Optimizing under- and out-of-warranty products’ decisions in the finite planning horizon

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Abstract

In this paper, we consider a manufacturer that produces products in a finite horizon time and sells products with non-renewing free replacement warranty policy. The manufacturer is responsible to provide spare parts for failed products, whether the products are under or out of warranty. Previous research on warranty optimization has focused on maximizing manufacturer profit without considering the spare part market for out-of-warranty products. This study proposes a novel nonlinear model that maximizes manufacturer profit by optimization of price, warranty length and spare part inventory for under- and out-of-warranty products in a manufacturing/remanufacturing system. Due to the model’s unique structure, we propose a new two-stage approach that combines metaheuristic and an exact method, in which the first stage is to determine product’s prices and warranty length with metaheuristic algorithm and in the second stage the remaining inventory related problem is transferred to a Minimum Cost Network Flow Problem which is solved for spare part inventory control. To illustrate the effectiveness of the suggested method, the model is solved for a case study of Iranian SANAM electronic company with two different metaheuristic algorithms and a sensitivity analysis is conducted to study the effect of various parameters on the optimal solution.

Keywords: non-renewing free replacement warranty; dynamic pricing; spare part Inventory control; remanufacturing.

1 Introduction

With tougher competition, technology advances, and shifting customer preferences, it is more crucial than ever that companies use warranty as a competition advantage in order to increase their market share. Warranty signifies the product’s quality in the eye of customers, therefore leading to growth in customer’s satisfaction and willingness to buy. Warranty has two main functions of protection and promotion. Regarding the former, warranty protects the manufacturer from excessive claims and protects the customer from purchase risks. Regarding the latter, warranty is a competitive advantage to differentiate the manufacturer from its competitors [1]. Relatively long warranty period will increase customer’s willingness to buy, but manufacturer can’t propose long warranty because they are responsible for product's failure during warranty period [2]. Also they must take into account product’s reliability, because undesirable reliability may leading to high cost for manufacturer [3]. Therefore, they should determine warranty length in order to optimize their profit and customer satisfaction.

In the literature, price and warranty length are mentioned as two key factors affecting manufacturer’s profit [4-7]. Obviously, longer warranty periods and lower price lead to increased sales but it also tends to decrease the manufacturer’s marginal profit. As a result, simultaneous decision about these two factors in order to optimize the profit is important and widely studied in the literature.
Glickman and Berger [8] proposed a model for maximizing manufacturer’s profit by determining price and warranty length, assuming that customers are homogeneous and their demand is an exponential function of price and warranty length. Nasrollahi and Asgharizadeh [9] solved a multi-objective problem with goal programming using this demand function for estimating warranty length. The problem in Lin and Shue [10] was determining price and warranty length where the demand is a function of price, warranty length, and cumulative sales and the objective is maximizing profit. They considered different product life distributions, while in [6, 7], the distribution of product lifetime is considered Normal and Gamma, respectively, and solution approaches based on maximum principle were presented. Huang, Liu [11] took account of reliability in addition to price and warranty length in modeling, and studied the problem both in stable and dynamic market scenarios. Manna [12] considered the problem with price and warranty length decision variables and proposed a method to extend the model to two-dimensional warranty. The model proposed in [5] incorporates price, warranty length, and production rate as decision variables with the objective of maximizing manufacturer’s profit. Wu, Chou [13] also developed a model with these decision variables where product lifetime is considered Weibull and the objective is maximizing profit. Zhou, Li [4] determined the optimal price and warranty length where the product is repairable and compared fixed-length warranty policy with dynamic policy. Fang and Huang [14] proposed a Bayesian decision model by which the integrated price of products, production quantity and term of warranty are determined under the situation that the manufacturer does not have sufficient historical data. Shafiee and Chukova [15] developed a mathematical optimization model to determine price, warranty length and upgrade strategy for second hand products. Faridimehr and Niaki [16] investigates optimal policies for price, warranty length and production rate in both static market and dynamic market. They calculate optimal policies by maximum principal approach. Mahmoudi and Shavandi [17] propose a bi-objective model for maximizing the manufacturer profit and minimizing the waiting time in queue. Also, they formulate the demand function as a fuzzy system. Tsao, Teng [18] considered the problem of determining retail price and inventory level for hi-tech products when the warranty length is predetermined. Wei, Zhao [19] proposed five decentralized models and determined equilibrium wholesale prices, and retail prices, and warranty periods using game theoretical approaches. The problem in [20] is to determine pricing policy for returned used products, along with their remanufacturing level, and pricing and warranty policy for remanufactured products.

Besides the profit obtained through the sale of the main products, aftermarket plays an important role in the manufacturer’s profitability. Selling spare parts to out-of-warranty products can lead to substantial profit, but it is very challenging to estimate the demand of spare parts due to its greater uncertainty compared to the uncertainty in the product’s demand [21]. To the best of our knowledge, there is not any research that considers selling spare parts to out-of-warranty products, while in many industries such as automobile and electronic devices, the company could increase its profits up to 25 percent from selling spare parts to out-of-warranty products [22].
Although pricing and warranty length are two key factors that affect spare parts’ inventory management, only a few researchers have considered this interdependency in the literature. Kim and Park [23] proposed a two-stage optimal control model to jointly determine product’s price, warranty length, and spare part inventory for under-warranty products. They divided the planning horizon into two parts: product’s life cycle and end of life period. Their study just considered under-warranty products, but a considerable portion of profit comes from selling spare parts for out-of-warranty products. Also, they assumed all spare parts were produced by manufacturer, but in practice, components can be refurbished by remanufacturing with lower amounts of cost. Chari, Diallo [24] developed a mathematical optimization model to maximize manufacturer’s total expected profit by optimization of warranty length, the sale price, the age of reconditioned components, and the proportion of reconditioned components to be used. They assumed renewing free replacement warranty and static pricing strategy for their model. Also, they didn’t consider the role of out-of-warranty products on manufacturer’s profit.

In order to present a compact review of previous studies and demonstrate the characteristics of the proposed approach as compared to the literature, Table 1 illustrates a state-of-the-art survey of Pricing and Warranty Inventory Optimization. Although the spare parts inventory decisions have a direct impact on warranty length and price decisions, (As shown in Table 1), many of the proposed approaches seek to optimize warranty length and price without considering inventory of spare parts. To the best of the authors’ knowledge, there is no research considering the effect of out-of-warranty products on manufacturers’ revenue.

The purpose of this paper is to develop a new mathematical model for optimization product price (in different stages of a product’s life cycle), warranty length, and spare part inventory control for under-warranty and out-of-warranty products in a manufacturing/remanufacturing system with the objective of maximizing manufacturer’s profit. Planning horizon consists of three main parts: (1) product life cycle, (2) end of life (EOL) and, (3) guarantee period for spare part availability. The product’s demand is considered a function of product’s price, time, and warranty length. The significant issue for producer is to determine the price in each pricing period (pp) of the product’s life cycle in order to gain the maximum profit. Another challenge is to determine the warranty length, where longer warranty length period tends to increase sales but at the same time increases warranty relevant costs. Although failures of under- and out-of-warranty products change stochastically in each period, the model can calculate a good estimation of failures in each inventory control period (ICP) of planning horizon in order to effectively manage the spare part inventory. In real word, a percent of failed items can rectify by remanufacturing; therefore, we assume spare parts can be obtained from two sources: (1) production by original manufacturer and (2) remanufacturing failed products.

Fundamentally, this paper aims to perform the following:

- Proposed a new model that considering out-of-warranty products as a main source of manufacturer revenue.
• Coordination between price, warranty length and spare part inventory decisions as an integrated model for under- and out-of-warranty products.
• Proposed a novel two stage approach with combination of metahueristic and exact method for solving the proposed model.
• Identifying how the changes of product’s life cycle affect warranty length.

Table 1 should be placed here

The rest of this paper is organized as follows: Section 2 presents the problem definition. Section 3 explains the mathematical modeling. A solution method is introduced in Section 4. Section 5 demonstrates the applicability of the presented mathematical model by a real-world numerical example taken from the Iranian SANAM electronic company along with sensitivity analyses. The paper concludes in Section 6.

2 Problem definition

The problem of this paper is defined for maximizing the profit of the manufacturer, which consists of a set of revenue and cost elements. The manufacturer’s revenue includes: (1) product’s sale in its life cycle, and (2) spare parts’ sale for out-of-warranty products. Moreover, the cost is composed of four main elements, which are: (1) production cost, (2) spare parts’ inventory cost, (3) spare parts’ remanufacturing cost, and (4) disposal cost. The product’s sale and its production cost are directly related to the market demand in its life cycle. The demand itself is dependent upon the time, sales price, and length of warranty. Sales price and warranty length are, respectively, inversely and directly proportional to the market demand. Therefore, the simultaneous decision about sales price and length of warranty is of considerable significance in order to maximize the profit.

Additionally, spare-part related elements affect the manufacturer’s revenue and cost. Effective spare-part inventory control decisions play an important role in reducing manufacturer’s cost. A challenge here is to estimate the number of product’s failure in each inventory control period (ICP) in order to optimize the inventory level.

Before presenting the model, the assumptions made for formulating the problem are given as below:

1) All ICP are equal and less than product’s life cycle,
2) All claims during the warranty period are valid,
3) Warranty policy for products is non-renewing Free Replacement Warranty (FRW),
4) The original manufacturer is also responsible for the remanufacturing of their used products,
5) The refurbished components return to as-good-as-new state,
6) Production capacity is unlimited in product life cycle,
7) The firm is a monopolist and Customers are myopic,
8) The amount of product’s sale is equal to the product’s demand,
9) Products have exponential failure distribution. This assumption was imposed by SANAM Electronic Company’s product development division,
10) Warranty length is a positive integer and multiple of the ICP,
11) The inventory delivery is assumed to be instantaneous (lead time is negligible),
12) Shortage is not allowed to avoid lost sales.

Assumptions 1 to 10 are common in reality of the problem (especially in electronic device manufacturers). However, assumptions 11 and 12 were set in order to make the problem technically more tractable. The notations used to formulate the problem are presented in Table 2.

Table 2 should be placed here

According to the above-mentioned assumptions, the planning horizon of the problem is divided into three segments (see Figure 1): (1) product’s life cycle \((0, L_T)\), (2) end of life (EOL) \((L_T, L_{T+w})\), and (3) guarantee period for spare part availability \((L_{T+w}, L_{T+g})\). In each of ICPs, an estimation of the number of products under warranty should be made. It is a challenging task because on one hand, these estimation conditions differ in each of the above-mentioned segments. On the other hand, in each ICP, a number of manufactured products are added to the products under warranty and some products become out of warranty. After calculating the number of under-warranty and out-of-warranty products in each ICP, the estimated number of failures can be calculated using product’s failure behavior, which is employed to estimate the demand for spare parts.

Figure 1 should be placed here

As can be seen in Figure 1, planning horizon is divided into \(T+g\) ICP’s, in which the spare part order size is determined based on the spare part’s demand, minus the remaining inventory from the previous period and the amount of remanufactured spare parts. In addition, each ICP is divided into \(l\) equal sub-periods in which the amount of production is a function of time, price and warranty length. Because the product’s price is determined in each of these sub-periods, they are named as pricing periods (PPs). If warranty length is equal to \(w\) ICPs, we may have products under warranty in the market until at most \(L_{T+w}\). Therefore, from \(L_T\) to \(L_{T+w}\), the decision is limited to spare part inventory control for under-warranty and out-of-warranty products. Finally, from \(L_{T+w}\) to \(L_{T+g}\), spare part inventory control for out-of-warranty products is the only decision to be made. This period is for ensuring the customers of the availability of spare parts for a fixed period of time after the end of all product’s warranty.

Spare parts inventory can be provided from two sources: (1) remanufactured failed components, and (2) manufactured components. Because the planning horizon is finite and the component’s holding and production costs could change with time, it is essential that the manufacturer optimizes the amount of component’s production in each ICP.
The relationship between production, market and spare part inventory is depicted in Figure 2. Manufacturer supplies the product according to market demand, which is a function of time, price, and warranty length. These products include $I$ critical components. The amount of failed products that enter the collection center in each ICP is dependent upon the product situation (to be defined later) and failure rate. Based on historical data, we can determine the proportion of times that product’s failure is due to each component. Then, based on the conditions of remanufacturing process, components that can be refurbished are sent to the refurbishing center and irreparable ones are disposed. Due to the assumption that the refurbishing is perfect, refurbished components are as good as new, so they can meet a proportion of spare part’s demand. Finally, the manufacturer satisfy spare market’s demand (demands from under- and out-of-warranty products) by spare parts that refurbished and produced in factory. Since, key elements of reverse logistics are defined in the literature as failed item remanufacturing, disposition, spare part inventory management, after-sale service, and product pricing [25, 26], we can say that the proposed framework is a part of reverse logistics. Therefore, manufacturer can overcome reverse logistics challenges related to under-warranty and out-of-warranty products by using the proposed model.

3 Mathematical model

In this section, we describe the problem’s mathematical model that is constructed based on the notations and assumptions which were mentioned.

3.1 Objective function

The objective function maximizes manufacturer profit that gains from selling products and spare parts. Manufacturer just can sell spare parts to out-of-warranty product.

$$\max z = \sum_{j=1}^{I} (P_j - c_j) S(j, P_j, w) + \sum_{j=1}^{I} p_{cj} \sum_{s=1}^{T} \left( \alpha_{ij} D_{ps}(s) \right)$$

$$-\sum_{j=1}^{I} c_{rij} \sum_{s=1}^{T} E_i(s) - \sum_{j=1}^{I} v_{ij} \sum_{s=1}^{T} V_i(s) - \sum_{j=1}^{I} h_{ij} \sum_{s=1}^{T} X_i(s) - \sum_{j=1}^{I} c_{oi} \sum_{s=1}^{T} Q_i(s)$$

In Eq. (1), the first term is the profit obtained from product’s sale, which is calculated by multiplying the net product’s profit and the demand in each PP. The revenue earned from spare part’s sale for out-of-warranty products is calculated in the second term. The third to fifth terms are the remanufacturing, disposal, and holding costs, respectively. The final term calculates the cost of spare part’s production in each ICP.
Product’s Market Demand or Number of Products Sold in Each Period

Eq. (2) shows the demand behavior according to time, which is increasing up to $\eta$ and decreasing afterwards. This is congruent with the product’s life cycle in which demand is rising until the maturity and is falling afterwards. Interested reader can find more information in [27]

$$
S(j) = \begin{cases} 
\frac{U}{1 + \Psi e^{-\lambda j}}, & 0 \leq j \leq \eta \\
\frac{U}{\lambda U(j - \eta) + \theta^2}, & \eta \leq j \leq T.l 
\end{cases}
$$

$$
\Psi = \frac{U}{D_0} - 1 \\
\theta = 1 + \phi e^{-k \eta}
$$

Eq. (3) calculates the amount of main product’s demand based on the assumption that the amount of product’s sale is equal to the product’s demand. Therefore, the manufacturer’s sale is a function of time ($j$), price ($p_j$), and warranty length ($w$). We assume that $S(j)$ is potential market demand in each PP. It is obvious that the sale is inversely proportional to the price and directly proportional to the warranty length. The element $-k_1 (p_j - P_{\min}) + k_2 w$ demonstrates this type of dependence In Eq. (3), $k_1$ is price coefficient and $k_2$ is warranty coefficient in the demand function. Constraint (4) show the lower bound and upper bound on the warranty length.

$$
S(j, p_j, w) = S(j) - k_1 (p_j - P_{\min}) + k_2 w \quad \text{for} \quad k_1 > 0, \ k_2 > 0.
$$

$$
W_{\min} \leq w \leq W_{\max}
$$

Because customers are considered myopic, the product’s price is decreasing in time. Constraints (5) take account of this fact, which is called markdown pricing. Constraints (6) are the lower bound and upper bound on the product prices in each PP.

$$
P_{j-1} \geq P_j \quad \forall j \in \{1, \ldots, (T + g) \times l\}
$$

$$
P_{\min} \leq p_j \leq P_{\max} \quad \forall j \in \{1, \ldots, (T + g) \times l\}
$$

3.2 Product situations during each inventory planning period

Each ICP, based on its position in the planning horizon, may include several types of situations for each product. Each situation will demonstrate how much time a product is under warranty or/and out of warranty in each ICP. Each product may have one or two of five situations in each ICP, where situations one, two, and three are for calculating the number of failed products under warranty and situations four and five are for computing the number of failed out-of-warranty products. These situations are as follows:

- **Situation 1 (for under-warranty products)**
The products that are manufactured in an ICP have the situation 1. For these products, the probability of failure in that ICP is proportional to the amount of time it lies in that ICP. It is obvious that all ICPs in product’s life cycle have products with situation 1 because there is a production in these ICPs.

- **Situation 2 (for under-warranty products)**

  This situation covers a case where the product is under warranty throughout the ICP.

- **Situation 3 (for under-warranty products)**

  A product with situation 3 falls in an ICP in which its warranty expires.

- **Situation 4 (for out-of-warranty products)**

  Situation 4 is the complement of the situation 3, i.e. it enables us to compute the probability of failure when the product becomes out of warranty.

- **Situation 5 (for out-of-warranty product)**

  Under this situation, the warranty has been expired in one of the prior ICPs, so the probability of failure is proportional to the whole ICP.

  Figure 3 summarizes situations 1 to 5 along with their features for products that are produced in \( t_i \). In this figure we assumed that the warranty length is equal to 4 ICPs, so the amount of products produced in \( t_i \) is \( S(1, P_i, 4) \). Products that are in \( s=1 \) are faced with situation 1 and the failure probability of these products in \( s=1 \) is proportional to \( (L - t_i) \). In \( s=\{2,3,4\} \) products are under situation 2, so their probability of failure is proportional to the whole length of ICP. Because their warranty will expire in \( s=5 \), as long as products are under warranty, they are under situation 3. After they become out of warranty, they get situation 4 until the end of the 5th ICP. Finally, these products will be in situation 5 in \( s=\{6,\ldots,T+g\} \). Therefore, it can be concluded that the products are in the situation 1 when they are produced, while they pass situation 2 to 5 sequentially throughout the planning horizon.

  **Figure 3 should be placed here**

3.3 **The number of failures for products under warranty**

  In order to calculate the number of failures of under-warranty products in each ICP, we must first compute how much time the product is under warranty in that ICP. As defined above, under-warranty products can have at most 3 situations 1, 2, and 3.
According to assumption 1, the probability of failure for product with situation 1 in the $s^{th}$ ICP that is produced in the $j^{th}$ PP ($L_{s-1} < t_j < L_s$) is equal to $P_{j1}^1 = P(f \leq L_s \mid f \geq t_j) = 1 - e^{-\lambda(L_s - t_j)}$. On the other hand, the number of products produced in the $j^{th}$ PP is denoted by $S(j, P_{j}, w)$, so the number of failures for products with situation 1 is binomially distributed as $y^1_{j} \sim b\left(S(j, P_{j}, w), P_{j1}^1 = 1 - e^{-\lambda(L_s - t_j)}\right)$. All products that are produced in the first ICP have the situation 1, therefore, using continuity correction, Eq. (7) calculates the maximum number of failures ($n^1_j$) in the $j^{th}$ PP with $\rho_w$ confidence which produced in the first ICP. Eq. (8) computes the total number of failures in the first ICP.

$$P(y^1_{j} \leq n^1_{j}) \geq \rho_w \Rightarrow P\left(z < \frac{n^1_{j} + 0.5 - S(j, P_{j}, w).P_{j1}^1}{\sqrt{S(j, P_{j}, w).P_{j1}^1(1-P_{j1}^1)}}\right) \geq \rho_w \quad \forall j \in \{1, 2, \ldots, I\}$$

$$n^1_{j} = \phi^{-1}(\rho_w).\sqrt{S(j, P_{j}, w).P_{j1}^1(1-P_{j1}^1) + S(j, P_{j}, w).P_{j1}^1 - 0.5}$$

$$D_w(l) = \sum_{j=1}^{I} n^1_{j}$$

Due to the fact that products with situation 2 are under warranty for the whole of ICP, their failure probability is proportional to the length of ICP, and is calculated with $P^2_{j} = P(f \leq L_s \mid f \geq L_{s-1}) = 1 - e^{-\lambda(L_s - L_{s-1})}$. Additionally, the random variable denoting the number of failed products with situation 2 follows a binomial distribution, i.e. $y^2_{j} \sim b\left(S(j, P_{j}, w), P_{j1}^2 = 1 - e^{-\lambda(L_s - L_{s-1})}\right)$. The characteristic of the $2^{nd}$ to $w^{th}$ ICP is that no product will get out of warranty, because it is assumed that the warranty length is equal to $w$ ICPs. Therefore, products in these ICPs only have situation 1 or 2, where all products produced before $s^{th}$ ICP have situation 2 and product produced in the $s^{th}$ ICP have situation 1. , Eq. (9) and Eq. (10) calculate the maximum number of failed products with situation 2 and 1, respectively, in the $s^{th}$ ICP. Finally, Eq. (11) shows the total number of failed products in the $s^{th}$ ICP ($1 < s \leq w$).

$$n^1_s = \phi^{-1}(\rho_w).\sqrt{S(j, P_{j}, w)P^2_{j} (1-P^2_{j}) + S(j, P_{j}, w)P^2_{j} - 0.5}, \quad \forall j \in \{1, 2, \ldots, (s-1)I\}, s \in \{2, \ldots, w\},$$

$$n^1_s = \phi^{-1}(\rho_w).\sqrt{S(j, P_{j}, w)P^2_{j} (1-P^2_{j}) + S(j, P_{j}, w)P^2_{j} - 0.5}, \quad \forall j \in \{(s-1)I, (s-1)I + 1, \ldots, sI\}, s \in \{2, \ldots, w\},$$

$$D_w(s) = \sum_{j=1}^{(s-1)I} n^2_{j} + \sum_{j=1}^{sI} n^1_{j}, \forall s \in \{2, \ldots, w\}$$

For ICPs between $w$ and $T$, product can have situations 1, 2, or 3. Products produced in $s^{th}$ ICP have situation 1, those produced in $s - w < j \leq s - 1$ have situation 2, and those produced in $s - w - 1 < j \leq s - w + 1$ have situation 3.
For products with situation 3, the probability of their failure in the $s^{th}$ ICP is
\[ P_{s}^{3} = P\left( j \leq t_{j} + w \mid f \geq L_{s-1}, s \right) = 1 - e^{-\lambda_{j}(t_{j} + w - L_{s-1})} \]
and the number of failures has binomial distribution, i.e.
\[ y_{s}^{3} \sim b\left( S(j, p_{j}, w), P_{s}^{3} = 1 - e^{-\lambda_{j}(t_{j} + w - L_{s-1})} \right) \].
Eq. (12), (13), and (14) calculate the maximum number of failed products with situation 3, 2, and 1, respectively; in the $s^{th}$ ICP when ICPs are between $k$ and $T$. Eq. (15) calculates the total number of failed under-warranty products in the $s^{th}$ ICP.

\[
\begin{align*}
\forall j & \in \{(s-k-1)J, (s-k-1)J + 1, \ldots, (s-k)J\}, s \in \{w+1, \ldots, T\}, \\
n_{s}^{3} & = \varphi^{-1}(\rho_{s}) \sqrt{S(j, p_{j}, w)P_{s}^{3}(1 - P_{s}^{3})} + S(j, p_{j}, w)P_{s}^{3} - 0.5,
\forall j & \in \{(s-k)J, (s-k)J + 1, \ldots, (s-1)J\}, s \in \{w+1, \ldots, T\}, \\
n_{s}^{3} & = \varphi^{-1}(\rho_{s}) \sqrt{S(j, p_{j}, w)P_{s}^{3}(1 - P_{s}^{3})} + S(j, p_{j}, w)P_{s}^{3} - 0.5,
\forall j & \in \{(s-1)J, (s-1)J + 1, \ldots, sJ\}, s \in \{w+1, \ldots, T\},
\end{align*}
\]
\[
D_{s}(s) = \sum_{j=(s-w-1)J}^{(s-w)J} n_{j}^{3} + \sum_{j=(s-w)J}^{(s-1)J} n_{j}^{3} + \sum_{j=(s-1)J}^{sJ} n_{j}^{3}, \quad \forall s \in \{w+1, \ldots, T\}.
\] (15)

Because the production is stopped at the end of $T$, all under-warranty products have situations 2 and 3 in $T < s \leq T + w$, specifically, products produced in $(s - w)J < j \leq TJ$ have situation 2 and products produced in $(s - w)J < j \leq (s - w)J$ have situation 3. Eq. (16) and (17) calculate the maximum number of failed products with situation 3 and 2, respectively, for $T < s \leq T + k$ and Eq. (18) calculates the total number of failed under-warranty products for the $s^{th}$ ICP.

\[
\begin{align*}
\forall j & \in \{(s-w-1)J, (s-w-1)J + 1, \ldots, (s-w)J\}, s \in \{T+1, \ldots, T+w\}, \\
n_{s}^{3} & = \varphi^{-1}(\rho_{s}) \sqrt{S(j, p_{j}, w)P_{s}^{3}(1 - P_{s}^{3})} + S(j, p_{j}, w)P_{s}^{3} - 0.5,
\forall j & \in \{(s-w)J, (s-w)J + 1, \ldots, sJ\}, s \in \{T+1, \ldots, T+w\},
\end{align*}
\]
\[
D_{s}(s) = \sum_{j=(s-w-1)J}^{(s-w)J} n_{j}^{3} + \sum_{j=(s-w)J}^{(s-1)J} n_{j}^{3}, \quad \forall s \in \{T+1, \ldots, T+w\}.
\] (18)

### 3.4 The number of failure for out of warranty products

The $(w+1)^{th}$ ICP is the first ICP in which the out-of-warranty products (that are produced in the $1^{st}$ ICP) have situation 4. The probability that products with situation 4 fail in the $s^{th}$ ICP is equal to
\[ P_{s}^{4} = P\left( j \leq t_{j} + w \mid f \geq L_{s-1}, s \right) = 1 - e^{-\lambda_{j}(t_{j} + w - L_{s-1})} \]
and the number of failed products is binomially distributed, i.e.
\[ y_{s}^{4} \sim b\left( S(j, p_{j}, w), P_{s}^{4} = 1 - e^{-\lambda_{j}(t_{j} + w - L_{s-1})} \right) \]. Eq. (19) calculates the maximum number of failed out-of-
warranty products with situation 4 that are produced in $j \in \{1, 2, \ldots, l\}$ and Eq. (20) calculates the total number of failed out-of-warranty products in $s = w+1$.

\[
  n^*_s = \varphi^{-1}(\rho_{pw}), \sqrt{S(j, P_j, w)P_{\mu}^s(1-P_{\mu}^*) + S(j, P_j, w)P_{\mu}^* - 0.5} \quad \forall j \in \{1, 2, \ldots, l\}, s = w+1, \tag{19}
\]

\[
  D_{pw}(w+1) = \sum_{j=1}^{n^*_s}, \quad s = w+1. \tag{20}
\]

In $w+1 < s \leq T+w$, out-of-warranty products have situations 4 and 5; so the products produced in $(s-w-1)l < j \leq (s-w)l$ have situation 4, and products produced in $1 < j \leq (s-w-1)l$ have situation 5.

The probability of failure of products with situation 5 is equal to $P_{\mu}^5 = P(f \leq L_5 | f \geq L_{\alpha-1}) = 1-e^{-\lambda(l_{\alpha-1})}$ and the random variable of the number of failed products with situation 5 is $y_{\mu}^5 \sim b(S(j, P_j, w), P_{\mu}^5 = 1-e^{-\lambda(l_{\alpha-1})})$. Using Eq. (21), we can compute the maximum number of failures for products with situation 5 that are produced in $j \in \{1, 2, \ldots, (s-(w+1))l\}$, while Eq. (22) calculates the number of failed out-of-warranty products with situation 4 that are produced in $\{(s-(w+1))l, (s-(w+1))l+1, \ldots, (s-w)l\}$. Finally, Eq. (23) calculates the total number of failed out-of-warranty products in $s \in \{w+2, \ldots, T+w\}$.

\[
  n^5_s = \varphi^{-1}(\rho_{pw}), \sqrt{S(j, P_j, w)P_{\mu}^5(1-P_{\mu}^5) + S(j, P_j, w)P_{\mu}^5 - 0.5} \quad \forall j \in \{1, 2, \ldots, (s-(w+1))l\}, s \in \{w+2, \ldots, T+w\}, \tag{21}
\]

\[
  n^4_s = \varphi^{-1}(\rho_{pw}), \sqrt{S(j, P_j, w)P_{\mu}^4(1-P_{\mu}^4) + S(j, P_j, w)P_{\mu}^4 - 0.5} \quad \forall j \in \{(s-(w+1))l, (s-(w+1))l+1, \ldots, (s-k)l\}, s \in \{w+2, \ldots, T+w\}, \tag{22}
\]

\[
  D_{pw}(s) = \sum_{j=1}^{[s-(w+1)]l} n^5_s + \sum_{j=[s-(w+1)]l}^{[s-w]l} n^4_s, \quad \forall s \in \{w+2, \ldots, T+w\}. \tag{23}
\]

Since at the end of $T+w$, all products have become out of warranty, so they have situation 5. For these products, the maximum number of failures can be calculated with Eq. (24), and the total number of failures in $s \in \{T+w+1, \ldots, T+g\}$ is calculated with Eq. (25).

\[
  n^5_s = \varphi^{-1}(\rho_{pw}), \sqrt{S(j, P_j, w)P_{\mu}^5(1-P_{\mu}^5) + S(j, P_j, w)P_{\mu}^5 - 0.5} \quad \forall j \in \{1, 2, \ldots, T\}, s \in \{T+w+1, \ldots, T+g\}, \tag{24}
\]

\[
  D_{pw}(s) = \sum_{j=1}^{T} n^5_s, \quad \forall s \in \{T+w+1, \ldots, T+g\}. \tag{25}
\]
3.5 Spare Part Inventory Control

In this section, we present the framework for spare part inventory control based on the number of failures in each ICP as calculated in the previous section. The number of components to send to refurbishing center is a proportion of the number of failed products.

As mentioned previously, failures are associated with under-warranty products until the \( w \)-th ICP. From \((w+1)\)-th ICP to \((T+w)\)-th ICP, failures are from both under- and out-of-warranty products. Finally, after \((T+w+1)\)-th ICP, failures are associated with just out-of-warranty products.

Eq.’s (26)-(28) calculate the number of component \( i \) to be sent to refurbishing center in each ICP.

\[
F_i(s) = \delta_i \times (\alpha_i D_v(s)) \quad \forall s \in [1, \ldots, w],\tag{26}
\]

\[
F_i(s) = \delta_i \times (\alpha_i (D_u(s) + D_{pw}(s))) \quad \forall s \in \{w + 1, \ldots, T + w\},\tag{27}
\]

\[
F_i(s) = \delta_i \times (\alpha_i D_{pw}(s)) \quad \forall s \in \{T + w + 1, \ldots, T + g - 1\}.\tag{28}
\]

In refurbishing center, some percentage of components are successfully refurbished and are taken into account as as-good-as-new component inventory, while others are disposed. Eq.’s (29) and (30) calculate the number of component \( i \) that is sent for refurbishment and disposal, respectively.

\[
E_i(s) = \kappa_i \times F_i(s) \quad \forall s \in \{1, 2, \ldots, T + g\},\tag{29}
\]

\[
V_i(s) = \left[ \frac{(1 - \delta_i)}{\delta_i} + (1 - \kappa_i) \right] \times F_i(s) \quad \forall s \in \{1, 2, \ldots, T + g\}.\tag{30}
\]

Figure 4 demonstrates the flow of spare part inventory for component \( i \) in each ICP of the planning horizon. In each ICP, the on-hand inventory consists of the remaining inventory of the previous ICP, the amount of products manufactured and remanufactured in the current ICP, minus the component’s demand in the current ICP.

**Figure 4 should be placed here**

Now we have completed the task of calculating different components of the profit function beside the required inventory balance equations for different ICP’s. The optimization problem is summarized as follows:

\[
\max z = \sum_{j=1}^{T} \left( P_j - c_j \right) S(j, P_j, w) + \sum_{i=1}^{I} p_{ci} \sum_{s=w+1}^{T+g} \left( \alpha_i D_{pw}(s) \right) + \sum_{i=1}^{I} p_{o_j} X_i(T + g)
\]

\[-\sum_{i=1}^{I} a_i \sum_{s=1}^{I} E_i(s) - \sum_{i=1}^{I} V_i(s) - \sum_{i=1}^{I} h_i \sum_{s=1}^{I} X_i(s) - \sum_{i=1}^{I} c_i \sum_{s=1}^{I} Q_i(s)\]

\[
P_{j-1} \geq P_j \quad \forall j \in \{1, \ldots, (T + g) \times l\},\tag{32}
\]

\[
P_{min} \leq P_j \leq P_{max} \quad \forall j \in \{1, \ldots, (T + g) \times l\},\tag{33}
\]
\[ W_{\min} \leq w \leq W_{\max} \] (34)

\[ X_i(0) = 0 \quad \forall i \in \{1, \ldots, I\}, \] (35)

\[ X_i(s) = X_i(s-1) - \alpha_i D_n(s) + E_i(s) + Q_i(s) \quad \forall i \in \{1, \ldots, I\}, \ s \in \{1, \ldots, w\} \] (36)

\[ X_i(s) = X_i(s-1) - \alpha_i \left( D_n(s) + D_m(s) \right) + E_i(s) + Q_i(s) \quad \forall i \in \{1, \ldots, I\}, \ s \in \{w+1, \ldots, T+w\} \] (37)

\[ X_i(s), Q_i(s) \geq 0 \quad \forall i \in \{1, \ldots, I\}, \ s \in \{1, \ldots, T+w\}. \] (38)

Eq. (35) states that the inventory of all components at the beginning of the planning horizon is zero. Eq.’s (36)-(38) are inventory balance equations for different ICP’s (see Figure 4). Finally, Eq. (39) ensures that no shortage is encountered in all ICP’s.

4 Solution Method

In this section we propose an effective algorithm to solve the problem already described. The objective function to be minimized and the constraints are all nonlinear with respect to prices \( P_j \) and warranty length \( w \) variables. Once we fix the value of these variables the remaining variables which are \( Q_i(s) \) can be found via translating the reduced problem to a minimum cost network flow problem. So, we can effectively reduce the search to the price-warranty length space to find good quality solutions of the problem.

Following the above notion, given known the prices (in each PP) and warranty length, the amount of products sold in each PP can be calculated. Then, the number of failed under- and out-of-warranty products in each ICP are calculated; this leads to determination of spare part demand in each ICP. The remaining is a spare part inventory control sub-problem for which we show it can be modeled as a network problem. Figure 5 demonstrates the network counterpart which can be solved by a minimum-cost network flow algorithm such as out-of-kilter. This method was first introduced by Fulkerson [28]. It works on both the primal problem (edges of the network) and the dual problem (nodes) in successive phases to find a feasible solution, and then to optimize the problem. The pseudo code of the out-of-kilter is as follows:
The out-of-kilter algorithm

Begin
\(\pi := 0;\)
Establish a feasible flow \(x\) in the network;
Define the residual network \(G(x)\) and compute the kilter number of arcs;
While the network contains an out-of-kilter arc do
  Begin
  Select an out-of-kilter arc \((p,q)\) in \(G(x)\)
  Define the length of each arc \((i,j)\) in \(G(x)\) as \(\max\{0,c\pi_{ij}\}\);
  Let \(d(.)\) denotes the shortest path distances from nodes \(q\) to all other nodes in \(G(x)\)-\{(\(q,p\)\)} and let \(P\) denote a shortest path from node \(q\) to node \(p\);
  Update \(\hat{\pi} = \pi(i) - d(i)\) for all \(i \in N\);
  If \(c\pi_{ij} < 0\) then
    Begin
    \(W = P \cup \{(p,q)\};\)
    \(\delta = \min\{r_{ij}: (i,j) \in W\};\)
    Augment \(\delta\) units of flow along \(W\);
    Update \(\delta\) units of flow along \(W\);
    Update \(x, G(x),\) and the reduced costs;
    End;
  End;
End;
End;

Now we can develop an algorithm for the problem which respectively scale the prices and warranty length with the aid of a search based algorithm e.g., the recently proposed Optics Inspired Optimization (OIO) and scale the production amount of each component’s spare parts, relevant to the given the prices and warranty length, by solving a minimum cost network flow problem optimally. This latter step is done to compute the fitness relevant to OIO’s individuals. To have a comparator algorithm, we also use from Improved Particle Swarm Optimization (IPSO) algorithm. Our justification behind using these algorithms comes from the point that PSO is a classic and popular algorithm for solving optimization problems. To include a relatively newer and modern algorithm, we use from the recently proposed OIO algorithm. OIO has proven as an effective algorithm and needs few parameters, so we think it may be useful to compare the results of an older algorithm like PSO beside a newer one.

4.1 Solution representation and fitness function

To solve the problem with the aid of Optics Inspired Optimization (OIO), an individual is a vector of length \(lT + 1\), for which the first \(lT\) elements are prices sorted in descending order, and the last element is the warranty length. All prices and warranty length should generate between their upper bound and lower bound. When computing the fitness function relevant to a given
individual, i.e., the objective function value, first the decision variables relevant to the production amount of each component’s spare parts in s\textsuperscript{th} ICP, are set optimally via solving the relevant minimum-cost network flow problem for spare part inventory control and then the fitness value is calculated. (see Fig 6).

Figure 5 should be placed here

Figure 6 should be placed here

4.2 The Optics Inspired Optimization (OIO) Algorithm

Optics Inspired Optimization (OIO) is an optics inspired population based evolutionary algorithm that was first proposed by Husseinzadeh Kashan [29]. The algorithm assumes that a number of artificial light points (points in R\textsuperscript{n+1} whose mapping in R\textsuperscript{n} are potential solutions to the problem) are sitting in front of an artificial wavy mirror reflecting their images. OIO treats the surface of the function to be optimized as the reflecting mirror composed of peaks and valleys. Each peak is treated as a convex reflective surface and each valley is treated as a concave reflective surface. In this way, the artificial ray glittered from the artificial light point is reflected back artificially by the function surface, given that the reflecting surface is a part of a peak or a part of a valley, and the artificial image point (a new point in R\textsuperscript{n+1} which is mapped in R\textsuperscript{n} as a new solution in the search domain) is formed upright (toward the light point position in the search space) or inverted (outward the light point position in the search space). Recently several studies use this algorithm for their problem optimization [30-33].

Figure 7 should be placed here

Figure 7 illustrates how the new solution is generated in OIO in the one dimensional search space. In this figure it is assumed that an artificial light point in the joint search and objective space (i.e., R\textsuperscript{n+1}) is in front of the function surface (mirror) in a particular distance from the vertex (values on the X-axis form the search/solution space and values on the f(X)-axis form the objective space. The set of all points in the X-f(X)-coordinate system forms the joint search and objective space). Using the mirror equations of Physics, the artificial image is formed in the joint search and objective space. Then, the new solution is generated in the search space through mapping the artificial image position into the search space. The procedure of generating new solutions is directly depends on the reflecting part of the function surface (convex or concave) and the position of the artificial light point in the joint search and objective space. Figure 7 shows four different situations which may occur in generating new solutions.

The above process for generation of a new solution can be translated in an algorithmic manner as follows. For a given individual solution O in the population, a different solution F (vertex point) is selected randomly from the population. If F has a worse fitness value than O, it is treated that the surface is convex and a new solution is generated upright somewhere toward O, on the line connecting O and F (See Figure 7a). If F has a better fitness value than O then it is assumed that the surface is concave and the new solution is generated upright toward (see Figure 7b) or inverted outward (see Figure 7c and 7d) O, on the line connecting O and F in the search space.
With the procedure of generating new solutions described conceptually in Figure 8, OIO is able to perform both exploration and exploitation during the search process. The exploration ability is achieved by adopting a larger jump in the solution space (see Figure 7b and 7c) while the exploitation is performed by adopting a smaller jump over the base solutions (see Figure 7a and 7d). The detailed and ready to implement flowchart of OIO has been shown in Figure 8. The notations used in Figure 8 are described as follows:

\[ O_j^t = [a_{j1}^t, a_{j2}^t, \ldots, a_{jn}^t] \]  
the position of artificial light point \( j \) in the \( n \) dimensional search space in iteration \( t \) (i.e., the \( j \)th solution in the population),

\[ F_j^t = [f_{j1}^t, f_{j2}^t, \ldots, f_{jn}^t] \]  
a different point in the search space (i.e., an individual in the population) which passes the artificial principal axis through itself,

\[ I_j^t = [i_{j1}^t, i_{j2}^t, \ldots, i_{jn}^t] \]  
an image position of the artificial light point \( j \) in the search space in iteration \( t \). The artificial image is formed by the artificial mirror whose principal axis passes through \( F_i^t \),

\[ s_{j,ki}^t \]  
the position of the artificial light point \( j \) (whose image is formed by the artificial mirror) on the function/objective axis (objective space) in iteration \( t \). The position of artificial light point \( j \) in the joint search and objective space is thus given by the vector \( [o_{j1}^t, o_{j2}^t, \ldots, o_{jm}^t] \),

\[ p_{j,ki}^t \]  
the distance between the position of artificial light point \( j \) on the function/objective axis and the position of artificial mirror vertex on the function/objective axis in iteration \( t \),

\[ q_{j,ki}^t \]  
the distance between the image position of the artificial light point \( j \) on the function/objective axis and the position of artificial mirror vertex on the function/objective axis in iteration \( t \),

\[ r_i^t \]  
the radius of curvature of the artificial mirror whose center of curvature is on the principal axis which passes through \( F_i^t \),

\[ m_{ki}^t \]  
the position of the center of curvature on the function/objective axis (objective space),

\[ HO_{j,ki}^t \]  
the height of the artificial light point \( j \) from artificial principal axis in iteration \( t \),

\[ HI_{j,ki}^t \]  
the image height of the artificial light point \( j \) from artificial principal axis in iteration \( t \),

\[ \kappa_{j,ki}^t \]  
the value of lateral aberration relevant to the artificial mirror which is reflecting the image of the artificial light point \( j \) in iteration \( t \).

Figure 8 should be placed here

4.3 The Improved Particle Swarm Optimization (IPSO) Algorithm

In the literature many studies can find that use algorithms based on PSO to solve similar problems [34-39]. Accordingly, we use an improved particle swarm optimization (IPSO), first proposed by Jiang, Hu [40], for comparing the results that calculating by OIO algorithm.

PSO is a population-based metaheuristic algorithm proposed by Kennedy and Eberhart [41]. Its concept originated from the social behavior of swarms. A particle in the swarm starts
from an initial position and moves in the search space according to the effects of two sources, namely the personal best \( p_{\text{best}} \) and global best \( g_{\text{best}} \). Specifically, each particle's velocity changes according to the effects of two sources, namely the personal best \( p_{\text{best}} \) and global best \( g_{\text{best}} \). The equations for updating velocity and position of each particle are:

\[
\begin{align*}
\dot{v}_{i,t} &= \omega \cdot v_{i,t} + C_1 \times (p_{\text{best}} - x_{i,t}) + C_2 \times (g_{\text{best}} - x_{i,t}) \quad (40) \\
\dot{x}_{i,t} &= x_{i,t} + v_{i,t} \quad (41)
\end{align*}
\]

The acceleration constants \( C_1 \) and \( C_2 \) in Eq. (40) are acceleration constants that control the effect of \( p_{\text{best}} \) and \( g_{\text{best}} \) on the velocity. On the other hand, \( \omega \) is the inertia factor, which is reduced throughout the search. The calculated velocities can be at most \( v_{\text{max}} \).

IPSO algorithm starts with a random population, which is clustered into some sub-populations. Then, PSO algorithm is applied to these sub-populations. At certain points in time, the sub-populations are merged in order to share information, and reclustered again.

The steps of IPSO algorithm are as follows:

**Step 1:** Choose \( p \geq 1, m \geq 1 \), where, \( p \) is the number of sub-swarms and \( m \) is the number of particles in each sub-swarm, set the Sample \( s=pm \) then calculate the objective function for each particle \( X_i \).

**Step 2:** Sort the function value of particles in ascending order and put them in an array \( E = \{ X_i, f_i \mid i = 1, \ldots, s \} \).

**Step 3:** Partition \( E \) into \( p \) sub-swarms \( A^1, A^2, \ldots, A^p \), such that:

\[
A^k = \{ X_j^k, f_j^k \mid X_j^k = X_{k+p(j-1)}, f_j^k = f_{k+p(j-1)}, j = 1, \ldots, m \}, k = 1, \ldots, p.
\]

**Step 4:** Evolve each \( A^k \) by particle swarm optimization (PSO).

**Step 4.1:** Determine the population size \( (q) \) and the maximum iteration \( (T) \).

**Step 4.2:** Select \( q \) particles \( Y_1^k, \ldots, Y_q^k \) from \( A^k \) by this strategy that the particles with better objective function have more probability to be selected. Store them in \( F^k = \{ Y_i^k, V_i^k, u_i^k \mid i = 1, \ldots, q \} \), where \( V_i^k \) is the velocity for particle \( Y_i^k \) and \( u_i^k \) is the corresponding function value. Set \( G^k \) the best individual of the whole swarm.

**Step 4.3:** Evaluate the function value of \( Y_i^k \) and \( P_i^k \). If \( Y_i^k \) is better, then put \( P_i^k = Y_i^k \). Evaluate the function value of \( Y_i^k \) and \( G^k \), and if \( Y_i^k \) is better, then put \( G^k = Y_i^k \).

**Step 4.4:** Update the position and velocity of each particle according to (40) and (41).

**Step 5:** substitute \( A^1, A^2, \ldots, A^p \) into \( E \).

**Step 6:** If convergence criteria are satisfied, stop. Otherwise, go to step 4.

5 Numerical Example and Sensitivity Analysis

In order to see and understand the patterns of key variables’ optimal dynamics, we conduct numerical analysis for LED 32" that is produced by SANAM Electronic Company. SANAM
Electronic is one of the leader companies in the electronic industry in Iran and began producing Color-Television sets named SANAM in 1993. The case company offers the following information: (1) Based on the price skimming strategy, the price starts from $280 (it is the maximum price) and prices decrease to $200. (2) The unit production cost for LED 32" is about $150. (3) According to historical data and experts’ opinions, the components are Main Board and Panel, which are two key components. Production cost, refurbishing cost, disposal cost, and holding cost for Main-Board are about $28 (per unit), $17 (per unit), $2 (per unit), and $2 (per unit/month) respectively, and these costs for Panel are $18 (per unit), $8 (per unit), $1 (per unit), and $1 (per unit/month) respectively. (3) Selling prices for Main Board and Panel for out-of-warranty products are $41 and $26, respectively. Other parameters are shown in Table 3.

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<td>In order to solve and analyze the case study problem, parameters of the OIO algorithm and IPSO algorithm are shown in Table 4 and Table 5, respectively.</td>
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After solving the problem, there are important dynamic relationships that can be derived from the numerical analysis. We first look into the relationship between $T_g$ and optimal warranty length and company profit. We also present the price trends, the production amount of the LED 32" product, and the production amount of spare parts. Finally, we show the impact of product’s failure rate on total profit and warranty length.

### 5.1 The optimal profit and warranty length for a given $T$ and $g$

Table 6 shows optimal profit for various $T$'s and $g$'s obtained by two algorithms, namely OIO and IPSO.

According to historical data, the product’s failure rate is 0.07 and the price and warranty coefficients ($k_1$ and $k_2$) are estimated by the company as 6 and 10. Currently, the company offers a two-year warranty for all products. We show that for each combination of $T$ and $g$, the company must choose different warranty periods. We run the proposed algorithm 10 times for each combination of $T$ and $g$ (720 problems in total), using MATLAB on a Pentium 4 computer with 8GB RAM and Corei7 3.61GHz CPU, and the results were reported in Table 3. The best objective function obtained, the worst, mean, standard deviation, and average time are reported in Table 3.

Table 6 shows that as product’s life cycle and guarantee period for spare parts availability increase, the manufacturer’s profit increases, but this does not necessarily mean that manufacturer can always select a longer life cycle or guarantee period. Because the competition situation is very complex and change in factors such as technology leads to changes in customer interest, the company must shift to new products.

As Figure 9 demonstrates, the best solutions for OIO algorithm are greater than or equal to best solutions obtained by IPSO algorithm, which shows the better performance of OIO in terms
of objective function (profit). Furthermore, it shows the impact of \( g \) and \( T \) changes on manufacturer profit.

**Figure 9 should be placed here**

**Table 6 should be placed here**

Figure 10 compared optimal warranty period for different life cycles and guarantee periods. It illustrates that optimal warranty period for \( g=34 \) is greater than or equal to other cases. On the other hand, longer life cycles have greater warranty periods. So it can be an important achievement for SANAM Company, because they often considered a fixed warranty period for products even if the products’ life cycles differ.

**Figure 10 should be placed here**

According to the current company’s policy, life cycle for LED 32” is 32 months \((T=32)\). So, the optimal prices for this \( T \) and different \( g \)’s is depicted in Figure 11.

**Figure 11 should be placed here**

It is worth noting that in final PPs, the slopes sharply increase. This is due to the fact that products produced in these periods have lower chance of failing as out-of-warranty products, so the manufacturer would better sell more products by decreasing the prices.

Total demand for spare parts are depicted in Figure. 12, where TD1 is for component 1 (Figure 11a) and TD2 is for component 2 (Figure 11b). This demand is met from two sources, namely, refurbishing (R1 and R2) and manufacturing. Q1 and Q2 represent the amount of product that should be produced. The increasing trend in these values are due to the cumulative demand of all products until a given time.

**Figure 12 should be placed here**

### 5.2 Sensitivity analysis and product’s failure rate changes

In this part, we focus on how the failure rate \((\lambda)\) changes affect optimal warranty period and total profit. In order to observe the dynamics more clearly, we need to use a fixed product life cycle \((T)\) and spare part availability period \((g)\). For the ensuing analysis, we consider \( T=32 \) and \( g=30 \). Table 7 shows the optimal value of warranty length and profit for various failure rate.

**Table 7 should be placed here**

As it can be observed in Figure. 13, manufacturer’s profits are increasing when failure rate increases or decreases. In situations the failure rate decreases, warranty costs decrease. So manufacturer can propose longer warranty length and sells more products. On the other hand with larger values of failure rate, manufacturer proposes smaller warranty length. Because, shorter warranty length leads to lower warranty cost and more profit from selling spare parts to out-of-warranty products; It should be noted that shorter warranty length and higher failure rate will
cause customer dissatisfaction, which will weaken the company’s competitiveness and lessen its market share.

Figure 13 should be placed here

6 Conclusion

The main purpose of the current study is to develop a new nonlinear model to integrate and optimize product’s price, warranty length and spare part inventory control decisions. Since the sale of spare parts to out-of-warranty product has sensible effects on a company profit, therefore a unique ability has been proposed in the presented model to calculate the number of out-of-warranty spare parts’ demand and optimize spare part inventory decisions which was not considered in previous studies. In order to solve the model, a new optimization approach was proposed that hybridizes the metaheuristic algorithm with a minimum cost network flow optimizer.

We solved the model to the real data of LED 32” by two type of algorithm. The first one was combination of OIO with MCNFP and the second one was combination of IPSO with MCNFP. Experimental analyses show that if the company decides to set a longer life cycle for the product, it is more profitable to propose longer warranty lengths, as compared with the cases where the life cycle is shorter. Additionally, it is recommended that the company decreases the prices more sharply in final periods of the life cycle compared to initial periods in order to benefit from increased sale in final periods. Finally, we found that product’s failure rate is inversely proportional to the warranty length, since increase in failure rate will lead to an increase in warranty costs, making it reasonable to decrease the warranty length.

As future direction it is interesting to conduct pricing for selling spare part for out-of-warranty products, since it will model the real-world condition more accurately. The model can also be extended to consider two-dimensional warranty case, which will make the model applicable to other fields such as automobile industry. Other considerations such as shortage and lost sales can also be incorporated in the inventory control problem, while minimizing shortage can be considered as another important objective along with maximizing the profit.

References


20. Yazdian, S.A., K. Shahanaghi, and A. Makui, "Joint optimisation of price, warranty and recovery planning in remanufacturing of used products under linear and non-linear demand, return and cost


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Mohsen Afsahi received his BS in Industrial Engineering from Kurdestan University in 2011. Also, he received MS and Ph.D. in Industrial Engineering for Tarbiat modares University (TMU)
in 2013 and 2017, respectively. Currently, he is lecturer in Science and Culture University and his research includes warranty inventory optimization, Pricing, Reliability, Maintenance and Simulation based Optimization.

**Table captions:**

Table 1. Relevant previous research works

Table 2. Indices, parameters, and decision variables

Table 3. Parameter values for LED 32"

Table 4. OIO parameters

Table 5. IPSO parameters

Table 6. Optimal profit for OIO and PSO algorithm

Table 7. Relationship between the failure rate, the warranty length and total profit

**Figure captions:**
Figure 1. Demand in product life cycle

Figure 2. The cycle of production, marketing and spare part inventory control

Figure 3. Products Situations

Figure 4. Spare part inventory system for under warranty and out-of warranty products

Figure 5. The min cost network flow problem for spare part inventory control

Figure 6. The solution representation

Figure 7. The idea behind generation of the new solutions in OIO

Figure 8. Flowchart of OIO algorithm

Figure 9. Comparison of OIO and PSO best solutions

Figure 10. Warranty length comparison for different T’s and g’s

Figure 11. Price trend for T=32

Figure 12. Spare part inventory

Figure 13. Failure Rate Analysis

### Tables

#### Table 1

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<th>References</th>
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<th>Planning horizon</th>
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Kim and Park [23]  
Wu, Chou [13]  
Lin, Wang [5]  
Faridimehr and Niaki [16]  
Tsao, Teng [18]  
Wei, Zhao [19]  
Yazdian, Shahanaghi [20]  
Chari, Diallo [24]  
Darghouth, Ait-kadi [45]  
This study

Table 2

| i  | Counter of key components for each product, \( i \in \{1,2,...,I\} \) |
| s  | Counter of Inventory Planning Periods (ICP), \( s \in \{1,2,...,T+g\} \) |
| j  | Counter of Pricing Periods (PP), \( j \in \{1,...,l(T+g)\} \) |

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<td>l</td>
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<td>( t_j )</td>
<td>The end of ( j^{th} ) PP, ( t_{j+1} - t_j = t_i \ \forall j \in {1,...,l(T+g)} )</td>
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<td>T</td>
<td>The number of ICP in product’s life cycle</td>
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<td>The number of ICP that guarantee the spare part availability after terminate product’s life cycle</td>
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<td>( L_s )</td>
<td>The calendar time for end of ( s^{th} ) ICP</td>
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\(^1\)WLO: Warranty Length Optimization, \(^2\)RO: Reliability Optimization, \(^3\)PRO: Production Rate Optimization, \(^4\)EOL: End of Life

This study
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<td>$h_{i}$</td>
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<td>$\rho_{pv}$</td>
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<td>$\lambda_d$</td>
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### Dependent variables

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<td>$y_{j,s}^{k}$</td>
<td>Random variable for number of failures for products with situation $k$ (situations will be described) that are produced in $j^{th}$ PP and failed in $s^{th}$ ICP</td>
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<td>$n_{j,s}^{k}$</td>
<td>Maximum number of failures for products with situation $k$ that are produced in $j^{th}$ PP and failed in $s^{th}$ ICP</td>
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<td>$S(j, P_{j}, w)$</td>
<td>Selling function in $j^{th}$ PP with price $P_{j}$ and warranty length $w$</td>
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<td>All components $i$ that have successful refurbishing in $s^{th}$ ICP</td>
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The number of on hand inventory at the end of $s^{th}$ ICP

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<td>Selling price during the $j^{th}$ PP</td>
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<td>$Q_{i}(s)$</td>
<td>Production amount of component i in $s^{th}$ ICP</td>
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Table 5

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*the parameter $\omega$ used is suggested by Shi and Eberhart [46]*

Table 6

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**Figures**

Figure 1
Calculate the objective function value as the fitness value.

Figure 5

An OIO individual

The variables optimally found by a minimum cost network flow solver

Figure 6
Figure 7

a) When the function surface serves as a convex surface and the artificial object/light point is between artificial focal point and the function surface.

b) When the function surface serves as a concave surface and the artificial object/light point is between artificial focal point and the function surface.

c) When the function surface serves as a concave surface and the artificial object/light point is between artificial focal point and centre of curvature.

d) When the function surface serves as a concave surface and the artificial object/light point is beyond centre of curvature.
FIGURE 8

Initialization
Generate a population of NO feasible solutions and evaluate them.

The best solution in the population named \( \hat{\mathcal{G}} \).
The value of the worst function value in the population named \( d_0 \).

\( t \leftarrow t + 1; \quad j \leftarrow 1; \quad k \leftarrow 1 \)

The \( j \)th member of the current population named \( \mathcal{G}_j \).

Generation of new solutions
Replace the \( j \)th member of the population with \( \mathcal{G}_j \).

Select \( c \) components randomly from \( \mathcal{G}_j \) and assign their values to their corresponding components in \( \mathcal{G}_j \).

Termination
If the stopping criteria are true,

\( j \leftarrow j + 1; \quad k \leftarrow k + 1 \)

Report \( \hat{\mathcal{G}} \) as the output of the algorithm and terminate.

FIGURE 9

Correction of spherical aberration
While:

\[ \left| \frac{\epsilon \epsilon'}{2} \right| > 0.01 \]

or

\[ \left| \frac{\epsilon \epsilon'}{2} \right| > 0.01 \]

the artificial mirror is concave

\[ m' \leftarrow m' - d_0 \]

No

Yes

\( d_0 \leftarrow -2d_0 \)

\[ m' \leftarrow m' + d_0 \]

Assume that the artificial mirror is convex:

\begin{align*}
\epsilon' & \leftarrow U[\epsilon, f(\hat{\mathcal{G}})] + d_0 \\
m' & \leftarrow U[f(\hat{\mathcal{G}}) - d_0, f(\hat{\mathcal{G}})]
\end{align*}

Assume that the artificial mirror is concave:

\begin{align*}
\epsilon' & \leftarrow U[f(\hat{\mathcal{G}}) + d_0, f(\hat{\mathcal{G}})] \\
m' & \leftarrow U[f(\hat{\mathcal{G}}) - d_0, f(\hat{\mathcal{G}})]
\end{align*}

Mirror type determination
Select an individual and named \( \mathcal{F}_i \).

\begin{align*}
(f(\mathcal{F}_i)) & > f(\hat{\mathcal{G}}) \\
\epsilon' & \leftarrow U[f(\hat{\mathcal{G}}), f(\hat{\mathcal{G}}) + d_0] \\
m' & \leftarrow U[f(\hat{\mathcal{G}}) - d_0, f(\hat{\mathcal{G}})]
\end{align*}

While:

\[ \left| \frac{\epsilon \epsilon'}{2} \right| > 0.01 \]

or

\[ \left| \frac{\epsilon \epsilon'}{2} \right| > 0.01 \]

the artificial mirror is concave

\[ m' \leftarrow m' + d_0 \]

No

Yes

\[ d_0 \leftarrow 2d_0 \]
Figure 10

Figure 11
a) Spare part inventory for component 1  
b) Spare part inventory for component 2

Figure 12

Figure 13