Novel Technique for Dynamic Analysis of Shear-Frames Based on Energy Balance Equations

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Abstract

In this paper, an efficient computational solution technique based on the energy balance equations is presented for the dynamic analysis of shear-frames, as an example of a multi-degree-of-freedom system. After deriving the dynamic energy balance equations for these systems, a new mathematical solution technique which is called Elimination of Discontinuous Velocities is proposed to solve a set of coupled quadratic algebraic equations. The method will be illustrated for the free vibration of a two-story structure. Subsequently, the damped dynamic response of a three-story shear-frame which is subjected to harmonic loading is considered. Finally the analysis of a three-story shear-building under horizontal earthquake load, as one of the most common problems in Earthquake Engineering, is studied. The results show that this method has acceptable accuracy in comparison with other techniques, but is not only faster compared with modal analysis but also does not require adjusting and calibrating the stability parameter as compared with a method of time integration like the Newmark method.

KEYWORDS: Numerical Technique; Dynamic Analysis; Shear-Frames; Energy Balance Equations; Coupled Equations; Elimination of Discontinuous Velocities
1. Introduction

Generally, in all engineering fields that deal with structural design, understanding the dynamic behavior of structures is very important [1]. In this context, although the applications of structural dynamics in aerospace engineering, civil engineering, engineering mechanics, and mechanical engineering are different, the principles and solution techniques are basically the same [2]. Accordingly, dynamic analysis plays a vital role in analyzing the dynamic response of buildings [3], dams [4,5], and bridges [6] to earthquakes. Control of very tall and slender buildings are among the most important issues for civil engineering researchers and have frequently been investigated in recent years.

Almost all practical structures are multiple-degree-of-freedom (MDF) systems, because of the distribution of dynamic properties such as mass in real systems, so many DOFs are required to determine the vibrational motion [7]. In addition, as we know, a greater number of DOFs will increase the complexity of solving a vibration problem. Thus, in engineering applications, we prefer to work with fewer DOFs without losing too much accuracy. For example, in the modeling of dynamical systems, simple structures (such as a water tank) when the majority of the mass of the system is concentrated in an area of the structure can be idealized as a system with a lumped mass (SDF\(^4\) systems) [8]. Also, under some condition such as when a mathematical function can express the variation of the mass and stiffness of structure, the real system is considered as a generalized single-degree-of-freedom system [3]. Furthermore, there are other techniques to reduce the dynamical DOFs of large order systems under some conditions (e.g., refer to [9-11]). However, in many cases in practical engineering works there is not the possibility of simplifying a real system to an SDF system, and performing an MDF dynamic analysis is essential.

There are various methods in order to evaluate the dynamic response of MDF systems. For example, in some special cases, by using the mathematical tools such as Fourier and Laplace integral transforms the exact solution of these problems can be obtained [12,13]. Moreover, modal analysis is a conventional approach to evaluate the response of MDF structures which are subjected to dynamic loads. One of the disadvantages of this method is its limitation for structures with non-linear behavior [1]. However some researchers have tried to modify the modal analysis in order to use it for nonlinear analyses; but, there is not still a comprehensive method for modal analysis of nonlinear structures (see, for example, [14-20]). Even though there exist some techniques to determine the eigenvalues and eigenvectors of large order systems (e.g. refer to [21,22]; as DOFs increase the calculation of eigenvalues and eigenvectors is particularly difficult, and that is another disadvantage of this approach.

In engineering analyses, the most general solution method for dynamic analysis is an incremental method or step-by-step direct time integration technique in which the equilibrium equations are solved at times \(\Delta t, 2\Delta t, 3\Delta t\), etc. [1]. In this category, Newmark [23], Houbolt [24] and Wilson [25] are some common implicit methods, and Central Difference method is one of the well-known explicit methods [26]. Stability and accuracy of these methods are essential in the practical analysis [27-31]; therefore, it is very important to use accurate and numerically efficient techniques in computer programs [32]. As a result of the large computational requirements, it can take a significant amount of time to solve structural systems with just a few hundred DOFs [26]. In addition, artificial or numerical damping must be added to most incremental solution methods to obtain stable solutions. For this reason, engineers must be very careful in the interpretation of the results [1]. Here, it should be noted that the artificial damping, which is defined as the reduction of the displacement amplitude with time for an undamped system [33], is different from the damping property of the structures.

Using energy balance equations, which is proposed in this study, can be an alternative approach to evaluate the dynamic response of a multi-dimensional system. In this context, as an instance, the energy conservation and dissipation properties of time-integration methods investigated by Acary [34] for the non-smooth elastodynamics with contact. Even though, several researchers in various fields such as hydrodynamic [35], aerospace [36,37] and CFD\(^5\) [38,39] have studied the energy method in order to determine the response of their dynamic systems, yet less attention has been paid to this topic in structural dynamics, except for a few studies that often have tried to use Hamilton’s Principle in order to calculate the frequency of simple SDF structures (see, for example, [40,41]).

Accordingly, this study aims to present a new numerical step-by-step method based on the energy equations for MDF shear-frames. The main idea of this approach is introduced first in [42] for linear and nonlinear SDF systems, and in the present paper, this technique is intended to be generalized to linear MDF structures. This method in the absence of damping leads to the de-coupled quadratic equations; and, when damping is considered, it leads to a set of coupled quadratic equation. According to the quadratic form of the algebraic equations at each time step, a novel mathematical technique called Elimination of Discontinuous Velocities is presented to detect the real velocity in every instance.

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4 Single-degree-of-freedom
5 computational fluid dynamics
In this study, shear frames are selected to illustrate the proposed method. The method largely eliminates the disadvantages of other methods, such as mathematical complexity and time-consuming calculation of a modal matrix in large order structures, as well as stability concerns and adjustment of the analytical coefficients in numerical integration methods. It should be noted that it is possible to extend this approach to other multi-dimensional structures as well. Furthermore, while in this investigation we assume that the structure will behave linearly; it is possible that the proposed method with some simple modifications can be used for nonlinear analyses in future studies.

2. Force vs. energy equations
In this section, shear-frames are introduced in brief. Subsequently, the mechanical energy relationships of these systems are expressed and by using the principle of conservation of energy, the equations of motions are derived from energy balance relationships of the system. The principal objective of these relations is to prove the equivalence of the force and energy approaches in structural dynamics. Finally, at the end of this section, the advantages and disadvantages of using these methods are compared with each other.

Fig.1 depicts an n-story shear-frame (or shear-building) as one of the simplest multi-degree-of-freedom systems that are widely used in civil engineering. In this idealization, the beams and floor systems are rigid in flexure, and several factors are neglected like axial deformation of the beams and columns, and the effect of axial force on the stiffness of the columns [8]. In this respect, the deflected building will have many of features of a cantilever beam that is deflected by shear force only, hence the name shear-building [43], where \( x_i \) denotes the displacement of \( i^{th} \) story. Moreover, \( k_i \) and \( m_i \) are respectively the stiffness and mass of \( i^{th} \) story. For these structures, the potential energy of the system (\( E_p \)) can be expressed as follows, by assuming a linear relationship between force and displacement.

\[
E_p = \frac{1}{2} m v^2 + \frac{1}{2} m v^2 + \ldots + \frac{1}{2} m v^2 + \ldots + \frac{1}{2} m v^2 + \frac{1}{2} m v^2
\]

Also, the kinetic energy of the structure regarding \( v_i = dx_i / dt \) (the velocity of \( i^{th} \) mass) is given by

\[
E_k = \frac{1}{2} k (x_i)^2 + \sum_{i=2}^{n} \frac{1}{2} k (x_i - x_{i-1})^2 + \sum_{i=1}^{n} \frac{1}{2} m v^2
\]

By neglecting the effects of energy dissipations and using the summation notation, the total energy of the system (\( E_T \)) (sum of the potential and kinetic energies) can be written as

\[
E_T = \frac{1}{2} k (x_i)^2 + \sum_{i=2}^{n} \frac{1}{2} k (x_i - x_{i-1})^2 + \sum_{i=1}^{n} \frac{1}{2} m v^2
\]

From the physical point of view, the law of conservation of energy states that the total energy of an isolated system remains constant, it is said to be conserved over time [44]. Hence, differentiating Eq.(3) with respect to time, we obtain

\[
\frac{dE_T}{dt} = 0
\]

alternatively,

\[
k_1 v_1 + \sum_{i=2}^{n} k (x_i - x_{i-1}) v_{i-1} + \sum_{i=1}^{n} m v_i a_i = 0
\]

where \( a_i \) is the acceleration of \( i^{th} \) mass, i.e.

\[
a_i = \frac{dv_i}{dt}
\]

Expanding the series in Eq.(5) leads to

\[
k_1 v_1 + k_2 (x_2 - x_1) v_2 + k_3 (x_3 - x_2) v_3 + k_4 (x_4 - x_3) v_4 + \ldots + m v a + m v_a + m v a + \ldots = 0
\]

By factoring \( v_1, v_2, v_3, \ldots \), one can write

\[
v_1 [k_1 x_1 + k_2 (x_2 - x_1) + m a] + v_2 [k_2 (x_2 - x_1) + m a_2] + \ldots + v_n [k_n (x_n - x_{n-1}) + m a_n] = 0
\]
Which corresponds to the following matrix form

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & m_n \\
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\vdots \\
\ddot{x}_n \\
\end{bmatrix}
+ \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 & 0 \\
-k_2 & k_2 + k_3 & -k_3 & 0 \\
0 & -k_3 & \ddots & -k_n \\
0 & 0 & \ldots & -k_n & k_n \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\] (9)

As we know, Eq.(9) represents the dynamic force equilibrium equations of an n-story shear-frame, and as previously mentioned, the primary objective of this part was the proof of the equality of energy and force balance equations. As a result, we can say that force equilibrium equations can be obtained from derivative of energy equations and mutually energy equations might be derived from the integration of force balance equations. It must be stated that although basically, these two equations are the same but each of them has its own advantages and disadvantages in practice. As an illustration, see Table.1

Here, it is important to note that the presented method in this study includes the linear behavior of shear frames (as a simple structural system). In other words, nonlinear response analysis of general structure like systems with hysteresis does not fall within the scope of this work. However, the idea presented in this research can be the basis for the ultimate goal of dynamic analysis of large-scale structures with different types of nonlinearities.

3. Methodology
3.1. Derivation of discretized energy balance scheme

Now, the energy balance approach is extended for general forced vibration problems to include the effects of damping, in which damping is assumed to be linear regarding of velocity (viscous damping) in this study. For this purpose, consider the equations of motion of an n-DOF shear-frame as follows

\[
m_i \ddot{x}_i + (c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_i + (k_i + k_{i+1}) x_i - k_i x_i = p_i
\]

\[
m_i \ddot{x}_i + (c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_i + (k_i + k_{i+1}) x_i + k_{i+1} x_{i+1} = p_{i+1}
\]

\[
m_i \ddot{x}_i + (c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_i + (k_i + k_{i+1}) x_i - k_i x_i - k_{i+1} x_{i+1} = p_i
\]

\[
m_{n-1} \ddot{x}_{n-1} + (c_{n-1} + c_n) \dot{x}_{n-1} - c_{n-1} \dot{x}_{n-1} + (k_{n-1} + k_n) x_{n-1} - k_{n-1} x_{n-1} = p_{n-1}
\]

\[
m_n \ddot{x}_n + c_n \dot{x}_n - c_n x_n - k_n x_n = p_n
\]

where \(c_i\) and \(p_i\) denote the damping coefficient and external force of \(i^{th}\) mass.

Integrating the \(i^{th}\) equation of Eq.(10) with respect to \(x_i\), we get

\[
\int \left( m_i \ddot{x}_i + (c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_i + (k_i + k_{i+1}) x_i - k_i x_i - k_{i+1} x_{i+1} \right) dx_i = \int p_i dx_i
\]

Each part of Eq.(11) according to the definition of various energies, \(E_i\), the area under the curve of the load-displacement, implies the changes of a specific type of energy.

\[
\int m_i \ddot{x}_i dx_i + \int [(c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_i - c_{i+1} \dot{x}_{i+1}] dx_i + \int [(k_i + k_{i+1}) x_i - k_i x_{i+1} x_{i+1}] dx_i = \int p_i dx_i
\]

The first integral in LHS represents the changes of kinetic energy \(\Delta E_k\), and by definition of velocity, it takes the following form

\[
\Delta E_k = \int m_i \ddot{x}_i dx_i \rightarrow \Delta E_k = \int m_i \dot{v}_i dv_i
\]

Integration from zero to arbitrary time gives

\[
\Delta E_k = \frac{1}{2} \int m_i \dot{v}_i^2 dy_i
\] (14)
The second integral in left-hand-side expresses the changes in damped energy ($\Delta E_D$), which is sometimes also called the energy loss, that is

$$\Delta E_D = \int_0^1 \left[ (c_i + c_{i+1}) \dot{x}_i - c_i \dot{x}_{i+1} - c_{i+1} \dot{x}_i \right] dx_i$$

(15)

Using the definition of velocity, Eq.(15) can also be written as

$$\Delta E_D = \int_0^1 \left[ (c_i + c_{i+1}) \ddot{v}_i - c_i \ddot{v}_{i+1} - c_{i+1} \ddot{v}_i \right] dt$$

(16)

Also, the change in potential energy ($\Delta E_P$) is

$$\Delta E_P = \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} - k_{i+1} x_i \right] dx_i$$

(17)

By operations equivalent to Eq.(17), it can be shown that

$$\Delta E_P = \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} - k_{i+1} x_i \right] v_i dt$$

(18)

and eventually, the changes in energy of the external loads $\Delta E_f$ is given by

$$\Delta E_f = \int_0^1 p_i v_i dt$$

(19)

Similarly, in order to have it in terms of velocity, Eq.(19) becomes

$$\Delta E_f = \int_0^1 p_i v_i dt$$

(20)

Therefore, energy balance equation for the $i^{th}$ mass is as follows

$$\Delta E_i + \Delta E_D + \Delta E_f = \Delta E_P$$

(21)

Here, energy balance equations are written for all of the masses

$$\frac{1}{2} m_{i+1} v_{i+1}^2 - \frac{1}{2} m_i v_i^2 + \int_0^1 \left[ (c_i + c_{i+1}) v_i^2 - c_i v_{i+1}^2 \right] dt + \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} \right] v_i dt = \int_0^1 \left[ p_i v_i \right] dt$$

$$\frac{1}{2} m_{i+1} v_{i+1}^2 - \frac{1}{2} m_i v_i^2 + \int_0^1 \left[ (c_i + c_{i+1}) v_i^2 - c_i v_{i+1}^2 \right] dt + \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} \right] v_i dt = \int_0^1 \left[ p_i v_i \right] dt$$

$$\vdots$$

$$\frac{1}{2} m_{i-1} v_{i-1}^2 - \frac{1}{2} m_i v_i^2 + \int_0^1 \left[ (c_i + c_{i+1}) v_i^2 - c_i v_{i+1}^2 \right] dt + \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} \right] v_i dt = \int_0^1 \left[ p_i v_i \right] dt$$

$$\vdots$$

$$\frac{1}{2} m_{1} v_1^2 - \frac{1}{2} m_2 v_2^2 + \int_0^1 \left[ (c_1 + c_2) v_1^2 - c_1 v_2^2 \right] dt + \int_0^1 \left[ (k_1 + k_2) x_1 - k_1 x_2 \right] v_1 dt = \int_0^1 \left[ p_1 v_1 \right] dt$$

(22)

Considering the $i^{th}$ mass

$$\frac{1}{2} m_i v_i^2 - \frac{1}{2} m_{i+1} v_{i+1}^2 + \int_0^1 \left[ (c_i + c_{i+1}) v_i^2 - c_i v_{i+1}^2 \right] dt + \int_0^1 \left[ (k_i + k_{i+1}) x_i - k_i x_{i+1} \right] v_i dt = \int_0^1 \left[ p_i v_i \right] dt$$

(23)

In principle, after discretizing Eq.(23) by using numerical integration methods, such as Trapezoidal and Simpson techniques [12], the correspondent energy equation of $i^{th}$ mass would be evaluated from Eq.(24). (See Appendix.A, for details)

$$A_i v_i^2 + B_i v_{i-1} v_i + C_i v_{i+1} v_i + D_i v_i v_{i+1} + E_i = 0$$

(24)

where, $A_i, B_i, C_i, D_i, E_i$ are constant coefficients that are determined from discretizing of integrals in energy balance relations; e.g., in the first time step that we have to use the Trapezoidal method, these coefficients take the following forms

5
\[
A_i = 0.5m_i + 0.5\Delta t(c_i + c_{i+1}), \quad B_i = -0.5\Delta t. c_i, \quad C_i = -0.5\Delta t. c_{i+1}
\]
\[
D_i = 0.5\Delta t\left[(k_{i} + k_{i+1})x_{i(\Delta t)} - k_{i} \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
\]
\[
E_i = -0.5m_i v_i^2 + 0.5\Delta t \cdot v_i \cdot \left[(c_i + c_{i+1})v_i - c_i \cdot v_{i-1(\Delta t)} + c_{i+1} \cdot v_{i+1(\Delta t)} + (k_i + k_{i+1})x_i - k_i \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
\]

In the time steps after the primary time step, to increase the accuracy of integration by using the Simpson method, one can write
\[
A_i = 0.5m_i + (\Delta t / 3)(c_i + c_{i+1}), \quad B_i = -(\Delta t / 3). c_i, \quad C_i = -(\Delta t / 3). c_{i+1}
\]
\[
D_i = (\Delta t / 3)\left[(k_{i} + k_{i+1})x_{i(\Delta t)} - k_{i} \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
\]
\[
E_i = -0.5m_i v_i^2 + (\Delta t / 3)\left[v_i \cdot \left[(c_i + c_{i+1})v_i - c_i \cdot v_{i-1(\Delta t)} + c_{i+1} \cdot v_{i+1(\Delta t)} + (k_i + k_{i+1})x_i - k_i \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
+ 4v_i \cdot \left[(c_i + c_{i+1})v_i - c_i \cdot v_{i-1(\Delta t)} + c_{i+1} \cdot v_{i+1(\Delta t)} + (k_i + k_{i+1})x_i - k_i \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
+ 2v_i \cdot \left[(c_i + c_{i+1})v_i - c_i \cdot v_{i-1(\Delta t)} + c_{i+1} \cdot v_{i+1(\Delta t)} + (k_i + k_{i+1})x_i - k_i \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]
+ \cdots + v_i \cdot \left[(c_i + c_{i+1})v_i - c_i \cdot v_{i-1(\Delta t)} + c_{i+1} \cdot v_{i+1(\Delta t)} + (k_i + k_{i+1})x_i - k_i \cdot x_{i-1(\Delta t)} - k_{i+1} \cdot x_{i+1(\Delta t)} - p_{i(\Delta t)} \right]\right]
\]

where \( j \) denotes the number of steps.

### 3.2. Solution procedure of coupled quadratic energy equations

As we saw in the previous section, after discretizing the energy balance equations, we are encountered with a set of equations in the following quadratic form
\[
a_1v_1^2 + c_1v_1v_2 + d_1v_2 + e_1 = 0
\]
\[
a_2v_2^2 + b_2v_1v_2 + c_2v_2v_1 + d_2v_2 + e_2 = 0
\]
\[
\vdots
\]
\[
a_nv_n^2 + b_nv_{n-1}v_n + c_nv_{n-1}v_n + d_nv_n + e_n = 0
\]

Mathematically, in solving the previous equations there are two main problems: A) these equations are coupled and that means they are dependent on each other and must be solved simultaneously; in other words, we cannot calculate \( v_j \) from \( j \)th equation directly. In addition, it should be noted that in the absence of damping the equations would be decoupled, viz., damping is the reason for coupling the equations. B) The quadratic form of equations implies that more than one velocity at each time step; i.e., from the physical point of view at every time step these relations provide some unreal velocities in the actual velocity of the structure. To better understand the above expressions, suppose that in a sample 2-DOF structure in a given time step we want to solve a mathematical equation of the form
\[
\begin{align*}
v_1^4 + v_2v_1 + v_1 + 1 &= 0 \\
v_2^4 + 2v_2^3 + v_2 + 2 &= 0
\end{align*}
\]

If the terms of \( v_2v_1 \) do not exist, one can obtain two values of each of \( v_1 \) and \( v_2 \) from solving the two uncoupled quadratic equation. However, with considering the \( v_2v_1 \), by combining the equations together and write them only in terms of one variable, we have
\[
\begin{align*}
v_1^4 + 2v_1^3 + 6v_1^2 + 3v_1 + 2 &= 0 \\
v_2^4 + 2v_2^3 + 8v_2^2 + 2v_2 + 4 &= 0
\end{align*}
\]

From Eq.(29), by solving the two fourth-degree equation, it is apparent that four values for each of \( v_1 \) and \( v_2 \) will be obtained. Note that at any moment the velocity of each mass is unique and only there is one value for real velocity of the system while in this case, three unrealistic velocities for each mass have appeared in the equation. Apparently, this
method (direct method) cannot be used to calculate the velocities at any instants, especially in large DOF systems, and we have to use a numerical method to calculate the velocities in each time step. In this context, as it is demonstrated in Appendix.B, well-known solution techniques like Newton-Raphson method are not efficient for the system of equations under consideration in this study. Two main reasons for the deficiency of these methods when applied to the considered equations in each time-step can be expressed as follows: a) Need for derivative of the system of equations b) Complex and time-consuming process of inverting the Jacobi matrix (specifically in large-scale structures).

3.3. Elimination of discontinuous velocities technique

The problem of coupled equations exists in many engineering fields, particularly in multi-dimensional systems, hence many researchers have studied how to solve these equations (e.g., refer to [45-51]). For the current study, a novel numerical method is presented in which the real velocities of the system at any time step can be easily calculated by removing the unrealistic velocities from the coupled equations.

In the proposed technique, first, the problem of coupled equations is resolved by neglecting of the coupling terms (terms that are the product of two different velocities). In this case, assuming a structure with n-DOF, we are faced with an n-quadratic equation in terms of velocity. To solve the problem of non-linear equations (quadratic in terms of velocity) and detect the actual velocities of system at any time, it is assumed that the variation of velocities with respect to time is continuous. Therefore, among the two velocities obtained at any time from the quadratic equations, the velocity closer to the velocity of the previous time step is selected as the real velocity of the structure. Therefore the name of the method is chosen as Elimination of Discontinuous Velocities Technique. At the beginning of this procedure, the coupling terms are ignored to obtain the velocities; here, the values of continuous velocities are substituted into them, and this iteration will be continued until the velocities in two subsequent iteration approach to each other. Table.2 gives a summary of the method.

4. Numerical examples and results

Various examples of multi-story shear-frame structures are analyzed by using the energy method in this section. In the first example, the vibration of a simple two-story shear-building has been investigated to describe the procedure of presented method in detail. In the next examples, some multi-story shear-frame structures subjected to harmonic and earthquakes loadings have been studied. Moreover, the results were compared with the exact solution and other common methods

Example 4.1. The free damped vibration of a two-story shear building.

Fig.2 shows a 2-DOF shear-frame in which for convenience the dynamic properties of the structure are chosen as: \( m_1=m_2=k_1=k_2=1 \), and \( c_1=0.06, c_2=0.16 \) (all units assumed to be compatible). Also, the following initial conditions will be considered in this example.

\[
x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]

(30)

In free-vibration cases, the equation of motion of these structures can be expressed as

\[
[m] \{\ddot{x}\} + [c] \{\dot{x}\} + [k] \{x\} = \{0\}
\]

(31)

where the mass, damping, and stiffness matrices are

\[
[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [c] = \begin{bmatrix} 0.22 & -0.16 \\ -0.16 & 0.16 \end{bmatrix}, \quad [k] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}
\]

(32)

Thus, by multiplying the matrices and vectors, we have the following governing equations

\[
\ddot{x}_1 + 0.22\dot{x}_1 - 0.16\dot{x}_2 + 2x_1 - x_2 = 0
\]

\[
\ddot{x}_2 - 0.16\dot{x}_1 + 0.16\dot{x}_2 - x_1 + x_2 = 0
\]

(33)

Here, the Laplace transform method is used to determine the exact solution of this problem (for details, see Appendix.C).

\[
\begin{aligned}
x_1 &= -0.336e^{-0.1733s} \cos(1.608t + 0.681) + 4.445e^{-0.0167s} \cos(0.618t - 1.283) \\
x_2 &= 0.210e^{-0.1733s} \cos(1.608t + 0.753) + 7.180e^{-0.0167s} \cos(0.618t - 1.3106)
\end{aligned}
\]

(34)
Applying the energy method
By substituting the assumed parameters of this example into the Eq.(23), the energy balance equations of the system would be as follows

\[
\begin{align*}
0.5v_1^2 - 0.5v_{1(0)}^2 + \int_0^t [0.22v_1^2 - 0.16v_2v_1] \, dt + \int_0^t [2x_1 - x_2] v_i \, dt &= 0 \\
0.5v_2^2 - 0.5v_{2(0)}^2 + \int_0^t [0.16v_2^2 - 0.16v_1v_2] \, dt + \int_0^t [x_2 - x_1] v_2 \, dt &= 0
\end{align*}
\]

(35)

Now, the integrals in above equations should be discretized to obtain the algebraic equations. Since Simpson rule needs at least three-point for integration, it cannot be used in the first time step. Hence trapezoidal method must be used in the first time step. For the problem at hand, the size of time intervals are assumed to be \( \Delta t=0.1s \), and according to Table.2, \( x_1 \) and \( x_2 \) (dynamic responses of floors at the time of \( t=0.1s \)) would be approximated by the Euler formula as follows

\[
\begin{align*}
x_{1(0.1)} &\approx x_{1(0)} + v_{1(0)} \times \Delta t = 1.3 \\
x_{2(0.1)} &\approx x_{2(0)} + v_{2(0)} \times \Delta t = 2.4
\end{align*}
\]

(36)

Discretized form of Eq.(36) is

\[
\begin{align*}
0.511v_1^2 - 0.008v_2v_1 + 0.01v_1 - 4.497 &= 0 \\
0.508v_2^2 - 0.008v_1v_2 + 0.055v_2 - 7.768 &= 0
\end{align*}
\]

(37)

Neglecting the coupling term, \( i.e. \) (-0.008\( v_1v_2 \)), we have two quadratic equations as follows

\[
\begin{align*}
0.511v_1^2 + 0.01v_1 - 4.497 &= 0 \\
0.508v_2^2 + 0.055v_2 - 7.768 &= 0
\end{align*}
\]

(38)

Roots of these quadratic equations are

\[
\begin{align*}
0.511v_1^2 + 0.01v_1 - 4.497 &= 0 \rightarrow v_1 = 2.9568, v_1 = -2.9763 \\
0.508v_2^2 + 0.055v_2 - 7.768 &= 0 \rightarrow v_2 = 3.5867, v_2 = -3.9649
\end{align*}
\]

(39)

By comparing the roots obtained from Eq.(39) with the velocity in the previous step, \( v_1=3, v_2=4 \), the closest velocities to previous step are selected and other ones are omitted.

\[
\begin{align*}
0.511v_1^2 + 0.01v_1 - 4.497 &= 0 \rightarrow v_1 = 2.9568, v_1 = -2.9763 \\
0.508v_2^2 + 0.055v_2 - 7.768 &= 0 \rightarrow v_2 = 3.5867, v_2 = -3.9649
\end{align*}
\]

(40)

Now, new values \( x_1 \) and \( x_2 \) at \( t=0.1s \) can be approximated by using the average of the velocity of this step and previous step, hence

\[
\begin{align*}
x_1 &\approx x_{1(0)} + \frac{v_{1(0)} + v_{1(0.1)}}{2} \times \Delta t = 1.2978 \\
x_2 &\approx x_{2(0)} + \frac{v_{2(0)} + v_{2(0.1)}}{2} \times \Delta t = 2.3928
\end{align*}
\]

(41)

Here, the coupling term (-0.008\( v_1v_2 \)), which was neglected previously, might be given by the substitution of \( v_1=2.9568 \) and \( v_2=3.5867 \).

\[
v_1 = 2.9568, v_2 = 3.5867 \rightarrow -0.008v_1v_2 = -0.0848
\]

(42)

And new values for velocities of the system can be determined as follows

\[
\begin{align*}
0.511v_1^2 - 0.008v_2v_1 + 0.01v_1 - 4.497 &= 0 \rightarrow -0.008v_1v_2 = -0.0848 \\
0.508v_2^2 - 0.008v_1v_2 + 0.055v_1 - 7.768 &= 0 \rightarrow 0.511v_1^2 + 0.01v_1 - 4.5882 = 0 \\
0.508v_2^2 + 0.055v_1 - 7.8592 &= 0
\end{align*}
\]

(43)
Similarly, the real velocities are
\[
\begin{align*}
0.511v_1 + 0.01v_1 - 4.5882 &= 0 \
0.508v_2 + 0.055v_1 - 7.8592 &= 0
\end{align*}
\]
\Rightarrow v_1 = 2.9867, v_1 = -3.0063 \times 10^{-2} \tag{44}

Now, the updated coupling term becomes
\[
v_1 = 2.9867, v_2 = 3.8795 \rightarrow -0.008v_1 v_2 = -0.0927 \tag{45}
\]

By introducing a relative error, \( e_i \), for velocities, as an absolute value of \((v_i' - v_i')/v_i'\) for a convergence criterion, where \( i \) is the number of story and \( j \) denotes the number of iteration. As shown in Table 3, the procedure can better be monitored by this definition. Note that in this case tolerance is chosen as \( 10^{-2} \). Also, Fig.3 illustrates the process of convergence which is shown the error of analysis vs. number of iteration.

If we use a computer program to continue the process to \( t=10s \), we can get the dynamic response of the system, as displayed in Fig.4. This figure compares the obtained results of the presented method against the exact solution of the problem. As shown in Fig.4, notwithstanding the size of time intervals \( \Delta t=0.1 \) selected is not very small in this analysis, it can be seen that the proposed method has excellent accuracy compared with the exact solution; in other words, the numerical solution can properly approach the exact solution of the problem in this case.

Now, by choosing a fixed time interval, which is deliberately not chosen too small, typically \( \Delta t=0.2 \) here, we are going to compare the accuracy and speed of analysis of presented method with other conventional methods such as modal [52], Newmark [23], and combined techniques like modal-Duhamel [53,54] and modal-Newmark (see Fig.5 and Fig.6). Adjustment factors in Newmark method, which are used to improve the accuracy and stability of the method, respectively are selected as: \( \beta=0.5 \) and \( \gamma=1/6 \) (typically these values which yield the linear acceleration method, are used in practice). In addition, combined modal methods also by converting an n-DOF structure to n-SDF systems and by using numerical techniques such as Duhamel and Newmark the structural analysis will be performed.

The tolerance of the proposed method is considered as \( 10^{-2} \).

From Fig.5 and Fig.6, it can be observed that the presented method compared to other conventional methods used in the dynamic analysis of MDF structure, has acceptable accuracy. In fact, (considering a constant \( \Delta t \)) among all methods, combined Modal-Duhamel and the proposed method have been closer to the exact response of problem. Furthermore, in engineering analysis, the time required to calculate the solution, or the speed of numerical technique, is one of the factors influencing the choice of method. So in this section, in accordance with the Table.4, the required times for analysis by the various numerical methods are compared. The results show that the methods using the modal techniques are very time-consuming compared to other methods; For example, the computational time for the proposed approach is less than half of the other modal methods.

It must be here stated that although the whole of damping matrices considered in this study are of a classical/proportional type, but in general, for non-classical damping, it is not easy to apply a conventional modal method. Because in this case the frequencies, the shape-modes, and damping ratios in addition to the mass and stiffness matrices, depending on the damping matrix of the system, and the complex modal coordinate must be used (for more details, see [55-57]). On the other hand, it is noteworthy that the energy-based method presented in this research has not any limitations in this regard and the classical or non-classical damping will be analyzed without a particular modification (it is another advantage of this technique).

**Example 4.2. The damped harmonic vibration of a three-story shear building.**

A two-DOF shear-frame is depicted in Fig.7 in which, like the previous example, for convenience, the dynamic properties of the structure are selected as: \( m_1=m_2=m_3=k_2=k_3=1, c_1=c_2=c_3=0.1 \), and the zero initial conditions will be assumed in this example. The structure is subjected to harmonic loads as: \( p_1=\cos t, p_2=\cos 2t, p_3=\cos 3t \). (All units are compatible)

In this case, the equation of motion is
\[
[m]\ddot{x} + [c]\dot{x} + [k]x = \{p\} \tag{46}
\]

where the mass, damping, and stiffness matrices are
\[
[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
[c] = \begin{bmatrix} 0.2 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix},
[k] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},
\{p\} = \begin{bmatrix} \cos t \\ \cos 2t \\ \cos 3t \end{bmatrix} \tag{47}
\]
Hence, the governing equations of this problem would be given by

\[
\begin{align*}
\ddot{x}_1 + 0.2\dot{x}_1 - 0.1\dot{x}_2 + 2x_1 - x_2 &= \cos t \\
\ddot{x}_2 - 0.1\dot{x}_1 + 0.2\dot{x}_2 - 0.1\dot{x}_3 + x_1 + 2x_2 - x_3 &= \cos 2t \\
\ddot{x}_3 - 0.1\dot{x}_2 + 0.1\dot{x}_3 - x_2 + x_3 &= \cos 3t
\end{align*}
\]

(48)

As for the previous example, the Laplace transform method is used to determine the exact solution of the problem (for details, see Appendix.D).

\[
\begin{align*}
x_1 &= 0.2131\exp(-0.01t)\cos(0.4449t + 0.0330) - 0.9089\exp(-0.0775t)\cos(1.2446t - 0.3013) + \\
& -0.5188\exp(-0.1625t)\cos(1.7946t + 0.5171) + 0.9674\cos(t - 0.2936) + 0.3286\cos(2t + 0.9803) + 0.002\cos(3t + 3.9267) \\
x_2 &= 0.3831\exp(-0.01t)\cos(0.4449t + 0.0282) + 0.4042\exp(-0.0775t)\cos(1.2446t + 2.8406) + \\
& 0.6476\exp(-0.1625t)\cos(1.7946t + 0.517) + 0.0962\cos(t + 4.4169) + 0.6581\cos(2t + 3.7258) + 0.0189\cos(3t + 0.4542) \\
x_3 &= 0.4772\exp(-0.01t)\cos(0.4449t + 0.0318) + 0.7288\exp(-0.0775t)\cos(1.2446t - 0.3012) + \\
& 0.4097\exp(-0.1625t)\cos(1.7946t + 4.0561) + 0.9618\cos(t + 2.9491) + 0.2235\cos(2t + 0.8471) + 0.1271\cos(3t + 3.1911)
\end{align*}
\]

(49)

\textbf{Applying the energy method}

Using Eq.(23), the energy balance equations of this system would be as follows

\[
\begin{align*}
0.5v_i^2 + \int_0^t \left[0.2v_i^2 - 0.1v_jv_j\right] dt + \int_0^t \left[2x_i - x_j\right] v_j dt &= \int_0^t v_i \cos t dt \\
0.5v_j^2 + \int_0^t \left[0.2v_j^2 - 0.1v_i v_i - 0.1v_j v_j\right] dt + \int_0^t \left[2x_j - x_i - x_i\right] v_i dt &= \int_0^t v_j \cos 2t dt \\
0.5v_i^2 + \int_0^t \left[0.1v_i^2 - 0.1v_jv_j\right] dt + \int_0^t \left[2x_i - x_i\right] v_j dt &= \int_0^t v_i \cos 3t dt
\end{align*}
\]

(50)

After discretizing the Eq.(50) and using Table.2, the same procedure as in the former example, must be performed. In this case, Fig.8 through Fig.10 gives the obtained results where the dynamic response of floors, assuming \(\Delta t=0.2s\) and \(e=0.01\), is plotted by using the various numerical methods vs. exact solution of the problem.

Fig.8-Fig.10 demonstrate that with a fixed size for time intervals, the proposed method in this study together with Modal-Duhamel technique are very close to the exact solution of the problem, and the methods using Newmark technique (with \(\beta=0.5\) and \(\gamma=1/6\)) are not appropriately converged. Here, similar to the previous example, the required time for analysis of this example is indicated in Table.5 where it can be observed that, like former analysis, the modal techniques are very time-consuming compared with others. In addition, note that although it is true that Newmark method has a good speed, in this case, its accuracy is not good compared to other methods.

After observing Figs.8-10, given that the modal-Duhamel method has been shown to be more accurate than other approaches. Here, the effect of the numerical technique used in the approximation of the Duhamel integral is investigated. In this regard, in addition to the Simpson rule which was used at first, the Trapezoidal rule for computing the Duhamel integral is also provided in Figs.11-13. Moreover, it must be mentioned that to prevent the cluttered graphs the results of the Newmark and Modal-Newmark methods are not represented in these figures.

Generally, Figs.11-13 show that the accuracy of the Duhamel method is strongly dependent on the numerical method (Simpson with trapezoidal) used in the approximation of this integral. As compared to the proposed method, the use of the Simpson method leads to more accurate results, and conversely, the application of the Trapezoidal rule leads to a reduction in the accuracy compared with the presented method.

In the following, to examine the efficiency of the proposed method in the case of large-scale structures. A high-rise 20-story shear-frame (as a generalized system of the structure studied in Example 4.2.) is considered with dynamic properties as below

\[
m_i = 1 , \quad c_i = 0.1 , \quad k_i = 1 , \quad p_i = \cos it , \quad i = 1,2,\ldots,20
\]

(51)

Now, by choosing the last node above the structure (as a control point) and then employing the proposed method, if we plot the roof’s velocity in 10 seconds versus two converged velocities: \(V_{\text{roof}}(1)\) and \(V_{\text{roof}}(2)\) at the end of iterations in quadratic energy equation, Eq.(24) as displayed in Fig.14.
From Fig.14 it can be seen that in this large-scale system, the roof velocity is properly calculated from the selection of right velocity based on the assumption of continues velocities in time. Though, it seems that future works especially by considering the nonlinear behavior in other high-rise building systems are essential to verify the efficiency of the given method in general problems of structural dynamics.

**Example 4.3. The forced damped vibration of a three-story shear building subjected to an earthquake.**

Consider a three-story shear-frame that is described in Fig.15, and subjected to ground motion, EL-Centro earthquake \((PGA^t=0.3g)\) as shown in Fig.15. Also, the dynamic characteristics of the system are: \(m_1=m_2=m_3=1\), \(c_1=c_2=c_3=0.05\), and \(k_1=k_2=k_3=10\). Moreover, the zero initial conditions are assumed in this case. (All units are compatible)

In this case, due to earthquake loading, by definition of effective force \((p_{eff})\) the equation of motion is

\[
[m] \{\ddot{x}\} + [c] \{\dot{x}\} + [k] \{x\} = \{p_{eff}\}
\]

where, \(p_{eff}\) defines as the negative product of the mass matrix \([m]\), influence coefficient vector \(\{l\}\), and the acceleration vector of ground motion \(\{\ddot{x}\}\), i.e.

\[
\{p_{eff}\} = -[m] \{l\} \{\ddot{x}\}
\]

For the problem at hand, influence coefficient vector is

\[
\{l\}^T = \{1,1,1\}
\]

where, mass, damping, and stiffness matrices are

\[
[m] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, 
[c] = \begin{bmatrix}
0.1 & -0.05 & 0 \\
-0.05 & 0.1 & -0.05 \\
0 & -0.05 & 0.05
\end{bmatrix},
[k] = \begin{bmatrix}
20 & -10 & 0 \\
-10 & 20 & -10 \\
0 & -10 & 10
\end{bmatrix}
\]

**Applying the energy method**

According to Table.2, and two examples mentioned before, via the Energy method, and assuming prior assumptions, except the value of tolerance that is equal to \(\varepsilon=10^{-4}\), the dynamic response of the structure can be plotted. As we know, in this kind of problems there is not a closed-form analytical solution which can be used to compare the results. Thus, only the results of various numerical methods (in a fixed time interval equal to 0.02) are plotted in Figs. 16 to 18.

It should be noted that in Fig.16-Fig.18 marked points on the figures are not indicative time steps and they are selected to distinguish the results better. In addition, the obtained results of Newmark and Modal-Newmark methods are overlapped, and cannot properly be identified. According to the figures, an acceptable agreement between the presented method and other methods can be seen. Hence, this method can be used for long-time dynamic analysis of shear-frames such as seismic analyses.

Once again for comparing the speed of analyses, Table.6 gives the required time to analyze the third example of this investigation. Similar to previous examples, it can be observed that, in a constant time interval, the proposed method regarding computational time is in second place after Newmark method.

At first glance, however, the times given in Table.6 may look great for a small 3-DOF structure. However, it should be here mentioned that the times presented in this table include executing all the commands written within the MATLAB program (e.g., time-consuming syntaxes like (xlsread). In other words, these values do not indicate the real time of the implementation of the integration schemes and are used only for comparison between different types of methods.

---

6 Peak Ground Acceleration
4.4. Stability and accuracy analysis
Here, the effects of time step size on the accuracy and stability of the presented method are discussed. In this regard, Bathe [58] has been proposed a technique based on the free response analysis of a simple SDF system, as shown in Fig.19. For simplicity, if we assume the following parameters in a compatible unit system: \( m=1 \); \( k=4\pi^2 \); \( x_0=0 \); and, \( v_0=1 \). The free response of this system (exact solution of the problem) can be written as follows
\[
x(t) = \frac{\sin(2\pi t)}{2\pi}
\]  
(56)
With respect to this exact response, the values for the period and amplitude of the vibrational motion are equal to \( T_{\text{Exact}} = 1 \) and \( A_{\text{Exact}} = 1/2\pi \), respectively. Obviously, the numerical solution obtained from the presented method will differ with these values. So, it would be appropriate to define two following parameters.
\[
R_T = \left| \frac{T_{\text{Exact}} - T_{\text{Num}}}{T_{\text{Exact}}} \right|
\]  
(57)
and,
\[
R_A = \left| \frac{A_{\text{Exact}} - A_{\text{Num}}}{A_{\text{Exact}}} \right|
\]  
(58)
where \( R_T \) and \( R_A \) respectively represent the numerical error in periodic and amplitude of the vibrational system. \( T_{\text{Num}} \) and \( A_{\text{Num}} \) also are the period and amplitude obtained from the numerical method, which are functions of the size of the time step, \( \Delta t \), used to discretize the time. Thus, taking into account different values for the time step, one can plot the parameters \( R_T \) and \( R_A \) as shown in Fig.20.
According to Fig.20, as the time step increases (with the rise of the numerical error in the system response), the accuracy of the solution is reduced, as expected. For example, when the time step size is \( \Delta t=0.1 \), the relative errors in the period and amplitude of the system respectively are equal to about 1.5% and 13%. In general, the results of this section show a greater sensitivity to the amplitude of motion than the period (this is in line with the results of the reference [27]).
Moreover, a key observation can be made from this figure; that is, by increasing the value of time step numerical errors increase significantly at a certain value (about 0.1 to 0.15), indicating that instability will occur in the numerical solution. For example, in the case of \( \Delta t=0.15 \), the use of about 6-7 points for the approximation of a complete sine wave created a significant error. Therefore, the selection of the appropriate value for \( \Delta t \) is essential in practice. Since large ones, by eliminating the precision of solution, can lead to instability. On the other hand, small \( \Delta t \) also increase the computational time. Consequently, an optimum size for time step should be used in practical dynamic analyses.

5. Conclusions
In this paper, a novel step-by-step solution technique based on energy method is presented for the dynamic analysis of shear-frames, as one of the applicable structures in practice. In this method rather than working with the equation of motions, we solve the energy balance relationships which have some advantage. For example, it leads to a reduction of unknowns. The proposed method is performed on various examples including harmonic and earthquake loading. The main implications of the study can be listed as follows:
  - The results show that the method has good accuracy compared with other common methods (e.g., it is more accurate vs. Newmark method).
  - Another advantage of this method compared to other time integration methods such as Newmark is avoiding selecting and calibrating the velocity and acceleration adjustment parameters such as \( \gamma \), \( \beta \).
  - Modal methods which have shown good accuracy in combination with Duhamel’s Integral, has complex mathematic relationships, particularly with increasing the degrees of freedom of the structure, and as was observed in this study, they are more time-consuming than other techniques.
  - The presented method, with a simple mathematical algorithm, has good accuracy and speed of analysis, and by selecting an allowable tolerance (usually in the range of 0.01-0.0001), it can be used in practical dynamic analyses of shear-frames.
Finally, it should be noted that the ideas expressed in this research have the capability to be applied to other engineering structures and also non-linear systems with some modifications.
Appendix A. Discretization of integral energy equations.
A.1. Trapezoidal rule

The value of \( \int_0^\Delta t f(t)\,dt \) can be evaluated by the Trapezoidal rule

\[
\int_0^\Delta t f(t)\,dt = \frac{\Delta t}{2} \left[ f(0) + f(\Delta t) \right] \tag{A.1}
\]

Recalling Eq.(23)

\[
\frac{1}{2} m v_i^2 - \frac{1}{2} m v_i(0)^2 + \int_0^\Delta t \left[ (c_i + c_{i+1}) v_i^2 - c_i v_{i-1}^2 v_{i+1} v_i - c_{i+1} v_{i+1}^2 v_i \right] \,dt + \int_0^\Delta t \left[ (k_i + k_{i+1}) x_i - k_{i-1} x_{i} - k_{i+1} x_{i+1} \right] v_i \,dt = \int_0^\Delta t p v_i \,dt \tag{23-rep.}
\]

Now, considering Eq.(23) and the use of the trapezoidal rule, the integrals in this expression can be discretized as follows.

\[
\int_0^\Delta t \left[ (c_i + c_{i+1}) v_i^2(t) - c_i v_{i-1}(t)v_i(t) - c_{i+1} v_{i+1}(t)v_i(t) \right] \,dt = \frac{\Delta t}{2} \left\{ (c_i + c_{i+1}) v_i^2(0) - c_i v_{i-1}(0)v_i(0) - c_{i+1} v_{i+1}(0)v_i(0) \right\} \tag{A.2}
\]

\[
\int_0^\Delta t \left[ (k_i + k_{i+1}) x_i(t) - k_{i-1} x_{i} - k_{i+1} x_{i+1}(t) \right] v_i(t) \,dt = \frac{\Delta t}{2} \left\{ (k_i + k_{i+1}) x_i(0) - k_{i-1} x_{i} - k_{i+1} x_{i+1}(0) \right\} \tag{A.3}
\]

\[
\int_0^\Delta t \left[ p(t)v_i(t) \right] \,dt = \frac{\Delta t}{2} \left\{ p(0)v_i(0) + p(\Delta t)v_i(\Delta t) \right\} \tag{A.4}
\]

Substituting Eqns.(A.2)-(A.4) into Eq.(23), and rearranging with regard to velocities, yields

\[
A_i v_i^2(\Delta t) + B_i v_{i+-1}(\Delta t) v_i(\Delta t) + C_i v_{i+1}(\Delta t) v_i(\Delta t) + D_i v_i(\Delta t) + E_i = 0 \tag{24-rep.}
\]

where,

\[
A_i = 0.5m_i + 0.5\Delta t(c_i + c_{i+1}) \quad , \quad B_i = -0.5\Delta t c_i \quad , \quad C_i = -0.5\Delta t c_{i+1}
\]

\[
D_i = 0.5\Delta t \left[ (k_i + k_{i+1}) x_i(\Delta t) - k_{i-1} x_{i+-1} - k_{i+1} x_{i+1} \right] \quad \text{(25-rep.)}
\]

\[
E_i = -0.5m_i v_i^2(0) + 0.5\Delta t \left[ (c_i + c_{i+1}) v_i(0) - c_i v_{i-1}(0) + c_{i+1} v_{i+1}(0) + (k_i + k_{i+1}) x_i(0) - k_{i-1} x_{i-1}(0) - k_{i+1} x_{i+1}(0) - p_{i}(0) \right]
\]
A.2. Simpson rule

Considering $j$th time step, i.e., $t=j\Delta t$ ($j=2,3,\ldots$).

In this case, $\int_0^t f(t)\,dt$ can be approximated by the composite Simpson rule as follows

$$\int_0^{jt} f(t)\,dt \approx \frac{\Delta t}{6} \left[ f(0) + 4f(\Delta t) + 2f(2\Delta t) + \ldots + 4f(j\Delta t) + f(jt) \right]$$

(A.5)

Similar to before, the discretized form of Eq.(23) using Simpson rule is given by

$$\int_0^{jt} \left[ (c_i + c_{i+1})v_i^2(t) - c_i v_i(t) v_i(t) - c_{i+1} v_{i+1}(t) v_i(t) \right] \,dt =$$

$$\frac{\Delta t}{6} \left[ ((c_i + c_{i+1})v_i^2(0) - c_i v_i(0) v_i(0) - c_{i+1} v_{i+1}(0) v_i(0)) + 4((c_i + c_{i+1})v_i^2(\Delta t) - c_i v_i(\Delta t) v_i(\Delta t) - c_{i+1} v_{i+1}(\Delta t) v_i(\Delta t)) \right]$$

$$+ \ldots + ((c_i + c_{i+1})v_{i+1}^2(j\Delta t) - c_i v_i(j\Delta t) v_i(j\Delta t) - c_{i+1} v_{i+1}(j\Delta t) v_i(j\Delta t)) \right]$$

(A.6)

$$\int_0^{jt} \left[ (k_i + k_{i+1})x_i(t) - k_i x_i(t) - k_{i+1} x_{i+1}(t) \right] \,v_i(t) \,dt =$$

$$\frac{\Delta t}{6} \left[ ((k_i + k_{i+1})x_i(0) - k_i x_i(0) + k_{i+1} x_{i+1}(0)) + 4((k_i + k_{i+1})x_i(\Delta t) - k_i x_i(\Delta t) - k_{i+1} x_{i+1}(\Delta t)) \right]$$

$$+ \ldots + ((k_i + k_{i+1})x_{i+1}(j\Delta t) - k_i x_i(j\Delta t) - k_{i+1} x_{i+1}(j\Delta t)) \right]$$

(A.7)

$$\int_0^{jt} \left[ p_i(t) v_i(t) \right] \,dt = \frac{\Delta t}{6} \left[ p_i(0) v_i(0) + 4p_i(\Delta t) v_i(\Delta t) + \ldots + p_i(j\Delta t) v_i(j\Delta t) \right]$$

(A.8)

Hence, by inserting Eqns.(A.6)-(A.8) in Eq.(23) and simplifying, one can obtain the discretized form of energy equations.

$$A_1 v_{i+1}(\Delta t) + B v_{i+1}(\Delta t) v_{i+1}(\Delta t) + C v_{i+1}(\Delta t) v_{i+1}(\Delta t) + D v_{i+1}(\Delta t) + E = 0 \quad \text{(24-rep.)}$$

where,

$A_1 = 0.5m_i + (\Delta t / 3)(c_i + c_{i+1})$, \quad $B = -(\Delta t / 3)c_i$, \quad $C_i = -(\Delta t / 3)c_{i+1}$

$D_i = (\Delta t / 3) \left[ (k_i + k_{i+1})x_{i+1}(\Delta t) - k_i x_i(\Delta t) - k_{i+1} x_{i+1}(\Delta t) - p_{i+1}(\Delta t) \right]$

$E_i = -0.5m_i v_i(\Delta t)^2 + (\Delta t / 3) \left[ v_i(\Delta t) \left[ (c_i + c_{i+1})v_i(\Delta t) - c_i v_{i+1}(\Delta t) + c_{i+1} v_{i+1}(\Delta t) + (k_i + k_{i+1})x_i(\Delta t) - k_i x_{i+1}(\Delta t) - k_{i+1} x_{i+1}(\Delta t) - p_{i+1}(\Delta t) \right] \right.$

$+ 4v_i(\Delta t) \left[ (c_i + c_{i+1})v_{i+1}(\Delta t) - c_i v_{i+1}(\Delta t) + c_{i+1} v_{i+1}(\Delta t) + (k_i + k_{i+1})x_i(\Delta t) - k_i x_{i+1}(\Delta t) - k_{i+1} x_{i+1}(\Delta t) - p_{i+1}(\Delta t) \right] \right.$

$+ \ldots + v_{i+1}(\Delta t) \left[ (c_i + c_{i+1})v_{i+1}(\Delta t) - c_i v_{i+1}(\Delta t) + c_{i+1} v_{i+1}(\Delta t) + (k_i + k_{i+1})x_i(\Delta t) - k_i x_{i+1}(\Delta t) - k_{i+1} x_{i+1}(\Delta t) - p_{i+1}(\Delta t) \right] \left. \right]$
Appendix B. The Newton-Raphson method for solving Eq.(37).

If we define two functions \( f_1, f_2 \), for each equation in Eq.(37) as follows
\[
\begin{align*}
  f_1 &= 0.511v_1^2 - 0.008v_2v_1 + 0.01v_1 - 4.497 = 0 \\
  f_2 &= 0.508v_2^2 - 0.008v_2v_1 + 0.055v_2 - 7.768 = 0
\end{align*}
\] (37-rep.)

By extending the Newton-Raphson method to the system of equations, to find the root of \( f_1 \) and \( f_2 \), one may write the following sequence [59]
\[
\{v\}_{i+1} = \{v\}_{i} - [J]^{-1}\{f\}_{i}, \quad i = 0, 1, 2, \ldots
\] (B.1)

Here, the subscript, \( i \), represent the iteration number to achieve the convergence criterion; \( \{v\} \) is the vector of unknowns (velocities); \( \{f\} \) is the vector of functions; and, \([J]\) denotes the Jacobian matrix.

\[
\{v\} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \{f\} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad [J] = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \end{bmatrix}
\] (B.2)

In this case, the inverse form of the Jacobian matrix can be expressed as
\[
[J]^{-1} = \begin{bmatrix} 1.022v_1 - 0.008v_2 + 0.01 & -0.008v_1 \\ -0.008v_2 & 1.016v_2 - 0.008v_1 + 0.055 \end{bmatrix}
\] (B.3)

Simplifying yields
\[
[J]^{-1} = \frac{1}{D} \begin{bmatrix} 500(8v_1^2 - 1016v_2^2 - 55) & -4000v_1 \\ -4000v_2 & -1000(511v_1^2 - 4v_2^2 + 5) \end{bmatrix}
\] (B.4)

\[ D = 4088v_1^2 + 4064v_2^2 - 51917v_1v_2 - 28065v_1 - 4860v_2 - 275 \]

Assuming the velocities of the previous step as the initial approximation, we can estimate the roots of Eq.(37) as
\[
\{v\}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\] (B.5)

then, Eq.(B.5) may be expressed as
\[
\overset{\text{iter}}{\longrightarrow} \quad \{v\}_i = \{v\}_0 - [J]^{-1}\{f\}_0
\] (B.6)

results in
\[
\{v\}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0.0026 & 0.2442 \\ 0.0036 & 0.4640 \end{bmatrix} \begin{bmatrix} 2.9873 \\ 3.8866 \end{bmatrix}
\] (B.7)

By inserting the obtained value in the sequence of Eq.(B.1) and continuing calculations until convergence is achieved, the accuracy of the solution can be increased. Note that this quantity, \([J]^{-1}\), within the Modified Newton Raphson is determined only once in the iteration and is assumed to be constant during the next iterations [60].
Appendix C. Derivation of the exact solution of Example.4.1 by Laplace transform

Considering Eq.(33)
\[
\begin{align*}
\dot{x}_1 + 0.22\dot{x}_1 - 0.16\dot{x}_2 + 2x_1 - x_2 &= 0 \\
\dot{x}_2 - 0.16\dot{x}_1 + 0.16\dot{x}_2 - x_1 + x_2 &= 0 
\end{align*}
\]
(33-rep.)

Taking Laplace transform, and let \( F = L(x_1) \) and \( G = L(x_2) \), then
\[
\begin{align*}
\mathcal{L}\{x_1\} &= \frac{s^2 - sx_{1(0)} - \dot{x}_{1(0)} + 0.22sF - 0.22x_{1(0)} - 0.16sG + 0.16x_{2(0)} + 2F - G}{s^2} = 0 \\
\mathcal{L}\{x_2\} &= \frac{s^2G - sx_{2(0)} - \dot{x}_{2(0)} - 0.16sF + 0.16x_{1(0)} + 0.16sG - 0.16x_{2(0)} - F + G}{s^2} = 0
\end{align*}
\]
(C.1)

where \( s \) is a transformed variable and the zero subscript denotes the initial value of at \( t=0 \).

Imposing the initial conditions and solving this system algebraically for \( F \) and \( G \), we obtain
\[
\begin{align*}
F &= \frac{s^3 + 3.38s^2 + 4.1296s + 7.06}{s^4 + 0.38s^3 + 3.0096s^2 + 0.22s + 1} \\
G &= \frac{2s^3 + 4.76s^2 + 6.3792s + 11.22}{s^4 + 0.38s^3 + 3.0096s^2 + 0.22s + 1}
\end{align*}
\]
(C.2)

Eq.(C.2) may be written in terms of partial fractions as follows
\[
\begin{align*}
F &= \frac{-0.1307 - 0.106i}{s - (-0.1733 + 1.608i)} + \frac{-0.1307 + 0.106i}{s - (-0.1733 - 1.608i)} + \frac{0.6307 - 2.1314i}{s - (-0.0167 + 0.618i)} + \frac{0.6307 + 2.1314i}{s - (-0.0167 - 0.618i)} \\
G &= \frac{0.0766 + 0.0718i}{s - (-0.1733 + 1.608i)} + \frac{0.0766 - 0.0718i}{s - (-0.1733 - 1.608i)} + \frac{0.9234 - 3.4692i}{s - (-0.0167 + 0.618i)} + \frac{0.9234 + 3.4692i}{s - (-0.0167 - 0.618i)}
\end{align*}
\]
(C.3)

Consequently, by taking the inverse transform and simplifying, displacements of the system \((x_1, x_2)\) will be given by
\[
\begin{align*}
x_1 &= -0.336e^{-1.608t}\cos(1.608t + 0.681) + 4.445e^{-0.618t}\cos(0.618t - 1.283) \\
x_2 &= 0.210e^{-1.608t}\cos(1.608t + 0.753) + 7.180e^{-0.618t}\cos(0.618t - 1.3106)
\end{align*}
\]
(C.4)

Appendix D. Derivation of the exact solution of Example.4.2 by Laplace transform

The following governing equations will be solved by the Laplace transform method.
\[
\begin{align*}
\dot{x}_1 + 0.2\dot{x}_1 - 0.1\dot{x}_2 + 2x_1 - x_2 &= \cos t \\
\dot{x}_2 - 0.1\dot{x}_1 + 0.2\dot{x}_2 - 0.1\dot{x}_3 - x_1 + 2x_2 - x_3 &= \cos 2t \\
\dot{x}_3 - 0.1\dot{x}_2 + 0.1\dot{x}_3 - x_2 + x_3 &= \cos 3t
\end{align*}
\]
(48-rep.)

Let \( F = L(x_1) \), \( G = L(x_2) \), and \( H = L(x_3) \), then taking Laplace transform, we have
\[
\begin{align*}
\mathcal{L}\{\dot{x}_1\} &= \frac{s^2F - sx_{1(0)} - \dot{x}_{1(0)} + 0.2sF - 0.2x_{1(0)} - 0.1sG + 0.1x_{2(0)} + 2F - G}{s^2} = \frac{s}{s^2 + 1} \\
\mathcal{L}\{\dot{x}_2\} &= \frac{s^2G - sx_{2(0)} - \dot{x}_{2(0)} - 0.1sF + 0.1x_{1(0)} + 0.2sG - 0.2x_{2(0)} - 0.1sH + 0.1x_{3(0)} + F + 2G - H}{s^2} = \frac{s}{s^2 + 4} \\
\mathcal{L}\{\dot{x}_3\} &= \frac{s^2H - sx_{3(0)} - \dot{x}_{3(0)} - 0.1sG + 0.1x_{2(0)} + 0.1sH - 0.1x_{3(0)} - G + H}{s^2} = \frac{s}{s^2 + 9}
\end{align*}
\]
(D.1)
By imposing the zero initial condition and solving for $F$, $G$, and $H$, we have

\[
(s^2 + 0.2s + 2)F + (-0.1s - 1)G + (0)H = \frac{s}{s^2 + 1}
\]

\[
(-0.1s - 1)F + (s^2 + 0.2s + 2)G + (-0.1s - 1)H = \frac{s}{s^2 + 4}
\]

\[
(0)F + (-0.1s - 1)G + (s^2 + 0.1s + 1)H = \frac{s}{s^2 + 9}
\]

Rearranging yields,

\[
F = \left\{ \begin{array}{l} 
1000s^9 + 400s^8 + 17030s^7 + 5500s^6 + 88280s^5 + 17300s^4 + 145490s^3 + 9800s^2 + 49000s \\
A
\end{array} \right.
\]

\[
G = \left\{ \begin{array}{l} 
1000s^9 + 500s^8 + 15050s^7 + 5800s^6 + 62430s^5 + 15300s^4 + 110620s^3 + 12400s^2 + 62000s \\
A
\end{array} \right.
\]

\[
H = \left\{ \begin{array}{l} 
1000s^9 + 500s^8 + 10060s^7 + 4200s^6 + 40480s^5 + 17300s^4 + 12100s^3 + 13200s^2 + 66000s \\
A
\end{array} \right.
\]

where,

\[
A = 1000s^{12} + 500s^{11} + 19060s^{10} + 8201s^9 + 125870s^8 + 41614s^7 + 369360s^6 + 81049s^5 + 491630s^4 + 57936s^3 + 266080s^2 + 10800s + 36000
\]

Eq.(D.4) may be written in terms of partial fractions as follows

\[
F = \left\{ \begin{array}{l} 
0.213s - 0.001 + -0.868s - 0.403 + -0.451s + 0.387 + 0.926s + 0.28 + 0.1830s - 0.546 + -0.002s + 0.006 \\
+ s^2 + 0.02s + 0.196 + s^2 + 0.155s + 1.555 + s^2 + 0.325s + 3.247 + s^2 + 1 + s^2 + 4 + s^2 + 9
\end{array} \right.
\]

\[
G = \left\{ \begin{array}{l} 
0.383s - 0.001 + -0.368s - 0.179 + 0.563s - 0.483 + -0.028s + 0.092 + -0.549s + 0.726 + 0.017s + -0.025 \\
+ s^2 + 0.02s + 0.196 + s^2 + 0.155s + 1.555 + s^2 + 0.325s + 3.247 + s^2 + 1 + s^2 + 4 + s^2 + 9
\end{array} \right.
\]

\[
H = \left\{ \begin{array}{l} 
0.477s - 0.002 + 0.696s + 0.323 + -0.25s + 0.215 + -0.944s - 0.184 + 0.148s - 0.335 + -0.127s + 0.019 \\
+ s^2 + 0.02s + 0.196 + s^2 + 0.155s + 1.555 + s^2 + 0.325s + 3.247 + s^2 + 1 + s^2 + 4 + s^2 + 9
\end{array} \right.
\]

Finally, by taking the inverse transform and simplifying, the displacements of the system $(x_1, x_2, x_3)$ will be given by

\[
\begin{align*}
  x_1 &= 0.2131\exp(-0.01t)\cos(0.449t + 0.0330) - 0.9089\exp(-0.0775t)\cos(1.2446t - 0.3013) + \\
      &-0.5188\exp(-0.1625t)\cos(1.7946t + 0.5171) + 0.9674\cos(t - 0.2936) + 0.3286\cos(2t + 0.9803) + 0.002\cos(3t + 3.9267) \\
  x_2 &= 0.3831\exp(-0.01t)\cos(0.449t + 0.0282) + 0.4042\exp(-0.0775t)\cos(1.2446t + 2.8406) + \\
      &0.6476\exp(-0.1625t)\cos(1.7946t + 0.517) + 0.0962\cos(t + 4.4169) + 0.6581\cos(2t + 3.7258) + 0.0189\cos(3t + 0.4542) \\
  x_3 &= 0.4772\exp(-0.01t)\cos(0.449t + 0.0318) + 0.7288\exp(-0.0775t)\cos(1.2446t - 0.3012) + \\
      &0.4097\exp(-0.1625t)\cos(1.7946t + 4.0561) + 0.9618\cos(t + 2.9491) + 0.2235\cos(2t + 0.8471) + 0.1271\cos(3t + 3.1911) \\
\end{align*}
\]
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Table 1

<table>
<thead>
<tr>
<th>Type of equations</th>
<th>Advantages</th>
<th>Disadvantages</th>
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<tbody>
<tr>
<td>Force equilibrium</td>
<td>Quadratic (non-linear) terms do not exist in these equations(^*)</td>
<td>Second order of derivative in the equations that leads to increasing the number of unknowns, including: displacement, velocity and acceleration</td>
</tr>
<tr>
<td>Energy equilibrium</td>
<td>First order of derivative in the equations that leads to reducing the number of unknowns, including: displacement, velocity</td>
<td>Existence of quadratic (non-linear) terms</td>
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</table>

\(^*\)Except in nonlinear analysis.

Table 2

A. initial calculations:
1. Form dynamic matrices: mass \(m\), damping \(c\), and stiffness \(k\)
2. Form the vectors of initial conditions: initial displacements \(x_0\), and velocities \(v_0\)
3. Select the time step \(\Delta t\)
4. Select the tolerance for each iteration \(e = 10^{-s}\) (\(s\) is a positive integer number)

B. for each time step:
5. Calculate a starting vector for \(x_i\), displacement vector at the time of \(t = i\Delta t\)
   \[
   x_i^{(1)} = x_0 + dx^{(1)} \quad , \quad dx^{(1)} = v_0 \Delta t
   \]
   (The superscripts and subscripts refer to the number of iteration and time step, respectively)
6. Calculate the coefficient of energy balance equation, \(i.e. A_i, B_i, C_i, D_i\), for all masses. Note that, the Trapezoidal rule for the first time step and subsequently Simpson rule must be used.
7. Neglect the coupling terms, \(i.e. B_i = C_i = 0\)
8. Solve the quadratic equation of energy balance for the velocity of \(i\)th mass, using the corresponding \(A_i\), and \(v_i = (-D_i \pm \Delta) / 2A_i\)
9. Select a velocity which is closer to the previous time step (call it \(v_j\)) (Elimination of Discontinuous Velocities)
10. Calculate a new approximated vector for \(x_i\), by using the average of new obtained velocities and initial velocities
    \[
    x_i^{(1)} = x_{i-1} + dx^{(j)} \quad , \quad dx^{(j)} = 0.5(v_{i-1} + v_j) \Delta t \quad , \quad j = 2, 3, \ldots
    \]
11. Determine the coupling terms, which were neglected at first
12. Iterate through step 6 to 11, except step 7, to convergence
13. Continue the procedure for subsequent time steps

Table 3

<table>
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<tr>
<th>Number of iterations</th>
<th>(v_1)</th>
<th>(e_1)</th>
<th>(v_2)</th>
<th>(e_2)</th>
<th>(e_{max} &lt; 0.01)</th>
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<td>3.5867</td>
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<td>3.8795</td>
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Table 4

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<tr>
<td>Modal-Newmark</td>
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<td>Modal-Duhamel</td>
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Table 5

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<tr>
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<td>Modal-Newmark</td>
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Table 6

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<td>25.035491</td>
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<td>Modal-Duhamel</td>
<td>28.462937</td>
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</table>

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