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Economic evaluation of investment projects under uncertainty: A probability theory perspective

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Abstract. In the current competitive economy, investors are constantly facing growing uncertainty when evaluating new investment projects. This uncertainty results from insufficient information, oscillating markets, unstable economic conditions, obsolescence of technology, and so on. Hence, uncertainty is inevitable in reality. In such conditions, the deterministic models, while easy to use, do not perfectly represent the real situations and might lead to misleading decisions. When the cash flows for an uncertain investment project over a number of future periods are discounted by the traditional deterministic approaches, investors may find it hard to have an accurate estimation of the project value. Therefore, this paper utilizes the probability theory tools to derive a closed-form Probability Distribution Function (PDF) and related expressions of the Net Present Worth (NPW), as a useful and frequently used criterion, for the cost-benefit evaluation of projects. The random cash flows follow normal, uniform or exponential distributions in our analysis. The PDF of the NPW is an important tool that helps investors to accurately estimate the probability of projects being economical; hence, it is an important tool for investment decision-making under uncertainty.

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1. Introduction

An appropriate evaluation and selection of investment projects is the most important financial factor in the success of investors and is vital to being economical in the current competitive environment [1]. Investors aim to not only prevent projects' failures but also select the best alternatives among available investment projects so as to gain more benefits and reach better results. In the economic evaluation of investment projects, the important parameters such as prices and quantities have been treated so far based on the deterministic and pre-known values. The values of these parameters used in the analysis and the resulting outcomes are single values with perfect certainty. An investor expects to achieve positive results from investment projects. Under deterministic settings, these results can be achieved unambiguously. However, under uncertainty in the real-world conditions, there is usually the lack of reliable information [2], and most investment parameters are uncertain [3]. The future outcomes have uncertainties, especially for long-term planning horizons, and variability in the value of the future items is inevitable. In the real world, uncertainty is always present and, in some particular cases, due to the lack of scientific information and technological innovation, it is more difficult to make reliable decisions. For example, the new products supplied to the market might not gain sales at the expected levels; the profits expected from international investments might not be gained satisfactorily, or the investment might encounter

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an unexpected delay and take a long time to achieve efficiency. The source of this uncertainty might be the behaviors of customers, suppliers or employees, or technical problems in processes. Due to complex and fast-changing factors in decision-making, many investment decisions contain remarkable uncertainty and, hence, suffer from high risks in outcomes. Therefore, these uncertainties should be taken into account in the economic evaluation of investment projects in order to ensure reliable decisions and long-term achievements.

The cash flow analysis has been utilized to assess the differences between investment projects and provide a basis for project evaluation. In order to financially evaluate investment projects, it is required to assess the possible beneficiaries due to their cash flows. To evaluate investment projects, the cash flow analysis is carried out by the classic methods such as Net Present Worth (NPW), Net Future Worth (NFW), Net Equivalent Uniform Annual (NEUA), Internal Rate of Return (ROR/IRR), payback period (PB), etc. Among them, the NPW is one of the most useful ones for determining the economic desirability of the projects. The appropriate criterion for monitoring and evaluating such projects is the NPW [4,5]. It is considered to be a major tool for analyzing the cash flow of projects during a long period [6]. It is the most common method used by banks and large-sized organizations to compare projects so as to select the most economical one among them. To perform such an analysis, the NPW of an investment project is defined as the sum of present income cash flow minus present cost cash flow during the project's time horizon. Hence, the NPW criterion can be calculated via:

$$NPW = \sum_{t=1}^{n} \frac{(R_t - C_t)}{(1+i)^t}, \tag{1}$$

where r_t and c_t indicate the values of income and cost during the period t respectively, i indicates the discount rate (minimum attractive rate of return), and n indicates the project's planning horizon. If the value of the NPW is equal to or higher than zero, the project will be economical and acceptable; if it is lower than zero, the project will be uneconomical and, hence, unacceptable. To incorporate the NPW in an uncertain environment, different approaches have been used in the literature. The most common methods are based on soft computing approaches such as fuzzy sets and simulation [2]. The fuzzy mathematical optimization [7], the fuzzy data envelopment analysis [8], and the fuzzy AHP and ANP [9] have attracted the most attention. Moreover, different hybrid approaches [9,10] attempt to combine these methods to reach better results. As another frequently used approach, the simulation is utilized in different ways. Some researchers have used the simulation tool incorporated with the transformation of fuzzy numbers to probability distributions [4], others have utilized Monte-Carlo simulation to achieve sample sizes [11], and others have applied fuzzy simulation directly [12]. Naimi Sadigh et al. [13] proposed a hybrid approach based on particle swarm optimization and Hopfield neural network for a cardinality constrained portfolio optimization problem. Salmasnia et al. [14] proposed a robust approach to project evaluation with time, cost, and quality considerations. Liu and Wu [15] presented a portfolio evaluation and optimization in electricity markets. Afshar-Nadjafi et al. [16] proposed a generalized resource investment problem with discounted cash flows and progress payment. Rębiasz and Macioł [17] employed a multi-criteria decision-making method and a fuzzy rule-based approach to evaluate investment projects. Dai et al. [18] discussed a model for renewable energy investment project evaluation based on an improved real option approach. Kilic and Kaya [19] investigated a decision-making methodology for an investment project evaluation based on type-II fuzzy sets. Kirkwood et al. [20] evaluated uncertainty in the integrated maintenance of the UK rail industry by net present value calculations. Tabrizi et al. [21] presented a novel project portfolio selection by using a fuzzy DEMATEL and goal programming. Fathallahi and Naja [22] designed a hybrid genetic algorithm to maximize the net present value of fuzzy project cash flows in the resource-constrained project scheduling problem. In addition, Dutta and Ashtekar [23] considered a system dynamics simulation tool for the project evaluation issue. Etemadi et al. [24] applied a goal programming capital budgeting model under uncertainty in the construction industry. Mohagheghi et al. [25] suggested a new interval type-II fuzzy optimization approach for R&D project evaluation and portfolio selection. Awasthi and Omrani [26] developed a scenario for the simulation approach to sustainable project evaluation based on fuzzy concepts.

All of the above soft methods can be applied if essential assumptions such as the existence of large samples and data accuracy are assured. However, another useful approach to handling uncertainty in economic evaluation is estimating some prediction functions like Probability Distribution Function (PDF). It is an analytical approach, in contrast to soft approaches, to evaluating investment projects based on the probability and statistical theory principles. In this way, the PDFs for the estimation and prediction of economic desirability NPW can be derived [27]. This paper aims to derive the PDF of economic desirability NPW for two classes of investment projects: (i) one-period cash flows (three cases) and (ii) two-period cash flows (six cases). In the deterministic environment, under certainty, a project is evaluated as "economical" or "uneconomical". However, in stochastic environment, under uncertainty, we can just talk about the

"probability of being economical" or "probability of being uneconomical", which can be calculated using the analytical PDF of performance criterion NPW. In deterministic economic evaluation problems, the investor has his/her own Minimum Attractive Rate of Return (MARR), which is used in his/her evaluations. In stochastic problems considered in this paper, the probability of being economical for an uncertain cash flow is calculated. When this parameter is calculated, the investor compares this parameter with his/her minimum probability of being economical. If the minimum probability of being economical is satisfied, the project is selected as economical; otherwise, it is rejected. Hence, this paper helps the investor achieve the analytical function for estimating the economic performance of investment projects. The probability theory tools such as moment-generating function and the transformation method will be used to derive the analytical and closed-form functions for PDF of the NPW in investment projects. This analysis will be performed for three classes of investment projects: (i) one-period cash flows, (ii) two-period cash flows, and (iii) multiple-period cash flows, all with normal, exponential, and uniform distributions of cash flows. The PDFs of NPW will be derived for each case separately, which can help investors accurately estimate the economic desirability of investment projects under uncertain cash flows. Since the cash flows follow continuous numbers, we should consider continuous random distribution functions. For this purpose, normal, exponential, and uniform distributions are considered as well-known and most frequently used continuous distributions for addressing random cash flows.

The rest of this paper is organized as follows. Section 2 provides one-period cash flow analysis, while Section 3 discusses two-period cash flow analysis. Section 4 investigates a multiple-period project cash flow under uncertainty. In Sections 2–4, the PDFs are derived for NPW in three distributions, i.e., normal, exponential, and uniform, separately. Moreover, some lemmas are proved to address the special cases of PDFs. Section 5 presents the analysis and simulation results to evaluate the performance of PDFs derived and calculate the probability of being economical for case examples in different situations. Finally, Section 6 concludes the paper and suggests some directions for future studies.

2. One-period cash flow analysis

This section introduces and investigates an investment problem consisting of a one-period project with the initial cost, A_0, and income at the end of the period A_1. The value of the income, A_1, is usually larger than that of the cost A_0; however, in the present paper, our analysis allows A_1 to be smaller than A_0. The economic desirability of the project NPW is a function of both cash flows A_1 and A_0, which can be calculated as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)}. \tag{2}$$

If the income, A_1, is random, the economic desirability NPW will be random too, and the mean and the variance of economic desirability NPW can be calculated as follows:

$$E[NPW] = -A_0 + \frac{\mu_{A_1}}{(1+i)},$$

$$Var[NPW] = \frac{\sigma^2_{A_1}}{(1+i)^2}, \tag{3}$$

where μ_{A_1} and $\sigma^2_{A_1}$ indicate the mean and the variance of the random income A_1, respectively.

2.1. Fixed cost A_0 and variable income A_1 with normal distribution

In this case, according to the assumptions $A_0 \geq 0, A_1 \sim Normal\left(\mu_1, \sigma_1^2\right)$, it can be concluded that the economic desirability of the project NPW is also a random variable with the following mean and variance:

$$E[NPW] = -A_0 + \frac{\mu_1}{(1+i)},$$

$$Var[NPW] = \frac{\sigma_1^2}{(1+i)^2}. \tag{4}$$

Therefore, according to the normal distribution, the PDF of the income A_1 can be calculated as follows:

$$PDF(A_1): f_{A_1}(a_1) = \frac{1}{\sqrt{2\pi}\sigma_1} EXP\left[\frac{-(a_1-\mu_1)^2}{2\sigma_1^2}\right]. \tag{5}$$

In addition, its Moment-Generating Function (MGF) can be calculated as follows:

$$MGF_{A_1}(t) = E(EXP[tA_1]) = EXP\left[t\mu_1 + \frac{t^2}{2}\sigma_1^2\right]. \tag{6}$$

By considering the fact that the economic desirability of the project NPW is a function of variable A_1, the moment-generating function of NPW can, therefore, be calculated as follows:

$$MGF_{NPW}(t) = MGF_{-A_0+\frac{A_1}{(1+i)}}(t). \tag{7}$$

According to the knowledge of probability theory, it has been proved for one random variable like Y and two fixed numbers like a and b that:

$$MGF_{aY+b}(t) = EXP[bt].MGF_Y\left(\frac{t}{a}\right). \tag{8}$$

Therefore, by using the above equation, the moment-generating function of the economic desirability of this project $MGF_{NPW}(t)$ is obtained as follows:

$$MGF_{NPW}(t) = EXP[-A_0 t].MGF_{A_1}\left(\frac{t}{1+i}\right). \quad (9)$$

By substituting the moment- generating function of the income into the above equation, $MGF_{NPW}(t)$ can be re-written as follows:

$$MGF_{NPW}(t) = EXP\left[t\left(\frac{\mu_1}{1+i} - A_0\right) + \frac{t^2}{2}\left(\frac{\sigma_1}{1+i}\right)^2\right]. \quad (10)$$

By comparing the above $MGF_{NPW}(t)$ with the moment-generating function of the normal distribution and based on the fact that the moment-generating function is unique for every random variable, it can be concluded that:

$$NPW \sim Normal\left(\frac{\mu_1}{1+i} - A_0, \left(\frac{\sigma_1}{1+i}\right)^2\right). \quad (11)$$

Therefore, the PDF of the economic desirability of this the project NPW is estimated as follows:

$$PDF(NPW): f_{NPW}(v) = \frac{(1+i)}{\sqrt{2\pi}\sigma_1} EXP \left[\frac{-(v(1+i) - \mu_1 + A_0(1+i))^2}{2\sigma_1^2}\right]. \quad (12)$$

The importance of this function is that it can help the investors estimate the economic desirability of such projects by calculating the probability of being economical. When the MARR of the investor is i, this probability can be calculated as follows:

$$P(NPW \geq 0|i) = 1 - \phi\left(\frac{A_0(1+i) - \mu_1}{\sigma_1}\right), \quad (13)$$

where $\phi(.)$ indicates the cumulative distribution function of the standard normal distribution. Obviously, the projects with a higher value of $P(NPW \geq 0|i)$ are more economical and preferable.

2.2. Fixed cost A_0 and variable income A_1 with exponential distribution

In this case, by taking into account the assumptions $A_0 \geq 0, A_1 \sim EXP(\beta_1)$, the mean and the variance of the income A_1 are $E[A_1] = \beta_1$ and $Var[A_1] = \beta_1^2$; consequently, the mean and the variance of the economic desirability of the project NPW can be calculated as follows:

$$E[NPW] = -A_0 + \frac{\beta_1}{(1+i)},$$

$$Var[NPW] = \frac{\beta_1^2}{(1+i)^2}. \quad (14)$$

In addition, the PDF of the income A_1, according to the PDF of the exponential random variable, is obtained as follows:

$$PDF(A_1): f_{A_1}(a_1) = \frac{1}{\beta_1} EXP\left[\frac{-a_1}{\beta_1}\right]. \quad (15)$$

Further, the moment-generating function of the income A_1 is achieved as follows:

$$MGF_{A_1}(t) = E(EXP[tA_1]) = \frac{1}{1-\beta_1 t}. \quad (16)$$

By considering the fact that the economic desirability of the project NPW is a function of random variable A_1 and using Eq. (9), the moment-generating function of the economic desirability NPW can be derived as follows:

$$MGF_{NPW}(t) = \frac{(1+i)}{1+i-\beta_1 t} EXP[-A_0 t]. \quad (17)$$

Since this function does not adapt to any of the known random variables, it can be understood that the economic desirability of such a project does not have a standard pattern. However, for the purpose of economic evaluation, we are looking for this unknown pattern. Therefore, another probability theory analysis, i.e., the transformation method in the sequel, is used [28]. To adopt this method, the reverse function of economic desirability NPW^{-1} is calculated as follows:

$$NPW^{-1}: A_1 = [NPW + A_0](1+i). \quad (18)$$

In addition, the derivative of the reverse function is calculated as $\left(NPW^{-1}\right)' = (1+i)$. Therefore, according to the transformation method, the PDF of the economic desirability of the project can be extracted as follows:

$$PDF(NPW): f_{NPW}(v) = f_{A_1}\left(NPW^{-1}\right).\left(NPW^{-1}\right)'. \quad (19)$$

After the mathematical simplification, the above PDF, $f_{NPW}(\nu)$, can be re-written in its final form as follows:

$$PDF(NPW): f_{NPW}(v) = \left(\frac{1+i}{\beta_1}\right) EXP\left[-\frac{(v+A_0)(1+i)}{\beta_1}\right]. \quad (20)$$

According to the above, if the income A_1 has its least amount as zero, the economic desirability of the project

NPW will be in its least amount, which is $-A_0$. In other words, the range of NPW variations is as follows:

$$-A_0 \leq NPW \leq +\infty. \tag{21}$$

Therefore, the condition of such a project being economical, when the investor has a particular MARR, i, can be calculated as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}(v)\, dv. \tag{22}$$

This probability is calculated and re-written as follows:

$$P(NPW \geq 0|i) = EXP\left[\frac{-A_0(1+i)}{\beta_1}\right]. \tag{23}$$

By using the above equation, investors can easily predict the economic desirability of such projects and make an appropriate economic decision.

2.3. Fixed cost A_0 and variable income A_1 with uniform distribution

In this case, it has been assumed that $A_0 \geq 0, A_1 \sim Uniform(\alpha_1, \beta_1)$; then, the mean and the variance of the income A_1 can be calculated as follows:

$$E[A_1] = \frac{\alpha_1 + \beta_1}{2}, \qquad Var[A_1] = \frac{(\beta_1 - \alpha_1)^2}{12}. \tag{24}$$

Therefore, the mean and variance of the economic desirability of the project NPW are achieved as follows:

$$E[NPW] = -A_0 + \frac{1}{2}\left(\frac{\alpha_1 + \beta_1}{1+i}\right),$$

$$Var[NPW] = \frac{1}{12}\left(\frac{\beta_1 - \alpha_1}{1+i}\right)^2. \tag{25}$$

Moreover, the PDF of the income A_1 can be attained as follows:

$$PDF(A_1) : f_{A_1}(a_1) = \frac{1}{\beta_1 - \alpha_1},$$

$$\alpha_1 \leq a_1 \leq \beta_1. \tag{26}$$

In addition, the moment-generating function of the income A_1 can be computed by:

$$MGF_{A_1}(t) = E(EXP[tA_1]) =$$

$$\frac{EXP[t\beta_1] - EXP[t\alpha_1]}{t(\beta_1 - \alpha_1)}. \tag{27}$$

Therefore, by using Eq. (9), the moment-generating function of the economic desirability of the project NPW is obtained as follows:

$$MGF_{NPW}(t) =$$

$$\frac{EXP\left[t\left(\frac{\beta_1}{1+i} - A_0\right)\right] - EXP\left[t\left(\frac{\alpha_1}{1+i} - A_0\right)\right]}{t\left(\frac{\beta_1}{1+i} - \frac{\alpha_1}{1+i}\right)}. \tag{28}$$

By comparing the above $MGF_{NPW}(t)$ with the moment-generating function of the uniform distribution and by considering the fact that the moment-generating function is unique for every random variable, it is revealed that:

$$NPW \sim Uniform\left(\frac{\alpha_1}{1+i} - A_0, \frac{\beta_1}{1+i} - A_0\right). \tag{29}$$

In addition, the PDF of the economic desirability of this project is estimated as follows:

$$PDF(NPW) : f_{NPW}(v) = \left(\frac{1+i}{\beta_1 - \alpha_1}\right),$$

$$\frac{\alpha_1}{1+i} - A_0 \leq v \leq \frac{\beta_1}{1+i} - A_0. \tag{30}$$

Consequently, the probability of such projects being economical, when the investor has a particular maximum attractive rate of return i, is calculated as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}(v)\, dv. \tag{31}$$

This probability can be calculated and re-written as follows:

$$P(NPW \geq 0|i) = \frac{\beta_1 - A_0(1+i)}{\beta_1 - \alpha_1}. \tag{32}$$

Further analysis of the above equation reveals that it is not valid for two special cases and more investigation is required. For this purpose, two lemmas are presented and discussed in the sequel.

Lemma 1: In a one-period project with a fixed cost A_0 and a variable income A_1 following uniform distribution $A_1 \sim Uniform(\alpha_1, \beta_1)$, if $\beta_1 < A_0(1+i)$, then this project is certainly uneconomical.

Proof: If the assumption of this lemma is true, it can be concluded that $\frac{\beta_1}{1+i} - A_0 < 0$ and, by using the fact $\alpha_1 < \beta_1$, it yields that $\frac{\alpha_1}{1+i} - A_0 < 0$. Therefore, both the upper bound and the lower bound of NPW are negative, and there is no chance for the economic viability of this project. We can represent the result of this lemma in the mathematical form as given below:

$$P(NPW \geq 0|i) = 0 \;\text{ if }\; \beta_1 < A_0(1+i). \tag{33}$$

Lemma 2: In a one-period project with a fixed cost A_0 and a variable income A_1 following uniform distribution $A_1 \sim Uniform(\alpha_1, \beta_1)$, if $\alpha_1 \geq A_0(1+i)$, then the project is certainly economical.

Proof: If the assumption of this lemma is true, it can be concluded that $\frac{\alpha_1}{1+i} - A_0 \geq 0$ and, by using the fact

that $\alpha_1 < \beta_1$, it yields that $\frac{\beta_1}{1+i} - A_0 > 0$. Therefore, both the upper bound and the lower bound of the NPW are positive, and the economic viability of this project is certainly economical. It means that:

$$P(NPW \geq 0|i) = 1 \text{ if } \alpha_1 \geq A_0 (1+i). \tag{34}$$

Therefore, according to both Lemmas 1 and 2, the probability of such projects being economical is re-written in the general form as follows:

$$P(NPW \geq 0|i) = \begin{cases} 0 & \text{if } \beta_1 < A_0 (1+i) \\ \frac{\beta_1 - A_0(1+i)}{\beta_1 - \alpha_1} & \text{if } \alpha_1 < A_0 (1+i) \ \& \ \beta_1 \geq A_0 (1+i) \\ 1 & \text{if } \alpha_1 \geq A_0 (1+i) \end{cases} \tag{35}$$

By using the above equation, the economic desirability of such a project can be simply estimated.

The aim of this section is to determine the probability of investment projects being economical with one-period cash flow by deriving the analytical equation of the NPW distribution function. For this purpose, three cases were analyzed. These cases are applicable in reality where the cost parameter A_0 is a fixed positive number and the income A_1 is a random variable with three distinct cases: (i) normal, (ii) exponential, and (iii) uniform distributions. For all of the cases, the analytical PDF of NPW was derived mathematically, which is a useful tool for investors to evaluate uncertain projects over a one-period cash flow.

3. Two-period cash flow analysis

This section investigates an investment problem consisting of a two-period project with an initial cost A_0, an income at the end of the first period A_1, and another income at the end of the second period A_2. In this project, the economic desirability NPW is a function of A_0, A_1, and A_2 cash flows, which can be calculated as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2}. \tag{36}$$

By taking into account the assumptions that the incomes A_1 and A_2 are random and independent variables, the economic desirability NPW will be, therefore, random with the following mean and the variance:

$$E[NPW] = -A_0 + \frac{\mu_{A_1}}{(1+i)} + \frac{\mu_{A_2}}{(1+i)^2},$$

$$Var[NPW] = \frac{\sigma^2_{A_1}}{(1+i)^2} + \frac{\sigma^2_{A_1}}{(1+i)^4}, \tag{37}$$

where μ_{A_1} and $\sigma^2_{A_1}$ indicate the mean and variance of the first random income A_1, respectively, and μ_{A_2} and $\sigma^2_{A_1}$ indicate the mean and variance of the second random income A_2. The aim is to determine the economic desirability of such a project through the determination of analytical equation for the PDF of NPW. To this end, six cases are analyzed in the sequel. In all of these cases, the initial cost A_0 is a fixed positive number, and the incomes A_1 or A_2 are random variables with (i) normal, (ii) exponential, and (iii) uniform distributions.

3.1. *Fixed cost A_0, fixed income A_1, and variable income A_2 with normal distribution*

In this case, by considering the assumptions $A_0 \geq 0$, $A_1 \geq 0$, and $A_2 \sim Normal(\mu_2, \sigma_2^2)$, it can be concluded that the economic desirability of the project NPW is also variable; then, the mean and variance of NPW can be calculated as follows:

$$E[NPW] = -A_0 + \frac{A_1}{(1+i)} + \frac{\mu_2}{(1+i)^2},$$

$$Var[NPW] = \frac{\sigma_2^2}{(1+i)^4}. \tag{38}$$

Moreover, the moment-generating function of the income A_2 can be calculated as follows:

$$MGF_{A_2}(t) = E(EXP[tA_2]) = EXP\left[t\mu_2 + \frac{t^2}{2}\sigma_2^2\right]. \tag{39}$$

and the moment-generating function of the NPW is a function of variable income A_2 as given below:

$$MGF_{NPW}(t) = MGF_{\left[-A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2}\right]}(t). \tag{40}$$

Therefore, by utilizing the probability theory principles, $MGF_{NPW}(t)$ can be calculated as follows:

$$MGF_{NPW}(t) = EXP\left[-A_0 t + \frac{A_1 t}{(1+i)}\right] . MGF_{A_2}\left(\frac{t}{(1+i)^2}\right). \tag{41}$$

which can be re-written by the mathematical simplifications as follows:

$$MGF_{NPW}(t) = EXP\left[t\left(-A_0 + \frac{A_1}{(1+i)} + \frac{\mu_2}{(1+i)^2}\right) + \frac{t^2}{2}\left(\frac{\sigma_2}{(1+i)^2}\right)^2\right]. \tag{42}$$

Therefore, by comparing this function with the moment-generating function of the normal distribution, it can be understood that:

$$NPW \sim Normal \left(-A_0 + \frac{A_1}{(1+i)} + \frac{\mu_2}{(1+i)^2}, \left(\frac{\sigma_2}{(1+i)^2}\right)^2\right). \quad (43)$$

Consequently, the probability of such projects being economical with MARR, i, is calculated as follows:

$$P(NPW \geq 0|i) = 1 - \phi\left(\frac{A_0(1+i)^2 - A_1(1+i) - \mu_2}{\sigma_2}\right). \quad (44)$$

3.2. Fixed cost A_0, variable income A_1 with normal distribution, and fixed income A_2

In this case, we have the assumptions $A_0 \geq 0$, $A_1 \sim Normal\left(\mu_1, \sigma_1^2\right)$, and $A_2 \geq 0$ similar to those in the previous part; thus, it can be found that:

$$NPW \sim Normal \left(-A_0 + \frac{\mu_1}{(1+i)} + \frac{A_2}{(1+i)^2}, \left(\frac{\sigma_1}{1+i}\right)^2\right). \quad (45)$$

Moreover, the probability of this project being economical with MARR, i, can be estimated as follows:

$$P(NPW \geq 0|i) = 1 - \phi\left(\frac{A_0(1+i) - \mu_1 - \frac{A_2}{1+i}}{\sigma_1}\right). \quad (46)$$

3.3. Fixed cost A_0, fixed income A_1, and variable income A_2 with exponential distribution

In this case, it is assumed that $A_0 \geq 0, A_1 \geq 0, A_2 \sim EXP(\beta_2)$; therefore, according to this, we have:

$$E[NPW] = -A_0 + \frac{A_1}{(1+i)} + \frac{\beta_2}{(1+i)^2},$$

$$Var[NPW] = \frac{\beta_2^2}{(1+i)^4}. \quad (47)$$

In addition, according to the exponential distribution, the moment-generating function of the income A_2 can be calculated as follows:

$$MGF_{A_2}(t) = E(EXP[tA_2]) = \frac{1}{1-\beta_2 t}. \quad (48)$$

By taking into account Eq. (9), the moment-generating function of the economic desirability of the project NPW can be computed as follows:

$$MGF_{NPW}(t) = EXP\left[\left(-A_0 + \frac{A_1}{(1+i)}\right)t\right] .MGF_{A_2}\left(\frac{t}{(1+i)^2}\right). \quad (49)$$

By using mathematical simplifications, this function can be expressed as follows:

$$MGF_{NPW}(t) = EXP\left[-A_0 t + \frac{A_1 t}{(1+i)}\right] \cdot \frac{(1+i)^2}{(1+i)^2 - \beta_2 t}. \quad (50)$$

Since the above function is not similar to any known moment-generating functions, it can be found that, in this case, the economic desirability of the project NPW does not follow a standard known distribution. Therefore, to reach the analytical PDF of the NPW in this case, the transformation method in the sequel is used. For this purpose, the reverse function of the NPW and its derivative are derived as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2}, \quad (51)$$

$$NPW^{-1} : A_2 = \left[NPW + A_0 - \frac{A_1}{(1+i)}\right](1+i)^2$$

$$\left(NPW^{-1}\right)' = (1+i)^2.$$

Therefore, according to the transformation method, the final form of the PDF for the economic desirability NPW can be obtained as follows:

$$PDF(NPW) : f_{NPW}(v) = \frac{(1+i)^2}{\beta_2} EXP \left[\frac{-\left[(v+A_0)(1+i)^2 - A_1(1+i)\right]}{\beta_2}\right]. \quad (52)$$

Since the value of the random income A_2 varies at an interval of $0 \leq A_2 < +\infty$, the economic desirability of the project NPW is also variable at the following interval:

$$-A_0 + \frac{A_1}{(1+i)} \leq NPW < +\infty. \quad (53)$$

Further, the probability of such projects being economical with MARR, i, can be calculated as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}(v)\, dv. \quad (54)$$

By performing mathematical computations, this probability can be simplified to the following:

$$P(NPW \geq 0|i) = EXP\left[\frac{A_1(1+i) - A_0(1+i)^2}{\beta_2}\right]. \quad (55)$$

Additional investigations show that the above equation is not valid in one special case, as expressed in Lemma 3.

Lemma 3: In a two-period project with fixed cost A_0, fixed income A_1, and variable income A_2 following exponential distribution $A_2 \sim EXP(\beta_2)$, if $\frac{A_1}{A_0} \geq (1+i)$, then the project is certainly economical.

Proof: If the assumption of Lemma 3 is true, it can be understood that the lower bound of the economic desirability of the project NPW is always nonnegative $\left(-A_0 + \frac{A_1}{(1+i)} \geq 0\right)$. Therefore, we always have $NPW \geq 0$; hence, this project is certainly economical. To analyze this lemma further, if we have $\frac{A_1}{A_0} \geq (1+i)$, then the income A_1 is enough by itself for the economic viability of the project and that the economic viability of the project is independent of the amount of the income A_2. By increasing the value of the income A_2, economic desirability will increase; however, by decreasing this desirability, the project does not quit the economic range at all.

By using the result of Lemma 3, the probability of such a project being economical would be re-written as follows:

$$P(NPW \geq 0|i) = \begin{cases} EXP\left[\frac{A_1(1+i)-A_0(1+i)^2}{\beta_2}\right] & \text{if } \frac{A_1}{A_0} < (1+i) \\ 1 & \text{if } \frac{A_1}{A_0} \geq (1+i) \end{cases} \quad (56)$$

By using the above equation, investors can predict the economic desirability of such two-period projects.

3.4. Fixed cost A_0, variable income A_1 with exponential distribution, and fixed income A_2

In this case, it is assumed that $A_0 \geq 0$, $A_1 \sim EXP(\beta_1)$, $A_2 \geq 0$; thus, we have:

$$E[NPW] = -A_0 + \frac{\beta_1}{(1+i)} + \frac{A_2}{(1+i)^2},$$

$$Var[NPW] = \frac{\beta_1^2}{(1+i)^2}. \quad (57)$$

In this case, similar to the previous case, for using the transformation method, the reverse function NPW^{-1} and its derivative can be derived as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2}, \quad (58)$$

$$NPW^{-1} : A_1 = \left[NPW + A_0 - \frac{A_2}{(1+i)^2}\right](1+i)$$

$$\left(NPW^{-1}\right)' = (1+i).$$

Therefore, by adopting the transformation method, the PDF of the economic desirability NPW will be simplified in its final form as follows:

$$PDF(NPW) : f_{NPW}(v) = \frac{(1+i)}{\beta_1} EXP \left[\frac{-[(v+A_0)(1+i) - A_2/(1+i)]}{\beta_1}\right]. \quad (59)$$

Since the income A_1 varies at an interval of $0 \leq A_1 < +\infty$, the economic desirability NPW varies in the following range:

$$-A_0 + \frac{A_2}{(1+i)^2} \leq NPW < +\infty. \quad (60)$$

Moreover, the probability of being economical with MARR, i, can be calculated and simplified as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}(v)\, dv = EXP\left[\frac{A_2/(1+i) - A_0(1+i)}{\beta_1}\right]. \quad (61)$$

However, additional analysis shows that the above equation is not valid in one special case, as expressed in Lemma 4.

Lemma 4: In a two-period project with fixed cost A_0, variable income A_1 with exponential distribution $A_1 \sim EXP(\beta_1)$, and fixed income A_2, if $\frac{A_2}{A_0} \geq (1+i)^2$, then this project is certainly economical.

Proof: If the assumption of Lemma 4 is true, it can be concluded that the lower bound of the economic desirability NPW is always nonnegative $\left(-A_0 + \frac{A_2}{(1+i)^2} \geq 0\right)$. Therefore, we always have $NPW \geq 0$; hence, the probability of being economical will be one. For further analysis of the result of this lemma, it can be mentioned that if we have $\frac{A_2}{A_0} \geq (1+i)^2$, the income A_2 is by itself enough for the economic viability of the project, and the economic viability of the project is independent of the value of the income A_1. In other words, with an increase in the value of A_1, the desirability will increase; however, with its decrease, the project will not quit the economic range at all.

By using the result of Lemma 4, the probability of such a project being economical is re-written as follows:

$$P(NPW \geq 0|i) = \begin{cases} EXP\left[\frac{A_2/(1+i)-A_0(1+i)}{\beta_1}\right] & \text{if } \frac{A_2}{A_0} < (1+i)^2 \\ 1 & \text{if } \frac{A_2}{A_0} \geq (1+i)^2 \end{cases} \quad (62)$$

The above equation predicts the economic desirability of the project and helps the investor with appropriate decision-making.

3.5. Fixed cost A_0, fixed income A_1, and variable income A_2 with uniform distribution

In this case, it is assumed that $A_0 \geq 0, A_1 \geq 0, A_2 \sim Uniform\,(\alpha_2, \beta_2)$; therefore, the mean and variance of the economic desirability of the project NPW can be calculated as follows:

$$E\,[NPW] = -A_0 + \frac{A_1}{(1+i)} + \frac{(\alpha_2+\beta_2)}{2\,(1+i)^2},$$

$$Var\,[NPW] = \frac{1}{12}.\frac{(\beta_2-\alpha_2)^2}{(1+i)^4}. \tag{63}$$

Furthermore, the moment-generating function of the income A_2 can be achieved as follows:

$$MGF_{A_2}\,(t) = E\,(EXP\,[tA_2])$$

$$= \frac{EXP\,[t\beta_2] - EXP\,[t\alpha_2]}{t\,(\beta_2-\alpha_2)}. \tag{64}$$

According to Eq. (9), the moment-generating function of the economic desirability can be calculated and summarized by Eq. (65) as shown in Box I.

By comparing $MGF_{NPW}\,(t)$ with the moment-generating function of the uniform distribution and utilizing the fact that the moment-generating function is unique for every random variable, it can be realized that:

$$NPW \sim Uniform$$

$$\left(\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0, \frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0\right), \tag{66}$$

and the PDF of the NPW can be estimated as follows:

$$PDF\,(NPW) : f_{NPW}\,(v) = \frac{(1+i)^2}{\beta_2-\alpha_2},$$

$$\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 \leq v \leq \frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0. \tag{67}$$

Therefore, the probability of such projects being economical with a particular MARR, i, is calculated as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}\,(v)\,dv. \tag{68}$$

This probability can be computed and summarized as follows:

$$P(NPW \geq 0|i) = \frac{\beta_2 + A_1\,(1+i) - A_0\,(1+i)^2}{\beta_2-\alpha_2}. \tag{69}$$

However, further analysis of the above formula reveals that it is not valid in two special cases. Lemmas 5 and 6 discuss these cases analytically.

Lemma 5: In a two-period project with fixed cost A_0, fixed income A_1, and variable income A_2 following the uniform distribution $A_2 \sim Uniform\,(\alpha_2, \beta_2)$, if $\beta_2 < A_0\,(1+i)^2 - A_1\,(1+i)$, then the project is certainly uneconomical.

Proof: If the assumption of this lemma is true, then $\frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 < 0$, and based on the fact that $\alpha_2 < \beta_2$, it is found that $\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 < 0$. Therefore, both the upper bound and the lower bound of the NPW are negative and, therefore, there is no possibility for the economic viability of such a project. It means that:

$$P(NPW \geq 0|i) = 0 \quad \text{if} \quad \beta_2 < A_0\,(1+i)^2 - A_1\,(1+i)\,. \tag{70}$$

Lemma 6: In a two-period project with fixed cost A_0, fixed income A_1, and variable income A_2 with uniform distribution $A_2 \sim Uniform\,(\alpha_2, \beta_2)$, if $\alpha_2 \geq A_0\,(1+i)^2 - A_1\,(1+i)$, then the project is certainly economical.

Proof: If the assumption of this lemma is true, it can be understood that $\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 \geq 0$, and based on the fact that $\alpha_2 < \beta_2$, it is found that $\frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 > 0$. Therefore, both the upper bound and the lower bound of the NPW are positive, and the project is certainly economical. It means that:

$$P(NPW \geq 0|i) = 1 \quad \text{if} \quad \alpha_2 \geq A_0\,(1+i)^2 - A_1\,(1+i)\,. \tag{71}$$

$$MGF_{NPW}\,(t) = EXP\left[\left(-A_0 + \frac{A_1}{(1+i)}\right)t\right].MGF_{A_2}\left(\frac{t}{(1+i)^2}\right)$$

$$= \frac{EXP\left[t\left(\frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0\right)\right] - EXP\left[t\left(\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0\right)\right]}{t\left(\frac{\beta_1}{(1+i)^2} - \frac{\alpha_1}{(1+i)^2}\right)}. \tag{65}$$

Box I

$$P(NPW \geq 0|i) = \begin{cases} 0 & \text{if } \beta_2 < A_0(1+i)^2 - A_1(1+i) \\ \frac{\beta_2 + A_1(1+i) - A_0(1+i)^2}{\beta_2 - \alpha_2} & \text{if } \alpha_2 < A_0(1+i)^2 - A_1(1+i) \ \& \ \beta_2 \geq A_0(1+i)^2 - A_1(1+i) \\ 1 & \text{if } \alpha_2 \geq A_0(1+i)^2 - A_1(1+i) \end{cases} \quad (72)$$

Box II

Therefore, according to the results of both Lemmas 5 and 6, the probability of such a project being economical is re-written by Eq. (72) as shown in Box II.

3.6. Fixed cost A_0, variable income A_1 with uniform distribution, and fixed income A_2

In this case, it is assumed that $A_0 \geq 0, A_1 \sim Uniform(\alpha_1, \beta_1)$, $A_2 \geq 0$; therefore, we have:

$$E[NPW] = -A_0 + \frac{(\alpha_1 + \beta_1)}{2(1+i)} + \frac{A_2}{(1+i)^2},$$

$$Var[NPW] = \frac{1}{12} \cdot \frac{(\beta_1 - \alpha_1)^2}{(1+i)^2}. \quad (73)$$

Similar to the previous case, the moment-generating function of the economic desirability can be calculated and summarized by Eq. (74) as shown in Box III. By comparing $MGF_{NPW}(t)$ with the moment-generating function of the uniform distribution and by utilizing the fact that the moment-generating function is unique for every random variable, it can be found that:

$$NPW \sim Uniform$$

$$\left(\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0, \frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0 \right). \quad (75)$$

Therefore, the PDF of the NPW can be estimated as follows:

$$PDF(NPW): f_{NPW}(v) = \frac{1+i}{\beta_1 - \alpha_1}, \quad (76)$$

$$\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0 \leq v \leq \frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0.$$

Consequently, when we have an investor with a particular MARR, i, the probability of such a project being economical can be calculated and simplified as follows:

$$P(NPW \geq 0|i) = \frac{A_2/(1+i) + \beta_1 - A_0(1+i)}{\beta_1 - \alpha_1}. \quad (77)$$

However, further analysis of the above formula shows that it is not valid for the two special cases, which are presented in Lemmas 7 and 8.

Lemma 7: In a two-period project with fixed cost A_0, variable income A_1 with uniform distribution $A_1 \sim Uniform(\alpha_1, \beta_1)$, and fixed income A_2, if $\beta_1 < A_0(1+i) - A_2/(1+i)$, then this project is certainly uneconomical.

Proof: If the assumption of this lemma is true, it can be understood that $\frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0 < 0$. By utilizing this result and based on the fact that $\alpha_1 < \beta_1$, we find that $\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0 < 0$; in this case, both the upper bound and the lower bound of the NPW are negative and, therefore, there is no chance for this project to be economical. This lemma can be represented in the following mathematical form.

$$P(NPW \geq 0|i) = 0 \text{ if } \beta_1 < A_0(1+i) - A_2/(1+i). \quad (78)$$

Lemma 8: In a two-period project with fixed cost A_0, variable income A_1 with uniform distribution $A_1 \sim Uniform(\alpha_1, \beta_1)$, and fixed income A_2, if $\alpha_1 \geq A_0(1+i) - A_2/(1+i)$, then this project is certainly economical.

Proof: If the assumption of this lemma is true, it can be understood that $\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0 \geq 0$. According to this result and based on the fact that $\alpha_1 < \beta$, it is revealed that $\frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0 > 0$. Thus, both

$$MGF_{NPW}(t) = EXP\left[\left(-A_0 + \frac{A_2}{(1+i)^2}\right)t\right] . MGF_{A_1}\left(\frac{t}{1+i}\right)$$

$$= \frac{EXP\left[t\left(\frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0\right)\right] - EXP\left[t\left(\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0\right)\right]}{t\left(\frac{\beta_1}{1+i} - \frac{\alpha_1}{1+i}\right)}. \quad (74)$$

Box III

the upper bound and the lower bound of the NPW are positive, and the project is certainly economical. In other words:

$$P(NPW \geq 0|i) = 1 \quad \text{if } \alpha_1 \geq A_0\,(1+i) - A_2/(1+i)\,. \tag{79}$$

Therefore, according to the results obtained by Lemmas 7 and 8, the probability of such a project being economical can be re-written by Eq. (80) as shown in Box IV. Eq. (80) can help the investor to simply estimate the economic desirability of projects in this case.

The aim of this section is to determine the probability of investment projects being economical with two-period cash flow by deriving the analytical equation of the NPW distribution function. For this purpose, six cases were analyzed. These cases are applicable in reality where the initial cost A_0 is a fixed positive number and A_1 incomes A_2 or are random variables with (i) normal, (ii) exponential, and (iii) uniform distributions. For all of the cases, analytical PDF of NPW was derived mathematically, which is a useful tool for investors to evaluate uncertain projects over a two-period cash flow.

4. Multiple-period cash flow analysis

This section extends the one- and two-period problems discussed in the previous sections to a general case and, then, studies an investment problem consisting of a multiple-period project with an initial cost A_0 and incomes at the end of the kth period A_k $(k = 1, 2, ..., n)$. The cash flow of such a project is shown in Figure 1.

In this project, the economic desirability NPW is a function of $A_0, A_1, A_2, ..., A_n$ cash flows, which can be calculated as follows:

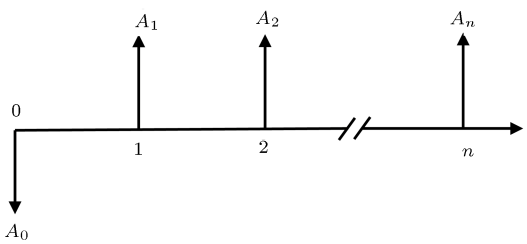

Figure 1. A multiple-period cash flow diagram.

$$NPW = -A_0 + \sum_{k=1}^{n} \frac{A_k}{(1+i)^k}. \tag{81}$$

By taking into account the assumptions that incomes $A_1, A_2, ..., A_n$ are random and independent variables, the economic desirability NPW is, therefore, random with the following mean and the variance:

$$E[NPW] = -A_0 + \sum_{k=1}^{n} \frac{\mu_{A_k}}{(1+i)^k},$$

$$Var[NPW] = \sum_{k=1}^{n} \frac{\sigma^2_{A_k}}{(1+i)^{2k}}. \tag{82}$$

where μ_{A_k} and $\sigma^2_{A_k}$ denote the mean and variance of the kth random income A_k, respectively. The aim is to determine the economic desirability of such a project by deriving the analytical equation for the PDF of NPW. To this end, three cases are analyzed in the sequel. In these cases, the initial cost A_0 is a fixed positive number, and the incomes A_k are random variables with (i) normal, (ii) exponential, and (iii) uniform distributions.

4.1. Variable income with normal distribution

In this case, by considering the assumptions $A_0 \geq 0$, $A_k \geq 0$, and for a special period l, $A_l \sim Normal(\mu_l, \sigma_l^2)$, it can be concluded that the economic desirability of the project NPW is also variable. By utilizing an approach similar to that in the previous sections, it can be concluded that:

$$NPW = -A_0 + \sum_{k=1}^{n} \frac{A_k}{(1+i)^k}$$

$$\sim Normal\left(-A_0 + \frac{\mu_{A_l}}{(1+i)^l} + \sum_{k=1(k\neq l)}^{n} \frac{A_k}{(1+i)^k}, \frac{\sigma^2_{A_l}}{(1+i)^{2l}}\right). \tag{83}$$

Consequently, the probability of such projects being economical with MARR, i, is calculated as follows:

$$P(NPW \geq 0|i) = \begin{cases} 0 & \text{if } \beta_1 < A_0\,(1+i) - A_2/(1+i) \\ \frac{A_2/(1+i)+\beta_1-A_0(1+i)}{\beta_1-\alpha_1} & \text{if } \alpha_1 < A_0\,(1+i) - A_2/(1+i) \ \& \ \beta_1 \geq A_0\,(1+i) - A_2/(1+i) \\ 1 & \text{if } \alpha_1 \geq A_0\,(1+i) - A_2/(1+i) \end{cases} \tag{80}$$

Box IV

$$P(NPW \geq 0|i) = 1 - \phi\left(\frac{A_0(1+i)^l - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}} - \mu_{A_l}}{\sigma_{A_l}}\right). \quad (84)$$

4.2. Variable income with exponential distribution

In this case, by considering the assumptions $A_0 \geq 0, A_k \geq 0$, and for a special period l, $A_l \sim EXP(\beta_l)$, it can be concluded that the economic desirability of the project NPW is also variable. Therefore, we have:

$$E[NPW] = -A_0 + \frac{\beta_l}{(1+i)^l} + \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k},$$

$$Var[NPW] = \frac{\beta_l^2}{(1+i)^{2l}}. \quad (85)$$

In this case, similar to the previous exponential cases, by using the transformation method, the reverse function NPW^{-1} and its derivative can be derived as follows:

$$NPW = -A_0 + \sum_{k=1}^{n} \frac{A_k}{(1+i)^k}$$

$$NPW^{-1} : A_l = \left[NPW + A_0 - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k}\right](1+i)^l,$$

$$\left(NPW^{-1}\right)' = (1+i)^l. \quad (86)$$

Therefore, by applying the transformation method, the PDF of the economic desirability NPW will be simplified into its final form as follows:

$$PDF(NPW) : f_{NPW}(v) = \frac{(1+i)^l}{\beta_l} EXP\left[\frac{-\left((v+A_0)(1+i)^l - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}}\right)}{\beta_l}\right]. \quad (87)$$

Since the income A_l varies at an interval of $0 \leq A_l < +\infty$, the economic desirability NPW varies in the following range:

$$-A_0 + \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} \leq NPW < +\infty. \quad (88)$$

Moreover, the probability of such a project being economic with MARR, i, can be calculated and simplified as follows:

$$P(NPW \geq 0|i) = \int_0^{+\infty} f_{NPW}(v)\, dv = EXP\left[\frac{\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}} - A_0(1+i)^l}{\beta_l}\right]. \quad (89)$$

However, the additional analysis shows that the above equation is not valid in one special case, which is expressed in Lemma 9.

Lemma 9: In a multiple-period project with fixed cost A_0, incomes $A_1, A_2, ..., A_n$ in which $A_l\,(l \in \{1, 2, \ldots, n\})$ is variable with exponential distribution $A_l \sim EXP(\beta_l)$, and the fixed income A_k $(k = 1, 2, \ldots, n|k \neq l)$, if $\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} \geq A_0$, it can be concluded that this project is certainly economical.

Proof: If the assumption of this lemma is true, it can be concluded that the lower bound of the economic desirability NPW is always nonnegative:

$$-A_0 + \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} \geq 0.$$

Thus, we always have $NPW \geq 0$ and the probability of this subject being economical will be one. According to further analysis of the result of this lemma, if:

$$\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} \geq A_0,$$

then the incomes $A_1, A_2, \ldots, A_{l-1}, A_{l+1}, \ldots, A_n$ may be enough for the economic viability of the project, and the economic viability of the project is independent of the value of the income A_l. In other words, with any increase in the value of A_l, the desirability will increase; however, with its decrease, the project will not quit the economic range at all.

By using the result of Lemma 9, the probability of such a project being economical is re-written by Eq. (90) as shown in Box V. Eq. (90) predicts the economic desirability of the project and helps the investor with appropriate decision-making.

4.3. Variable income with uniform distribution

In this case, by considering the assumptions $A_0 \geq 0$, $A_k \geq 0$, and for a special period l, $A_l \sim Uniform(\alpha_l, \beta_l)$, it can be concluded that the economic desirability of the project NPW is also variable. Therefore, we have:

$$E[NPW] = -A_0 + \frac{(\alpha_l + \beta_l)}{2(1+i)^l} + \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k},$$

$$P(NPW \geq 0|i) = \begin{cases} EXP\left[\frac{\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}} - A_0(1+i)^l}{\beta_l}\right] & \text{if} \quad \sum\limits_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} < A_0 \\ 1 & \text{if} \quad \sum\limits_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} \geq A_0 \end{cases} \tag{90}$$

Box V

$$Var\left[NPW\right] = \frac{1}{12} \cdot \frac{\left(\beta_l - \alpha_l\right)^2}{\left(1+i\right)^{2l}}. \tag{91}$$

Furthermore, the moment-generating function of the income A_2 can be achieved as follows:

$$MGF_{A_l}\left(t\right) = E\left(EXP\left[tA_l\right]\right) = \frac{EXP\left[t\beta_l\right] - EXP\left[t\alpha_l\right]}{t\left(\beta_l - \alpha_l\right)}. \tag{92}$$

Therefore, the moment-generating function of the economic desirability can be calculated and summarized by Eq. (93) as shown in Box VI. By comparing $MGF_{NPW}(t)$ with the moment-generating function of the uniform distribution and by utilizing the fact that the moment-generating function is unique for every random variable, it can be found that:

$$NPW \sim Uniform \left(\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^k} + \frac{\alpha_l}{\left(1+i\right)^l} - A_0, \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^k} + \frac{\beta_l}{\left(1+i\right)^l} - A_0 \right). \tag{94}$$

Therefore, the PDF of the NPW can be estimated as follows:

$$PDF\left(NPW\right) : f_{NPW}\left(v\right) = \frac{\left(1+i\right)^l}{\beta_l - \alpha_l}, \tag{95}$$

$$\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^k} + \frac{\alpha_l}{\left(1+i\right)^l} - A_0 \leq v \leq \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^k} + \frac{\beta_l}{\left(1+i\right)^l} - A_0.$$

Consequently, when there is an investor with a particular MARR, i, the probability of such a project being economical can be calculated and simplified as follows:

$$P(NPW \geq 0|i) = \frac{\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}} + \beta_l - A_0\left(1+i\right)^l}{\beta_l - \alpha_l}. \tag{96}$$

However, further analysis of the above formula shows that it is not valid for two special cases, which are presented in the following Lemmas 10 and 11.

Lemma 10: In a multiple-period project with the fixed cost A_0 and incomes $A_1, A_2, .., A_n$ in which $A_l\,(l \in \{1, 2, \ldots, n\})$ is variable with uniform distribution $A_l \sim Uniform(\alpha_l, \beta_l)$, if $\beta_l < A_0\left(1+i\right)^l - \sum\limits_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}}$, then this project is certainly uneconomical.

Proof: If the assumption of this lemma is true, then $\sum\limits_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\beta_l}{(1+i)^l} - A_0 < 0$. By utilizing this result and based on the fact that $\alpha_1 < \beta_1$, it is found that $\sum\limits_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\alpha_l}{(1+i)^l} - A_0 < 0$. Therefore, both upper and lower bounds of the NPW are negative and, therefore, there is no chance for this project to be economical. This lemma can be represented in the following mathematical form.

$$P(NPW \geq 0|i) = 0 \;\; \text{if} \;\; \beta_l < A_0\left(1+i\right)^l - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^{k-l}}. \tag{97}$$

$$MGF_{NPW}\left(t\right) = EXP\left[\left(-A_0 + \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{\left(1+i\right)^k}\right)t\right].MGF_{A_l}\left(\frac{t}{\left(1+i\right)^l}\right) = \frac{EXP\left[t\left(\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\beta_l}{(1+i)^l} - A_0\right)\right] - EXP\left[t\left(\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\alpha_l}{(1+i)^l} - A_0\right)\right]}{t\left(\frac{\beta_l}{(1+i)^l} - \frac{\alpha_l}{(1+i)^l}\right)}. \tag{93}$$

Box VI

Lemma 11: In a multiple-period project with the fixed cost A_0, and incomes $A_1, A_2, ..., A_n$ in which $A_l\,(l \in \{1, 2, \ldots, n\})$ is variable with uniform distribution $A_l \sim Uniform\,(\alpha_l, \beta_l)$, if:

$$\alpha_l \geq A_0\,(1+i)^l - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}},$$

then this project is certainly economical.

Proof: If the assumption of this lemma is true, it can be understood that:

$$\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\alpha_l}{(1+i)^l} - A_0 \geq 0.$$

Utilizing this result and the fact that $\alpha_1 < \beta_1$ reveals that:

$$\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} + \frac{\beta_l}{(1+i)^l} - A_0 > 0.$$

Therefore, both upper and lower bounds of the NPW are positive, and the project is certainly economical. It means that:

$$P(NPW \geq 0|i) = 1 \;\text{ if }\; \alpha_l \geq A_0\,(1+i)^l - \sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^{k-l}}. \tag{98}$$

Therefore, according to the results obtained from Lemmas 10 and 11, the probability of such a project being economical can be re-written by Eq. (99) as shown in Box VII.

Eq. (99) can help the investor to simply estimate the economic desirability of projects in this case.

The aim of this section is to determine the probability of investment projects being economical with two-period cash flow by deriving the analytical equation of the NPW distribution function. For this purpose, three cases were analyzed. These cases are applicable in reality, where the initial cost A_0 is a fixed positive number and income at the lth period A_l is random variables with (i) normal, (ii) exponential, and (iii) uniform distributions. For all of the cases, the analytical PDF of NPW was derived mathematically, which is a useful tool for investors in evaluating uncertain projects over multiple-period cash flows.

5. Analyses and simulation results

This section investigates the validity of the proposed analytical equations derived in previous sections by comparing them to the results of the simulation approach. For this purpose, numerical examples in each following section are discussed separately:

One-period cash flows:

- Fixed cost A_0 and variable income A_1 with normal distribution;
- Fixed cost A_0 and variable income A_1 with exponential distribution;
- Fixed cost A_0 and variable income A_1 with uniform distribution.

Two-period cash flows:

- Fixed cost A_0, fixed income A_1, and variable income A_2 with normal distribution;
- Fixed cost A_0, variable income A_1 with normal distribution, and fixed income A_2;
- Fixed cost A_0, fixed income A_1, and variable income A_2 with exponential distribution;
- Fixed cost A_0, variable income A_1 with exponential distribution, and fixed income A_2;
- Fixed cost A_0, fixed income A_1, and variable income A_2 with exponential distribution;
- Fixed cost A_0, variable income A_1 with uniform distribution, and fixed income A_2.

5.1. Numerical examples for one-period cash flows

- *Fixed cost A_0 and variable income A_1 with normal distribution*: Assume that in a project with a one-period cash flow, the initial cost is $A_0 = 125$, the

$$P(NPW \geq 0|i) = \begin{cases} 0 & \text{if } \beta_1 < A_0\,(1+i)^l - \sum\limits_{k=1|k\neq l}^{n} \frac{A_k}{(1+i)^{k-l}} \\ \frac{\sum\limits_{k=1|k\neq l}^{n} \frac{A_k}{(1+i)^{k-l}} + \beta_l - A_0\,(1+i)^l}{\beta_l - \alpha_l} & \text{if } \alpha_l < A_0\,(1+i)^l - \sum\limits_{k=1|k\neq l}^{n} \frac{A_k}{(1+i)^{k-l}},\;\; \beta_l \geq A_0\,(1+i)^l - \sum\limits_{k=1|k\neq l}^{n} \frac{A_k}{(1+i)^{k-l}} \\ 1 & \text{if } \alpha_l \geq A_0\,(1+i)^l - \sum\limits_{k=1|k\neq l}^{n} \frac{A_k}{(1+i)^{k-l}} \end{cases} \tag{99}$$

Box VII

variable income is $A_1 \sim Normal(\mu_1 = 150, \sigma_1^2 = 2)$, and an investor with MARR is $i = 0.1$ while evaluating such a project. In this case, the economic desirability of such a project will be a function of the variable income as $NPW = -125 + 0.9091A_1$. According to this, the mean and the variance of the economic desirability of the project are calculated as $E[NPW] = 11.3636$ and $Var[NPW] = 1.6529$. By using the analytical equations derived in Section 3.1, the NPW of this project is a random variable with distribution $NPW \sim Normal(11.3636, 1.6529)$ and PDF as follows:

$$PDF(NPW) : f_{NPW}(v) = 0.3103 * EXP\left[\frac{-(1.1v - 12.5)^2}{4}\right].$$

Figure 2 indicates the behavior of this analytical equation graphically, and Figure 3 depicts the simulation results. To achieve the simulation results, 200000 random samples have been simulated for the income A_1 from distribution $Normal(\mu_1 = 150, \sigma_1^2 = 2)$, and the values of the economic desirability NPW have been calculated according to the observed values for A_1 through $NPW = -A_0 + A_1/(1+i)$. Finally, the frequency of simulated NPW values, called `SimulatedNPW`, is drawn in the format of histogram diagram. The pseudo code of the simulation procedure is given below:

```
A1=zeros(200000,1);
SimulatedNPW=zeros(200000,1);
for j=1:200000
   A1(j,1)=normrnd(mu1, sigma1);
   SimulatedNPW(j,1)=-A0+A1(j,1)/(1+i);
end
hist(SimulatedNPW,20).
```

As is obvious, the pattern of simulation results in Figure 2 completely coincides with the analytical equation depicted in Figure 3. It confirms the appropriateness of the analytical equation derived in Section 3.1 in two terms: (i) the type of probability distribution and (ii) the mean and variance values.

Furthermore, the probability of such a project being economical by considering the assumption of having MARR, $i = 0.1$, can be achieved as follows:

$$P(NPW \geq 0 | i = 0.1) = 1 - \phi(-8.8388) \approx 1.$$

- *Fixed cost A_0 and variable income A_1 with exponential distribution*: Assume that, in a project with one-period cash flow, the initial cost is $A_0 = 125$, the variable income is $A_1 \sim EXP(\beta_1 = 135)$, and an investor with MARR is $i = 0.2$ while evaluating such a project. In this case, the economic desirability function will be computed as $NPW = -125 + 0.4545A_1$. According to this, the mean and variance of the economic desirability of this project are calculated as $E[NPW] = -12.50$ and $Var[NPW] = 12656$. In addition, according to the analytical equation derived in Section 3.2, the NPW of this project is a random variable with PDF, $PDF(NPW)$, as follows:

$$PDF(NPW) : f_{NPW}(v) = 0.0089 * EXP[-0.0089(v + 125)],$$

where the variation range of the NPW is $-125 \leq v \leq +\infty$. Figure 4 depicts the analytical equation of $f_{NPW}(v)$ derived in Section 3.2, and Figure 5 indicates the pattern of simulation results of 200000 random samples. As is obvious from the figures, both analytical and simulated diagrams are completely in agreement with each other, which confirms the validity of our analytical equation. The pseudo code of the simulation procedure is shown below:

```
A1=zeros(200000,1);
SimulatedNPW=zeros(200000,1);
```

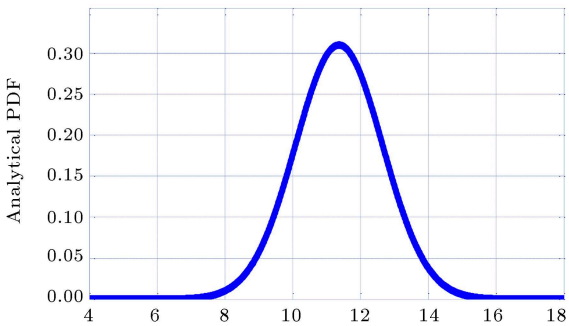

Figure 2. The Net Present Worth (NPW) distribution function according to the analytical equation derived for the project with fixed cost A_0 and variable income A_1 with normal distribution.

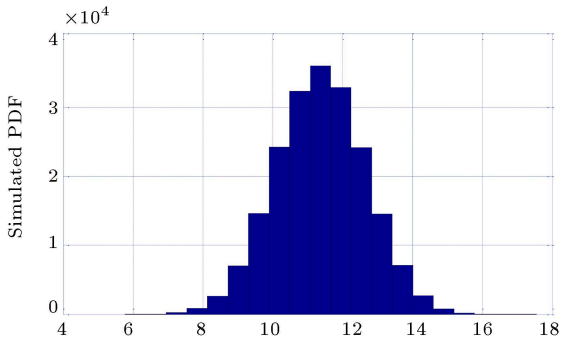

Figure 3. The Net Present Worth (NPW) distribution histogram according to the simulation results for the project with fixed cost A_0 and variable income A_1 with normal distribution.

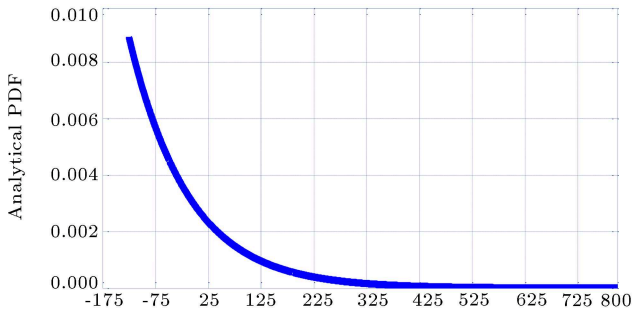

Figure 4. The Ner Present Worth (NPW) distribution function according to the analytical equation derived for the project with fixed cost A_0 and variable income A_1 with exponential distribution.

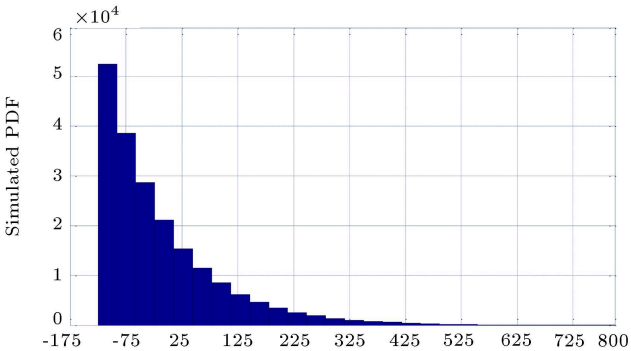

Figure 5. The Net Present Worth (NPW) distribution histogram according to the simulation results for the project with fixed cost A_0 and variable income A_1 with exponential distribution.

```
for j=1:200000
    A1(j,1)=exprnd(Beta1);
    SimulatedNPW(j,1)=-A0+A1(j,1)./(1+i);
end
hist(SimulatedNPW,30).
```

Moreover, the probability of such a project being economical by taking into account the assumption of having MARR, $i = 0.2$, can be computed as follows:

$$P(NPW \geq 0|i = 0.2) = EXP\left[\frac{-125\,(1+0.2)}{135}\right]$$

$$= 0.3292.$$

As is obvious from Figures 4 and 5, the economic desirability of this project NPW is less than zero in some cases $(-125, 0)$ and, for this reason, the probability of this project being economical is computed as 32.92%.

- *Fixed cost A_0 and variable income A_1 with uniform distribution:* In this case, we take into account a project with an initial cost $A_0 = 100$, a variable income $A_1 \sim Uniform(\alpha_1 = 125, \beta_1 = 155)$, and an investor with MARR, $i = 0.35$. In this case, the function of economic desirability will be as $NPW = -100 + 0.7407A_1$. Therefore, the mean and the variance of the economic desirability of this project are calculated as $E[NPW] = 3.7037$ and $Var[NPW] = 41.1523$. Moreover, according to the analytical equation derived in Section 3.3, the NPW of this project has a uniform distribution with PDF, $PDF(NPW)$, as follows:

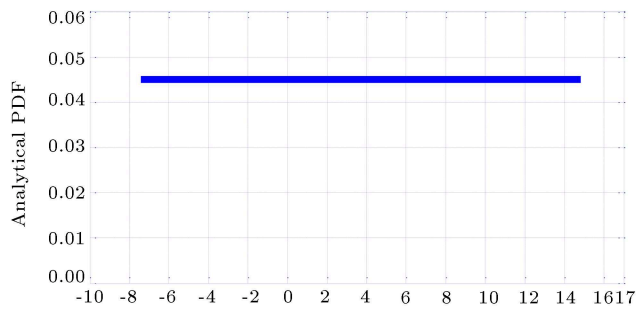

Figure 6. The Net Present Worth (NPW) distribution function according to the analytical equation derived for the project with fixed cost A_0 and variable income A_1 with exponential distribution.

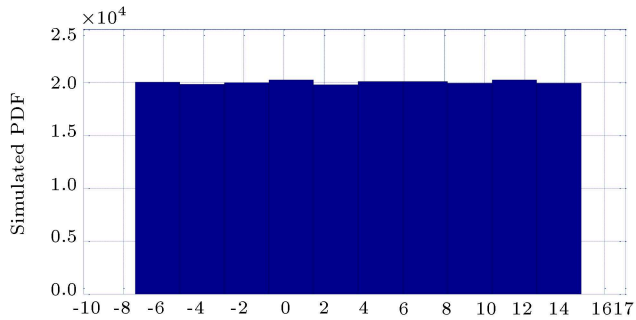

Figure 7. The Net Present Worth (NPW) distribution histogram according to the simulation results for the project with fixed cost A_0 and variable income A_1 with exponential distribution.

$$NPW \sim Uniform$$

$$\left(\frac{\alpha_1}{1+i} - A_0 = -7.4074, \frac{\beta_1}{1+i} - A_0 = 14.8148\right)$$

$$PDF\,(NPW) : f_{NPW}\,(v) = 0.0450,$$

$$-7.4074 \leq v \leq 14.8148.$$

To validate this analytical equation, the simulation approach is utilized. Figures 6 and 7 confirm the adoption of the analytical equation of $f_{NPW}\,(v)$ and the simulated results of 200000 samples. The pseudo code for the simulation procedure is shown as follows:

```
A1=zeros(200000,1);
SimulatedNPW=zeros(200000,1);
```

```
for j=1:200000
   A1(j,1)=random('unif',Alfa1,Beta1);
   SimulatedNPW(j,1)=-A0+A1(j,1)./(1+i);
end
hist(SimulatedNPW,10).
```

Moreover, the probability of such a project being economical is determined as follows:

$$P(NPW \geq 0|i) = \frac{\beta_1 - A_0(1+i)}{\beta_1 - \alpha_1} = 0.6667.$$

In this project, if $i = 0.6$, then $\beta_1 < A_0(1+i)$; according to Lemma 1, then this project is certainly uneconomical, and the probability of it being economical equals 0. Besides, since $\alpha_1 \geq A_0(1+i)$, if $i = 0.2$, according to Lemma 2, then this project is certainly uneconomical, and the probability of it being economical equals 1.

5.2. Numerical examples for two-period cash flows

- *Fixed cost A_0, fixed income A_1, and variable income A_2 with normal distribution:* Assume that, in a two-period project, the cash flow is as $A_0 = 160, A_1 = 130, A_2 \sim Normal\left(\mu_2 = 100, \sigma_2^2 = 20\right)$ and the MARR equals $i = 0.25$. In this project, the function of the economic desirability NPW is derived as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2} = -56 + 0.64A_2,$$

$$E[NPW] = 8, \quad Var[NPW] = 8.1920,$$

and, according to the results derived in Section 4.1, it can be concluded that:

$$NPW \sim Normal(8, 8.1920).$$

Similar to one-period cases, the comparison between the derived analytical equation and simulation results was carried out, and the results confirmed the existence of this equation completely. For the sake of simplicity and with the aim of preventing repetitive materials, the simulation procedure is disregarded hereafter. Further, the probability of this project being economical with MARR, $i = 0.25$, is obtained as follows:

$$P(NPW \geq 0|i = 0.25) = 1 - \phi\left(\frac{160(1+0.25)^2 - 130(1+0.25) - 100}{(20)^{0.5}}\right) = 0.9974.$$

- *Fixed cost A_0, variable income A_1 with normal distribution, and fixed income A_2:* Assume that, in the previous example, the cash flow would be as $A_0 = 160, A_1 = Normal\left(\mu_1 = 130, \sigma_1^2 = 20\right)$, $A_2 \sim 100$ and the MARR would be $i = 0.25$. In this case, the function of the economic desirability, NPW, can be achieved as follows:

$$NPW = -A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2} = -96 + 0.8A_1,$$

$$E[NPW] = 8, \quad Var[NPW] = 12.80.$$

Therefore, according to the analytical results derived in Section 4.2, it can be concluded that:

$$NPW \sim Normal(8, 12.80).$$

Similar to the previous cases, the simulation results confirm the existence of this equation completely. In addition, the probability of this project being economical with MARR, $i = 0.25$ is attained as follows:

$$P(NPW \geq 0|i = 0.25) = 1 - \phi\left(\frac{160(1+0.25) - 130 - 100/(1+0.25)}{(20)^{0.5}}\right) = 0.9873.$$

- *Fixed cost A_0, fixed income A_1, and variable income A_2 with exponential distribution:* Consider a two-period project with cash flow as $A_0 = 160, A_1 = 130, A_2 \sim EXP(\beta_2 = 100)$ and MARR as $i = 0.25$. In this case, the function of the economic desirability would be as NPW $= -56 + 0.64\mathrm{A}_2$ with $E[NPW] = 8$ and Var[NPW] $= 4096$. Therefore, according to the analysis performed in Section 4.3, it can be concluded that:

$$PDF(NPW): f_{NPW}(v) = 0.0156 * EXP\left[\frac{-[1.5625(v+160) - 162.5]}{100}\right],$$

where the variation range of the NPW is $-56 \leq v \leq +\infty$. Similar to the one-period cases, the simulation results confirm the existence of this equation. Furthermore, the probability of this project being economical with MARR, $i = 0.25$, can be gained as follows:

$$P(NPW \geq 0|i = 0.25) = EXP\left[\frac{130(1+0.25) - 160(1+i)^2}{100}\right] = 0.4169.$$

However, since $\frac{A_1}{A_0} \geq (1+i)$, if the initial cost is $A_0 = 100$ instead of $A_0 = 160$, according to Lemma 3, then this project is certainly economical in this case, and the probability of it being economical will be 1.

- *Fixed cost A_0, variable income A_1 with exponential distribution, and fixed income A_2*: Assume that the cash flow of a two-period project is $A_0 = 160, A_1 \sim EXP(\beta_1 = 130), A_2 = 100$ and the MARR is $i = 0.25$. In this case, the function of the economic desirability would be as $NPW = -96 + 0.8A_1$ with $E[NPW] = 8$ and $Var[NPW] = 10816$. Therefore, according to the analysis performed in Section 4.4, it can be concluded that:

$$PDF(NPW): f_{NPW}(v) = 0.0096 *EXP\left[\frac{-[1.25(v+160)-80]}{100}\right],$$

where the variation range of the NPW is $-96 \leq v \leq +\infty$. Similar to the previous cases, simulation analysis was carried out, and the results confirmed the existence of this equation. In addition, the probability of it being economic with MARR, $i = 0.25$, can be calculated for this project as follows:

$$P(NPW \geq 0|i = 0.25) = EXP\left[\frac{100/(1+0.25) - 160(1+0.25)}{130}\right] = 0.3973.$$

However, according to what has been proved in Lemma 4, if the initial cost would be $A_0 = 60$ instead of $A_0 = 160$, we have $\frac{A_2}{A_0} \geq (1+i)^2$; therefore, the project is certainly economical, and the probability of it being economical is 1.

- *Fixed cost A_0, fixed income A_1, and variable income A_2 with uniform distribution:* Consider a two-period project with cash flow as $A_0 = 160$, $A_1 = 130$, $A_2 \sim Uniform(\alpha_2 = 80, \beta_2 = 120)$ and the MARR as $i = 0.25$. In this case, the economic desirability function would be as $NPW = -56 + 0.64A_2$ with $E[NPW] = 8$ and $Var[NPW] = 54.6133$. According to the analyses performed in Section 4.5, it can be concluded that:

$$NPW \sim Uniform\left(\frac{\alpha_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 = -4.80, \frac{\beta_2}{(1+i)^2} + \frac{A_1}{1+i} - A_0 = 20.80\right),$$

$$PDF(NPW): f_{NPW}(v) = 0.0391,$$

$$-4.8 \leq v \leq 10.80.$$

Similar to the one-period cases, the simulation analysis was carried out and the results confirmed the existence of such a function completely. In addition, the probability of this project being economical with MARR, $i = 0.25$, is calculated as follows:

$$P(NPW \geq 0|i) = \frac{120 + 130(1+0.25) - 160(1+0.25)^2}{120 - 80} = 0.8125.$$

However, according to what has been presented in Lemma 5, if the upper bound of the income A_2 is $\beta_2 = 87$, then we have $\beta_2 < A_0(1+i)^2 - A_1(1+i)$, and the project is certainly uneconomical with the probability of it being economical as 0. On the other hand, if the lower bound of the income A_2 is $\alpha_2 = 90$, then $\alpha_2 \geq A_0(1+i)^2 - A_1(1+i)$; according to Lemma 6, the project is certainly economical with the probability of it being economical as 1.

- *Fixed cost A_0, variable income A_1 with uniform distribution, and fixed income A_2*: Consider the cash flow of a two-period project is as $A_0 = 160, A_1 \sim Uniform(\alpha_1 = 110, \beta_1 = 150), A_2 = 100$ and the MARR is $i = 0.25$. In this case, the function of the economic desirability is $NPW = -96 + 0.8A_1$ with $E[NPW] = 8$ and $Var[NPW] = 85.3333$. In this case, according to the results derived in Section 4.6, it can be concluded that:

$$NPW \sim Uniform\left(\frac{A_2}{(1+i)^2} + \frac{\alpha_1}{1+i} - A_0 = -8, \frac{A_2}{(1+i)^2} + \frac{\beta_1}{1+i} - A_0 = 24\right),$$

$$PDF(NPW): f_{NPW}(v) = 0.0313,$$

$$-8 \leq v \leq 24.$$

Similar to the one-period cases, the simulation results confirm the existence of this function completely. In addition, the probability of this project being economical with MARR, $i = 0.25$, is obtained as follows:

$$P(NPW \geq 0|i) = \frac{100/(1+0.25) + 150 - 160(1+0.25)}{150 - 110} = 0.75.$$

However, according to what has been proved in Lemma 7, if the upper bound of the income A_2 is $\beta_1 = 115$, then we have $\beta_1 < A_0(1+i) - A_2/(1+i)$, and the project is certainly uneconomical. On the other hand, if the lower bound of the income A_2 is $\alpha_3 = 130$, we have $\alpha_1 \geq A_0(1+i) - A_2/(1+i)$ and, according to Lemma 8, the project is certainly economical and the probability of being economical will be 1.

$$P(NPW \geq 0|i=0.25) = 1 - \phi\left(\frac{400\,(1+0.25)^3 - \frac{150}{(1+0.25)^{1-3}} - \frac{200}{(1+0.25)^{2-3}} - \frac{50}{(1+0.25)^{4-3}} - \frac{150}{(1+0.25)^{5-3}} - 100}{10}\right)$$
$$= 0.00.$$

Box VIII

$$PDF\,(NPW) : f_{NPW}\,(v) = \frac{(1+0.25)^3}{350} EXP$$
$$\left[\frac{-\left((v+400)\,(1+0.25)^3 - \frac{150}{(1+0.25)^{1-3}} - \frac{200}{(1+0.25)^{2-3}} - \frac{50}{(1+0.25)^{4-3}} - \frac{150}{(1+0.25)^{5-3}}\right)}{350}\right].$$

Box IX

5.3. Numerical examples for multiple-period cash flows

- *Variable income with normal distribution:* Consider an investment project with five periods, where the cash flow is as $A_0 = 400, A_1 = 150, A_2 = 200, A_3 \sim Normal\left(\mu_3 = 100, \sigma_3^2 = 10\right)$, $A_4 = 50$, and $A_5 = 150$. In addition, assume that the MARR equals $i = 0.25$. In this project, the function of the economic desirability NPW is derived as follows:

$$NPW = -A_0 + \sum_{k=1}^{5} \frac{A_k}{(1+i)^k} = -82.4 + 0.51A_3,$$

$$E\,[NPW] = -31.16, \quad Var\,[NPW] = 2.62.$$

In addition, according to the results derived in Section 5.1, it can be concluded that:

$$NPW \sim Normal\,(-31.16, 2.62)\,.$$

Similar to the one- and two-period cases, the comparison between derived NPW equation and simulation results was made, and the results confirm this NPW equation completely. Moreover, the probability of this project being economical for an investor with MARR, $i = 0.25$, is obtained by the expression shown in Box VIII. It means that this project has no chance to be economical.

- *Variable income with exponential distribution:* Assume that the cash flow in the third period is $A_3 \sim EXP\,(\beta_3 = 350)$ and all of the other parameters are unchanged in the previous example. In this case, the mean and variance of economic desirability NPW are determined as follows:

$$E\,[NPW] = 96.83, \quad Var\,[NPW] = 32112.64.$$

Thus, according to the analytical results derived in Section 5.2, the expression shown in Box IX is concluded.

Similar to the previous cases, the simulation results confirm the existence of this equation completely. Further, the probability of this project being economical is attained by the expression shown in Box X.

Since the condition $\sum_{k=1|(k\neq l)}^{n} \frac{A_k}{(1+i)^k} < A_0$ $(317.63 < 400)$ is true for this example, we can conclude that the probability of this project being economical is 0.6315.

- *Variable income with uniform distribution:* Consider the previous example, in which the cash flow in the third period is $A_3 \sim Uniform\,(\alpha_3 = 500, \beta_3 = 600)$

$$P(NPW \geq 0|i=0.25) = EXP\left[\frac{\frac{150}{(1+0.25)^{1-3}} + \frac{200}{(1+0.25)^{2-3}} + \frac{50}{(1+0.25)^{4-3}} + \frac{150}{(1+0.25)^{5-3}} - 400\,(1+0.25)^3}{350}\right] = 0.6315.$$

Box X

and all of the other parameters are unchanged. In this case, the mean and variance of the economic desirability NPW are determined as follows:

$$E[NPW] = 199.23, \quad Var[NPW] = 218.45.$$

Thus, according to the analytical results derived in Section 5.3, the PDF of NPW can be concluded that:

$$PDF(NPW) : f_{NPW}(v) = \frac{(1+0.25)^3}{600-500} = 0.0195,$$

$$173.63 \leq v \leq 224.83.$$

As can be seen, both upper and lower bounds of the NPW are greater than zero and, then, $P(NPW \geq 0|i = 0.25)$ is 1, meaning that this project is certainly economical.

6. Conclusions

This paper utilized the probability theory tools such as the moment-generating function and transformation method to derive the analytical and closed-form function for Probability Distribution Function (PDF) of the Net Present Worth (NPW) in investment projects. The random cash flows follow normal, uniform, and exponential distributions in our analysis. This analysis was performed for three classes of investment projects: (i) one-period cash flows including three different cases (fixed cost A_0 and variable income A_1 with normal, exponential, and uniform distributions), (ii) two-period cash flows including six different cases (fixed cost A_0, fixed income A_1, and variable income A_2 with normal, exponential, and uniform distributions; fixed cost A_0, variable income A_1 with normal, exponential, and uniform distributions, and fixed income A_2), and (iii) multiple-period cash flows with normal, exponential, and uniform distributions of incomes. The PDFs of the NPW were derived for each case separately. To validate the analytical PDFs derived, numerical examples were presented, and a set of comprehensive simulation analysis was conducted. The comparisons were performed between simulation results and analytical functions. The simulated results demonstrated that both analytical and simulated results of derived PDFs were completely in agreement with each other, which confirmed the validity of our analytical equations. The importance of such PDF functions is that the investor can simply calculate the probability of investment projects being economical and make more reliable decisions. In the case of deterministic economic evaluation problems, the investor has his/her own Minimum Attractive Rate of Return (MARR), which is used in project evaluations. In our stochastic problems, the probability of the project being economic for an uncertain cash flow was calculated. After calculating this parameter, the investor can compare this parameter with his/her minimum probability of the project being economical. If the minimum probability of the project being economical is satisfied, the project is selected as economical; otherwise, it is rejected. In other words, this paper presented a decision-making tool for the evaluation of investment projects with uncertain cash flows.

An interesting direction for future study is to consider other performance criteria like Net Future Worth (NFW), Net Equivalent Uniform Annual (NEUA), or even payback period, with other probability distributions like Weibull or Gamma.

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