result in Theorem 2 is sharp. In order to show this, we construct a graph $G = K_{\frac{3m+1}{2}} \vee 2(m+1)K_2$, where $m \geq 1$ is an odd integer and $\vee$ means "join". It is obvious that $t(G) = \frac{\frac{3m+1}{2}}{2(m+1)} = \frac{3m+1}{4(m+1)} = \frac{3}{4} - \frac{1}{2(m+1)}$. Set $E' \subseteq E(2(m+1)K_2)$ with $|E'| = m$ and $G' = G - E'$. We choose $S = V(K_{\frac{3m+1}{2}}) \subseteq V(G')$, and so $|S| = \frac{3m+1}{2}$. Thus, we have:

$$sun(G'-S) = (m+2) + 2m$$
$$= 3m + 2 > 3m + 1 = 2|S|.$$

In terms of Theorem 1, G' has no $P_{\geq 3}$-factor. Combining this with the definition of $(P_{\geq 3}, m)$-factor deleted graph, G is not a $(P_{\geq 3}, m)$-factor deleted graph.

Theorem 3. Let $k \geq 0$ be an integer. A graph G with $\kappa(G) \geq k+2$ is a $(P_{\geq 3}, k)$-factor critical graph if $t(G) \geq \frac{k+1}{2}$.

Remark 2. Let $k \geq 0$ be an integer. We now show that the conditions $t(G) \geq \frac{k+1}{2}$ and $\kappa(G) \geq k+2$ in Theorem 3 cannot be replaced by $t(G) \geq \frac{k+1}{3}$ and $\kappa(G) \geq k+1$. We consider the graph $G = K_{k+1} \vee 3K_2$, and choose $U \subseteq V(K_{k+1})$ with $|U| = k$. Obviously, $t(G) = \frac{k+1}{3}$ and $\kappa(G) = k+1$. Let $G' = G - U$. For $S = V(K_{k+1}) \setminus U$, we have:

$$sun(G'-S) = 3 > 2 = 2|S|.$$

In view of Theorem 1, G' has no $P_{\geq 3}$-factor; thus, G is not a $(P_{\geq 3}, k)$-factor critical graph.

2. Proof of Theorem 2

If G is a complete graph, then it is obvious that G is a $(P_{\geq 3}, m)$-factor deleted graph by $\lambda(G) \geq \kappa(G) \geq \frac{3m+1}{2}$ and the definition of $(P_{\geq 3}, m)$-factor deleted graph. Hence, we may assume that G is a non-complete graph.

For any $E' \subseteq E(G)$ with $|E'| = m$, we write $G' = G - E'$. It is obvious that $V(G') = V(G)$ and $E(G') = E(G) \setminus E'$. To prove the theorem, we only need to verify that G' admits a $P_{\geq 3}$-factor. By contradiction, we assume that G' has no $P_{\geq 3}$-factor. Then it follows from Theorem 1 that there exists some subset $S \subseteq V(G')$ satisfying:

$$sun(G' - S) > 2|S|. \tag{1}$$

It follows from $\lambda(G) \geq \kappa(G) \geq \frac{3m+1}{2}$ that $G' = G - E'$ is connected; therefore:

$$sun(G') \leq \omega(G') = 1. \tag{2}$$

Claim 1. $S \neq \emptyset$.
Proof. If $S = \emptyset$, then by Eqs. (1) and (2) we obtain:

$$1 = \omega(G') \geq sun(G') \geq 1,$$

that is:

$$sun(G') = \omega(G') = 1. \tag{3}$$

Since $m \geq 1$ is an integer, we have:

$$\lambda(G) \geq \kappa(G) \geq \frac{3m+1}{2} \geq 2,$$

which implies:

$$|V(G')| = |V(G)| \geq 3, \tag{4}$$

and:

$$\lambda(G') = \lambda(G - E') \geq \lambda(G) - |E'| \geq \frac{3m+1}{2} - m = \frac{m+1}{2}. \tag{5}$$

If $m = 1$, then $G' = G - E'$ is a big sun by Eq. (3), Eq. (4), and the definition of a big sun. Let R denote the factor-critical graph in $G' = G - E'$. It is obvious that there exists $x \in V(R)$ with $\omega(G - \{x\}) = 2$. Thus, we obtain:

$$\frac{1}{2} = \frac{3}{4} - \frac{1}{2(m+1)} < t(G) \leq \frac{|\{x\}|}{\omega(G - \{x\})} = \frac{1}{2},$$

for $m = 1$, which is a contradiction.

If $m \geq 2$, then from Eq. (5) we have:

$$\lambda(G') \geq \frac{m+1}{2} > 1.$$

In view of the integrity of $\lambda(G')$, we obtain $\lambda(G') \geq 2$, and so $sun(G') = 0$, which contradicts Eq. (3). Claim 1 is proved.□

In the following, we shall consider two cases.

Case 1. S is not a vertex cut set of G.

Since $\kappa(G) \geq \frac{3m+1}{2}$, G is a connected graph. Thus, we have $\omega(G-S) = \omega(G) = 1$. After deleting an edge in a graph, the number of its components increases by at most 1. Hence, we have:

$$sun(G' - S) \leq \omega(G' - S) = \omega(G - S - E') \leq \omega(G - S) + m = m + 1. \tag{6}$$

According to Eqs. (1) and (6), we obtain:

$$2|S| < sun(G' - S) \leq m + 1,$$

that is:

$$|S| < \frac{m+1}{2}. \tag{7}$$

It follows from Eq. (7), $\lambda(G - S) \geq \kappa(G - S)$ and $\kappa(G) \geq \frac{3m+1}{2}$ that:

$$\lambda(G - S) \geq \kappa(G - S) \geq \kappa(G) - |S| > \frac{3m+1}{2} - \frac{m+1}{2} = m.$$

In terms of the integrity of $\lambda(G - S)$, we have:

$$\lambda(G - S) \geq m + 1. \tag{8}$$

Using Eq. (8), we obtain:

$$\lambda(G' - S) \geq \lambda(G - S) - m \geq 1,$$

and so:

$$sun(G' - S) \leq \omega(G' - S) = 1,$$

which contradicts Eq. (1) by Claim 1.

Case 2. S is a vertex cut set of G.

Since S is a vertex cut set of G and $\kappa(G) \geq \frac{3m+1}{2}$, we obtain:

$$\omega(G - S) \geq 2, \tag{9}$$

and:

$$|S| \geq \frac{3m+1}{2}. \tag{10}$$

Note that $sun(G'-S) \leq \omega(G'-S) \leq \omega(G-S) + m$. Combining this with Eq. (1), we have:

$$2|S| + 1 \leq sun(G' - S) \leq \omega(G - S) + m,$$

that is:

$$\omega(G - S) \geq 2|S| - m + 1. \tag{11}$$

In view of Eqs. (9) and (11), $t(G) > \frac{3}{4} - \frac{1}{2(m+1)}$, and the definition of $t(G)$, we obtain:

$$\frac{3}{4} - \frac{1}{2(m+1)} < t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{2|S|-m+1},$$

which implies:

$$2(m-1)|S| < (m-1)(3m+1). \tag{12}$$

If $m = 1$, then by Eq. (12) we have $0 < 0$, which is impossible. If $m \geq 2$, then it follows from Eq. (12) that:

$$|S| < \frac{3m+1}{2},$$

which contradicts Eq. (10). Theorem 2 is proved.□

3. Proof of Theorem 3

It is obvious that a complete graph G with $\kappa(G) \geq k+2$ is a $(P_{\geq 3}, k)$-factor critical graph. Hence, we may assume that G is not a complete graph.

For any $U \subseteq V(G)$ with $|U| = k$, we write $G' = G - U$. To verify the theorem, we only need to prove that G' has a $P_{\geq 3}$-factor. On the contrary, we assume that G' has no $P_{\geq 3}$-factor. Then, there exists some subset $S \subseteq V(G')$ such that:

$$sun(G'-S) \geq 2|S|+1, \tag{13}$$

by Theorem 1.

If $S = \emptyset$, then we have:

$$sun(G') = sun(G-U) = 0,$$

by $|U| = k$ and $\kappa(G) \geq k+2$, which contradicts Eq. (13). Hence, we may assume that $S \neq \emptyset$. In the following, we consider two cases by the value of $sun(G'-S)$.

Case 1. $sun(G'-S) \leq 1$.

It follows from $S \neq \emptyset$ and Eq. (13) that:

$$sun(G'-S) \geq 2|S|+1 \geq 3,$$

which contradicts $sun(G'-S) \leq 1$.

Case 2. $sun(G'-S) \geq 2$.

Note that $\omega(G-(S\cup U)) = \omega(G'-S) \geq sun(G'-S) \geq 2$. Combining this with $t(G) \geq \frac{k+1}{2}$ and the definition of $t(G)$, we obtain:

$$\frac{k+1}{2} \leq t(G) \leq \frac{|S\cup U|}{\omega(G-(S\cup U))} \leq \frac{|S|+k}{sun(G'-S)}$$

$$= \frac{|S|}{sun(G'-S)} + \frac{k}{sun(G'-S)} \leq \frac{|S|}{sun(G'-S)} + \frac{k}{2},$$

which implies:

$$sun(G'-S) \leq 2|S|,$$

which contradicts Eq. (13). This completes the proof of Theorem 3.□

4. Conclusion

In this paper, we investigated the existence of $(P_{\geq 3}, m)$-factor deleted graphs and $(P_{\geq 3}, k)$-factor critical graphs and obtained two results for them, which were two extensions of the previous studies. Path factors in graphs or networks have attracted a great deal of attention due to their applications in network design, statistical mechanics, information transmission in networks, and so on. Hence, there is theoretical and practical significance in investigating the problem of path factors in graphs or networks.

Acknowledgments

The authors would like to thank the anonymous referees for their kind help and valuable suggestions, which led to an improvement of this paper. This work was sponsored by 333 Project of Jiangsu Province and Six Big Talent Peak of Jiangsu Province (Grant No. JY–022), and supported by the National Natural Science Foundation of China (Grant No. 11501256, 61503160) and the Natural Science Foundation of Xinjiang Province of China (Grant No. 2015211A003).

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