Hartley-Ross Type Unbiased Estimators of Population Mean Using Two Auxiliary Variables

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Abstract: In survey sampling, it is well established phenomenon that the efficiency of estimators increases with the help of information of auxiliary variable(s). Keeping this fact in mind, we utilized the information of two auxiliary variables to propose a family of Hartley-Ross type unbiased estimators for estimating population mean under simple random sampling without replacement. Minimum variance of the new estimators is derived up to first degree of approximation. Three real data sets are used to verify that the new family acts efficiently than the usual unbiased, Hartley and Ross and other competing estimators.

Keywords: Auxiliary variable, Hartley-Ross type Estimator, Unbiased, Variance.

1. Introduction

In sample surveys, researchers normally used ratio type estimators when there is need to estimate the unknown population parameters of the study variable with the help of known population parameters of a correlated auxiliary variable(s). An eye view of literature, on the estimation of population mean under simple random sampling without replacement (SRSWOR), with the help of information on two auxiliary variables includes Abu-Dayyeh et al. [1], Kadilar and Cingi [2], Singh and Tailor [3], Lu and Yan [4], Lu et al. [5], Vishwakarma and Kumar [6], Sharma and Singh [7], Yasmeen et al. [8], Lu [9], Muneer et al. [10] and Shabbir and Gupta [11].

One eminent disadvantage of using ratio type estimators is that they are typically biased. Hartely and Ross [12] initiated a concept of unbiased estimators for estimating population mean. Similar efforts for unbiased estimators were carried out by Robson [13], Murthy and Nanjamma [14], Biradar and Singh [15-17], Sahoo et al. [18], Singh et al. [19], Cekim and Kadilar [20] and Khan et al. [21]. Under different sampling techniques they used the information of single auxiliary variable. Continuing these efforts, we proposed a new optimal family of Hartley-Ross type unbiased estimators with the novelty that information on two auxiliary variables is used.

Consider $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_N\}$ be $N$ units of a finite population. Let the $i^{th}$ unit of population, $y_i$ be the value of study variable $y$ and $x_i, z_i$ be the values of auxiliary variables $x$ and $z$ respectively. $n$ is the size of sample selected from the population under SRSWOR scheme given that $n < N$. Some usual measures related to study variable and auxiliary variables are presented in Table 1.

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Following relative error terms are used to derive the expressions for the bias, variance and minimum variance of the existing and suggested estimators.

\[
\begin{align*}
\varepsilon_0 &= \frac{y}{Y} - 1, \\
\varepsilon_1 &= \frac{x}{X} - 1, \\
\varepsilon_2 &= \frac{z}{Z} - 1, \\
\varepsilon_3 &= \frac{s_{yx}}{S_{yx}} - 1, \\
\varepsilon_4 &= \frac{X^{(j)}}{X} - 1, \quad \text{for } j = 1, 2, 3, \\
\varepsilon_5 &= \frac{r^{(k)}}{R} - 1, \quad \text{for } k = 0, 1, 2, 3,
\end{align*}
\]

(1)

Table 2 presents a detailed description of some terms used in expression (1).

**Remark 1.1.** \(s_{yx}, \overline{X}^{(j)}\) and \(r^{(k)}\) are the unbiased estimators of their population parameters \(S_{yx}, \overline{X}^{(j)}\) and \(\overline{R}^{(k)}\) respectively.

**Remark 1.2.** The expectations of the relative errors given in (1) are such that (see Singh et al. [19] and Cekim and Kadilar [20]).

\[
\begin{align*}
E(\varepsilon_i) &= 0 \quad \text{for } i = 0, 1, 2, 3, 4, 5, \\
E(\varepsilon_0^2) &= \psi C_y^2, \\
E(\varepsilon_1^2) &= \psi C_x^2, \\
E(\varepsilon_2^2) &= \psi C_z^2, \\
E(\varepsilon_3^2) &= \psi \left(\frac{\theta_{2x}}{\rho_{yx}} - 1\right), \\
E(\varepsilon_4^2) &= \psi C_{x(j)}^2, \\
E(\varepsilon_5^2) &= \psi C_{r(k)}^2, \\
E(\varepsilon_0 \varepsilon_1) &= \psi \rho_{yx} C_x C_y = \psi s_{yx}, \\
E(\varepsilon_0 \varepsilon_2) &= \psi \rho_{yx} C_y C_z = \psi C_{yz}, \\
E(\varepsilon_0 \varepsilon_3) &= \psi \left(\frac{C_x \theta_{21x}}{\rho_{yx}}\right), \\
E(\varepsilon_1 \varepsilon_2) &= \psi \rho_{x} C_x C_z = \psi C_{xz}, \\
E(\varepsilon_1 \varepsilon_3) &= \psi \left(\frac{C_x \theta_{12x}}{\rho_{yx}}\right), \\
E(\varepsilon_2 \varepsilon_3) &= \psi \left(\frac{C_x \theta_{12z}}{\rho_{yz}}\right)
\end{align*}
\]

where

\[
\psi = \left(\frac{1}{n} - \frac{1}{N}\right) \quad \text{finite population correction factor}
\]

\[
C_x^{(j)} = \left(\overline{X}^{(j)}\right)^{-1} S_{x(j)} \quad \text{coefficient of variation for transformed } x
\]

\[
C_r^{(k)} = \left(\overline{R}^{(k)}\right)^{-1} S_{r(k)} \quad \text{coefficient of variation for ratio of } y \text{ to the transformed } x
\]
\[ S_{x(j)} = \sqrt{(N-1)^{-1} \sum_{j=1}^{N} (x_j(i) - \bar{X}(j))^2} \] standard deviation of transformed \( x \)

\[ S_{j(i)} = \sqrt{(N-1)^{-1} \sum_{i=1}^{N} (x(i) - \bar{R}(k))^2} \] standard deviation of ratio of \( y \) to the transformed \( x \)

\[ S_{y(x)} = \sqrt{(N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(x(i) - \bar{X}(j))} \] covariance between \( y \) and transformed \( x \)

(For all above expressions \( j = 1, 2, 3 \) and \( k = 0, 1, 2, 3 \))

\[ \rho_{yx} = \left( S_y S_x \right)^{-1} S_{yx} \] correlation coefficient between \( y & x \), \( y & z \) and \( x & z \) respectively

\[ \rho_{yz} = \left( S_y S_z \right)^{-1} S_{yz} \]

\[ \rho_{xz} = \left( S_x S_z \right)^{-1} S_{xz} \]

\[ S_{yx} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) \]

\[ S_{yz} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(z_i - \bar{Z}) \] covariance between \( y & x \), \( y & z \) and \( x & z \) respectively

\[ S_{xz} = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})(z_i - \bar{Z}) \]

Values of \( \theta_{21x}, \theta_{12x}, \theta_{22x} \) and \( \theta_{12z} \) can be obtained with the help of following expressions.

\[ \theta_{pqx} = \left( \frac{p}{\mu_{pqx}} \right) \left( \frac{\mu_{20x}}{\mu_{02x}} \right)^{\frac{q}{2}} \]

\[ \mu_{pqx} = \frac{\sum_{i=1}^{N} (y_i - \bar{Y})^p (x_i - \bar{X})^q}{N} \] for \( p, q = 0, 1, 2 \)

\[ \theta_{pxz} = \left( \frac{p}{\mu_{pxz}} \right) \left( \frac{\mu_{20z}}{\mu_{02z}} \right)^{\frac{q}{2}} \]

\[ \mu_{pxz} = \frac{\sum_{i=1}^{N} (y_i - \bar{Y})^p (z_i - \bar{Z})^q}{N} \] for \( p, q = 0, 1, 2 \)

2. Review of Literature

This section presents some estimators from the literature when estimating population mean under simple random sampling.

2.1 The traditional unbiased estimator for unknown population mean along with its variance is

\[ \bar{y}_0 = y \] (2)
\[ \text{Var}\left(\bar{y}_{0}^{(u)}\right) = \psi \bar{Y}^2 C_y^2 \]  
(3)

2.2 Hartley and Ross [12] initiated an unbiased ratio type estimator as given below

\[ \bar{y}_{HR} = \bar{y} - \frac{n(N-1)}{N(n-1)} \left( \bar{y} - \bar{r} \right) \]  
(4)

where \( \bar{r} = n^{-1} \sum_{i=1}^{n} r_i^{(0)} \), \( r_i^{(0)} = \frac{Y_i}{x_i} \)

The variance of this estimator, to the first order of approximation, is equal to the mean square error of the usual ratio estimator (see Singh and Mangat [22]).

\[ \text{Var}\left(\bar{y}_{HR}^{(u)}\right) \approx \psi \bar{Y}^2 \left[ C_y^2 + C_x^2 - 2 \rho_{yx} C_y C_x \right] \]  
(5)

2.3 Grover and Kaur [23] followed the lines of Gupta and Shabbir [24] and Shabbir and Gupta [25] to suggest a generalized class based on ratio-type exponential estimators as follows

\[ \bar{y}_{GR} = \left[ q_1 \bar{y} + q_2 (\bar{X} - \bar{x}) \right] \exp \left[ \frac{\alpha (\bar{X} - \bar{x})}{\alpha (\bar{X} + \bar{x}) + 2 \beta} \right] \]  
(6)

Where \( q_1 \) and \( q_2 \) are the suitable weights to be chosen. \( \alpha (\neq 0) \) and \( \beta \) are either known quantities or functions of any known population parameters, including coefficient of variation, coefficient of skewness, coefficient of Kurtosis and coefficient of correlation etc.

Following are the optimal values of \( q_1 \) and \( q_2 \)

\[ q_{1(\text{opt})} = \frac{8 - \psi \lambda^2 C_x^2}{8 \left(1 + \psi C_y^2 (1 - \rho_{yx}^2)\right)} \]

and

\[ q_{2(\text{opt})} = \frac{\bar{Y} \left[ \psi \lambda^3 C_x^3 + 8C_y \rho_{yx} - \psi \lambda^2 C_x^2 C_s \rho_{yx} - 4 \lambda C_x \left\{1 - \psi C_y^2 (1 - \rho_{yx}^2)\right\}\right]}{8 \bar{X} C_x \left[1 + \psi C_y^2 (1 - \rho_{yx}^2)\right]} \]

where \( \lambda = \frac{\alpha \bar{X}}{\bar{X} + \beta} \)

We get the minimum MSE by utilizing the above optimal values. Following is the expression for minimum MSE, up to first degree of approximation.
\[
MSE_{\text{min}} \left( \bar{y}_{gk} \right) \approx \frac{\psi \bar{y}^2}{64 \left[ 2C_y^2 \left( 1 - \rho_x^2 \right) - \psi \lambda^2 C_x^4 - 16\psi \lambda^2 C_x C_y^2 \left( 1 - \rho_x^2 \right) \right]} \frac{64}{1 + \psi C_y^2 \left( 1 - \rho_x^2 \right)}
\]  

(7)

2.4 Singh et al. [19] suggested two Hartley-Ross type estimators by considering the estimators of Kadilar and Cingi [26] and Upadhyaya and Singh [27].

\[
\bar{y}_{s1} = r^{(1)} \bar{X} + \frac{n(N-1)}{N(n-1)} \left( \bar{y} - r^{(1)} x^{(1)} \right)
\]  

(8)

\[
\bar{y}_{s2} = r^{(2)} \bar{X} + \frac{n(N-1)}{N(n-1)} \left( \bar{y} - r^{(2)} x^{(2)} \right)
\]  

(9)

where \( r^{(1)} = \frac{1}{n} \sum_{i=1}^{n} r_i^{(1)} \), \( r_i^{(1)} = \frac{y_i}{C_x x_i + \rho_{yx}} = \frac{y_i}{x_i^{(1)}} \), \( \bar{X} = C_x \bar{X} + \rho_{yx} \)

\[
\bar{y}_{s2} = r^{(2)} \bar{X} + \frac{n(N-1)}{N(n-1)} \left( \bar{y} - r^{(2)} x^{(2)} \right)
\]

\[
\bar{X} = C_x \bar{X} + \beta_{2(x)}
\]

Here \( \rho_{yx} \) is the coefficient of correlation between study variable \( y \) and auxiliary variable \( x \) and \( \beta_{2(x)} \) is the coefficient of kurtosis of auxiliary variable \( x \).

Given below are the variances of \( \bar{y}_{s1} \) and \( \bar{y}_{s2} \)

\[
\text{Var} \left( \bar{y}_{s1} \right) \equiv \psi \left[ S_y^2 + \left( \bar{R}^{(1)} S_{x^{(1)}} \right)^2 - 2\bar{R}^{(1)} S_{yx^{(1)}} \right]
\]  

(10)

\[
\text{Var} \left( \bar{y}_{s2} \right) \equiv \psi \left[ S_y^2 + \left( \bar{R}^{(2)} S_{x^{(2)}} \right)^2 - 2\bar{R}^{(2)} S_{yx^{(2)}} \right]
\]  

(11)

Where \( \bar{R}^{(1)} \) and \( \bar{R}^{(2)} \) are defined in Table 2 and \( S_{x^{(1)}} \), \( S_{x^{(2)}} \), \( S_{yx^{(1)}} \) and \( S_{yx^{(2)}} \) are defined in Remark 1.2.

2.5 A special version of estimators of Khoshnevisan et al. [28] was utilized by Cekim and Kadilar [20] to form a general family of Hartley-Ross type unbiased estimators as below

\[
\bar{y}_{ck1} = r^{(3)} \bar{X} + \frac{n(N-1)}{N(n-1)} \left( \bar{y} - r^{(3)} x^{(3)} \right)
\]  

(12)

where \( r^{(3)} = n^{-1} \sum_{i=1}^{n} r_i^{(3)} \), \( r_i^{(3)} = \frac{y_i}{\alpha x_i + \beta x_i^{(3)}} = \frac{y_i}{x_i^{(3)}} \), \( \bar{X} = \alpha \bar{X} + \beta \)

\( \alpha \) and \( \beta \) are as explained earlier.

The variance of \( \bar{y}_{ck1} \) is given by

\[
\text{Var} \left( \bar{y}_{ck1} \right) \equiv \psi \left[ S_y^2 + \left( \bar{R}^{(3)} S_{x^{(3)}} \right)^2 - 2\bar{R}^{(3)} S_{yx^{(3)}} \right]
\]  

(13)

Where \( \bar{R}^{(3)} \) is defined in Table 2 and \( S_{x^{(3)}} \) and \( S_{yx^{(3)}} \) are defined in Remark 1.2.
Remark 2.1. Some important considerations can be noticed here. If we have

i) \( \alpha = C_x \) and \( \beta = \rho_{yx} \) in \( r_i^{(3)} \), then
\[
r_i^{(3)} = r_i^{(1)} \quad \text{and} \quad \bar{y}_{\bar{y}X} = \bar{y}_{S_1}
\]

ii) \( \alpha = C_x \) and \( \beta = \beta_{2\varphi} \) in \( r_i^{(3)} \), then
\[
r_i^{(3)} = r_i^{(2)} \quad \text{and} \quad \bar{y}_{\bar{y}X} = \bar{y}_{S_2}
\]

2.6 Another family of Hartley-Ross type unbiased estimators, from special version of Koyuncu and Kadilar [29] is proposed by Cekim and Kadilar [20] defined as

\[
\bar{y}_{\bar{y}X} = q_3 \int \frac{\alpha X + \beta}{\gamma (\alpha X + \beta) + (1 - \gamma)(\alpha X + \beta)} - q_3 \sqrt{\frac{t(t+1)}{2} \gamma^2 \lambda^2 C_x^2 - t\gamma \lambda \frac{s}{\gamma X}} - (q_3 - 1) \bar{y}
\]

(14)

Where \( q_3 \) is the suitable weight that minimize the variance, \( t = 1, \gamma = 1, \lambda = \frac{\alpha X}{\alpha X + \beta} \) and \( \alpha \) and \( \beta \) are as defined earlier.

\[
\text{Var}(\bar{y}_{\bar{y}X}) \approx \bar{Y} \sqrt{\Psi}\left[ t^2 \gamma^2 \lambda^2 \left[ C_{yx} \left\{ 1 - \frac{\psi \theta_{21x}}{\rho_{yx}} \right\} + \frac{(t+1)}{2} \psi \lambda C_x^2 C_y^2 \right] - \psi \left\{ q_3 t \gamma \lambda \left( \frac{t}{2} \gamma \lambda C^2_x - C_{yx} \right) \right\}^2 \right]
\]

(15)

The optimal value of \( q_3 \) is given as

\[
q_3^{(opt)} = \frac{\Psi}{\Delta}
\]

where

\[
\Psi = t\gamma \lambda \left( C_{yx} \left\{ 1 - \frac{\psi \theta_{21x}}{\rho_{yx}} \right\} + \frac{(t+1)}{2} \psi \lambda C_x^2 C_y^2 \right)
\]

\[
\Delta = t^2 \gamma^2 \lambda^2 \left[ C_x^2 + \psi \left( C_{yx} \left\{ (t+1) \gamma \lambda C^2_x - 2 \frac{C_x \theta_{21x}}{\rho_{yx}} \right\} - \frac{(t+1)}{2} \gamma \lambda C^2_x - C_{yx} \right) \right]^2
\]

We have the minimum variance by placing the optimal value of \( q_3 \) in (15) as below

\[
\text{Var}_{\min}(\bar{y}_{\bar{y}X}) \approx \bar{Y} \sqrt{\psi \left( C_y^2 - \frac{\Psi^2}{\Delta} \right)}
\]

(16)

2.7 Muneer et al. [10] followed the lines of Gupta and Shabbir [24] and Singh and Singh [30] to propose the following estimators. These estimators utilized the information of two auxiliary variables.
\[ \bar{y}_{MU,y} = \left[ q_4 \bar{y} + q_5 \left( \bar{X} - \bar{x} \right) \right] \left[ \gamma \left( 2 - \exp \left( \frac{\bar{z} - \bar{Z}}{\bar{z} + \bar{Z}} \right) \right) + (1 - \gamma) \exp \left( \frac{\bar{Z} - \bar{z}}{\bar{Z} + \bar{z}} \right) \right] \] (17)

By putting \( \gamma = 0 \) and 1, we obtained two estimators \( \bar{y}_{MU,0} \) and \( \bar{y}_{MU,1} \) respectively and \( q_4 \) and \( q_5 \) are the constants to be determined.

The optimal values of \( q_4 \) and \( q_5 \) are as follows:

\[ q_{4(\text{opt})} = \frac{1 + \left( \frac{3}{8} - \frac{\gamma}{4} \right) \psi C_z^2 - \frac{1}{2} \psi C_y^2 - \frac{\psi C_{xz} (C_{xz} - C_{yx})}{2C_x^2}}{1 + \psi C_y^2 + \left( 1 - \frac{\gamma}{2} \right) \psi C_z^2 - 2 \psi C_y^2 + \frac{\psi (C_{xz} - C_{yx})^2}{C_x^2}} \]

\[ q_{5(\text{opt})} = \frac{\bar{Y}}{\bar{X}} \frac{C_{xz}^2}{2C_x^2} - \frac{1 + \left( \frac{3}{8} - \frac{\gamma}{4} \right) \psi C_z^2 - \frac{1}{2} \psi C_y^2 - \frac{\psi C_{xz} (C_{xz} - C_{yx})}{2C_x^2}}{1 + \psi C_y^2 + \left( 1 - \frac{\gamma}{2} \right) \psi C_z^2 - 2 \psi C_y^2 + \frac{\psi (C_{xz} - C_{yx})^2}{C_x^2}} \left[ C_{xz} - C_{yx} \right] \]

Using \( q_{4(\text{opt})} \) and \( q_{5(\text{opt})} \), we get the \( MSE_{\text{min}} \left( \bar{y}_{MU,y} \right) \) as follows

\[ MSE_{\text{min}} \left( \bar{y}_{MU,y} \right) \cong \bar{Y}^2 \left[ \frac{1 - \psi C_z^2}{4C_x} - \left( 1 + \psi C_y^2 + \left( 1 - \frac{\gamma}{2} \right) \psi C_z^2 - 2 \psi C_y^2 + \frac{\psi (C_{xz} - C_{yx})^2}{C_x^2} \right) \right] \] (18)

2.8 Recently Shabbir and Gupta [11] used the linear combination of two auxiliary variables to propose a difference cum exponential ratio type estimator. This concept was initiated by Gupta and Shabbir [24] and Grover and Kaur [31].

\[ \bar{y}_{SG} = \left[ q_6 \bar{y} + q_7 \left( \bar{X} - \bar{x} \right) + q_8 \left( \bar{Z} - \bar{x} \right) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \] (19)

Where \( q_6, q_7 \) and \( q_8 \) are the feasible constants defined below

\[ q_{6(\text{opt})} = \frac{1 - \frac{1}{8} \psi C_y^2}{1 + \psi C_y^2 \left( 1 - R_{y,xz}^2 \right)} \]

\[ q_{7(\text{opt})} = \frac{1}{2} C_x \left( 1 - \rho_{xz}^2 \right) \left( \psi C_y^2 (1 - R_{y,xz}^2) - \frac{1}{4} \left( 1 - \frac{1}{8} \psi C_x^2 \right) \right) + C_y \left( \rho_{yx} - \rho_{yz} \rho_{xz} \right) \left( 1 - \frac{1}{8} \psi C_x^2 \right) \]

\[ q_{8(\text{opt})} = \frac{1}{C_x} \left( 1 - \rho_{xz}^2 \right) \frac{1 + \psi C_y^2 (1 - R_{y,xz}^2)}{1 + \psi C_y^2 \left( 1 - R_{y,xz}^2 \right)} \]
Proposed Family of Estimators

It is valuable to mention that the estimator $\bar{y}_{SG}$ has limited application due to the restricted transformation of auxiliary variable. In this section, we consider a general linear transformation of auxiliary information and use a special version of Shabbir and Gupta [11] estimator to present a family of Hartley-Ross type unbiased estimators. Minimum variance of the new class is also derived up to first order of approximation.

Using a general linear transformation of auxiliary information in the Shabbir and Gupta [11] estimator, we get

\[
\bar{y}_{SG,Gen} = \left[ q_0 \bar{y} + q_{10} (\bar{X} - \bar{x}) + q_{11} (\bar{Z} - \bar{z}) \right] \exp \left[ \frac{\alpha (\bar{X} - \bar{x})}{\alpha (\bar{X} + \bar{x}) + 2\beta} \right]
\]

Bias, up to first degree of approximation, of the generalized family presented in eq. (21) is obtained as

\[
Bias(\bar{y}_{SG,Gen}) \equiv (q_9 - 1) \bar{Y} + q_9 \psi \bar{Y} \left( \frac{3}{8} \lambda^2 C_x^2 - \frac{1}{2} \lambda C_x C_y \rho_{yx} \right) + \frac{1}{2} q_{10} \psi \lambda \bar{X} C_x^2 + \frac{1}{2} q_{11} \psi \lambda \bar{Z} C_z \rho_{xz}
\]

Subtracting $Bias(\bar{y}_{SG,Gen})$ from (21), we get

\[
\left[ q_0 \bar{y} + q_{10} (\bar{X} - \bar{x}) + q_{11} (\bar{Z} - \bar{z}) \right] \exp \left[ \frac{\alpha (\bar{X} - \bar{x})}{\alpha (\bar{X} + \bar{x}) + 2\beta} \right] - \left[ (q_9 - 1) \bar{Y} + q_9 \psi \bar{Y} \left( \frac{3}{8} \lambda^2 C_x^2 - \frac{1}{2} \lambda \frac{S_{yx}}{XY} \right) + \frac{1}{2} q_{10} \psi \lambda \bar{X} C_x^2 + \frac{1}{2} q_{11} \psi \lambda \bar{Z} C_z \rho_{xz} \right]
\]
After some simplifications and replacing the parameters $\bar{Y}$ and $S_{yx}$ by their unbiased estimators $\bar{y}$ and $s_{yx}$ in equation (22), we proposed a new family of Hartley-Ross type unbiased estimators as follows

$$
\bar{y}_p^{(u)} = [q_9 \bar{y} + q_{10} (\bar{X} - \bar{x}) + q_{11} (\bar{Z} - \bar{z})] \exp \left[ \frac{\alpha (\bar{X} - \bar{x})}{\alpha (\bar{X} + \bar{x}) + 2\beta} \right] + \bar{y} - \frac{1}{2} q_{10} \psi \lambda \bar{X} C_x^2 \\
- \frac{1}{2} q_{11} \psi \lambda C_{xz} \bar{Z} - q_9 \bar{y} - \frac{3}{8} q_9 \psi \lambda^2 \bar{y} C_x^2 + \frac{1}{2} q_9 \psi \lambda \frac{s_{yx}}{\bar{X}}
$$

(23)

Where $q_9$, $q_{10}$ and $q_{11}$ are the suitable weights to be chosen. $\alpha (\neq 0)$ and $\beta$ are either known constants or functions of any known population parameters, including coefficient of variation $C_x$ or $C_z$, coefficient of skewness $\beta_{(x)}$, coefficient of Kurtosis $\beta_{(x)}$ and coefficient of correlation $\rho_{yx}$ or $\rho_{xy}$ etc. and $s_{yx}$ is an unbiased estimator of $S_{yx}$.

To express the equation (23) in terms of $\varepsilon$'s, we use the relative error terms defined in eq. (1). Expression is expanded up to first order of approximation.

$$
\bar{y}_p^{(u)} \cong \left[ q_9 \bar{y} (1 + \varepsilon_0) - q_{10} \bar{X} \varepsilon_1 - q_{11} \bar{Z} \varepsilon_2 \right] \left[ 1 - \frac{1}{2} \lambda \varepsilon_1 + \frac{3}{8} \lambda^2 \varepsilon_1^2 \right] + \bar{y} (1 + \varepsilon_0) - \frac{1}{2} q_{10} \psi \lambda X \bar{X} C_x^2 \\
- \frac{1}{2} q_{11} \psi \lambda C_{xz} \bar{Z} - q_9 \bar{y} (1 + \varepsilon_0) - \frac{3}{8} q_9 \psi \lambda X^2 C_x^2 (1 + \varepsilon_0) + \frac{1}{2} q_9 \psi \lambda \frac{S_{yx}}{\bar{X}} (1 + \varepsilon_3)
$$

Solving above, we have

$$
\left( \bar{y}_p^{(u)} - \bar{Y} \right) \cong \bar{Y} \left( 1 - \frac{3}{8} q_9 \psi \lambda^2 C_x^2 \right) \varepsilon_0 - \left( \frac{1}{2} q_9 \bar{Y} \lambda + q_{10} \bar{X} \right) \varepsilon_1 - q_{11} \bar{Z} \varepsilon_2 + \frac{1}{2} q_9 \psi \lambda \frac{S_{yx}}{\bar{X}} \varepsilon_3 \\
- \frac{1}{2} q_9 \bar{Y} \lambda \varepsilon_1 \varepsilon_1 + \frac{1}{2} q_{11} \lambda \bar{Z} \varepsilon_1 \varepsilon_2 + \left( \frac{3}{8} q_9 \bar{Y} \lambda^2 \varepsilon_1^2 + \frac{1}{2} q_{10} \bar{X} \lambda \right) \varepsilon_1^2 \\
- \frac{1}{2} \psi \lambda \left( q_{10} \bar{X} C_x^2 + q_{11} C_{xz} \bar{Z} + \frac{3}{4} q_9 \bar{Y} \lambda C_x^2 + q_9 \bar{Y} C_{xy} \right)
$$

(24)

We get approximately zero bias by taking expectation on both sides of (24). It indicates that proposed class generates Hartley-Ross type unbiased estimators.

$$
Bias \left( \bar{y}_p^{(u)} \right) = E \left( \bar{y}_p^{(u)} - \bar{Y} \right) \cong 0
$$

To obtain the variance of proposed estimators up to first degree of approximation, squaring both sides of (24) and taking the expectation
\[
\text{Var} \left( \bar{y}^{(i)}_p \right) \equiv \bar{Y}^2 \psi C_{x}^2 - \bar{Y}^2 \psi \lambda V_9 - 2 \bar{X} \bar{Y} \psi C_{x_9} q_{10} - 2 \bar{Y} \bar{Z} \psi C_{x_9} q_{11} + \bar{Y}^2 \lambda^2 \psi V_9 q_9^2 + \bar{X}^2 \psi C_{x}^2 V_9 q_{10}^2 \\
\quad + \bar{Z}^2 \psi C_{x}^2 V_9 q_{11}^2 + \bar{X} \bar{Y} \psi \lambda C_{x_9} V_9 q_{10} + \bar{Y} \bar{Z} \psi \lambda V_9 q_{11} + 2 \bar{X} \bar{Z} \psi C_{x_9} V_9 q_{10} q_{11}
\]  

(25)

where

\[
V_1 = \frac{3}{4} \psi \lambda C_{x_9}^2 + C_{x_9} - \frac{\psi C_{x_9} C_{x_9} \theta_{1x}}{\rho_{x_9}}
\]

\[
V_2 = \frac{1}{4} C_{x_9}^2 - \frac{9}{64} \psi \lambda^2 C_{x_9}^2 - \frac{1}{4} \psi C_{x_9}^2 - \frac{\psi C_{x_9} C_{x_9} \theta_{2x}}{2 \rho_{x_9}} + \frac{3}{4} \psi \lambda^2 C_{x_9}^2
\]

\[
V_3 = 1 - \frac{1}{4} \psi \lambda^2 C_{x}^2
\]

\[
V_4 = 1 - \frac{1}{4} \psi \lambda^2 C_{x}^2 \rho_{x_9}^2
\]

\[
V_5 = C_{x_9} + \frac{5}{4} \psi \lambda C_{x_9} - \frac{\psi C_{x_9} C_{x_9} \theta_{12x}}{\rho_{x_9}} - \frac{3}{8} \psi \lambda^2 C_{x_9}^2
\]

\[
V_6 = \frac{3}{4} \psi \lambda^2 C_{x_9}^2 + C_{x_9} - \frac{\psi C_{x_9} C_{x_9} \theta_{12x}}{\rho_{x_9}} + \frac{1}{2} \psi \lambda C_{x_9} C_{x_9} - \frac{3}{8} \psi \lambda^2 C_{x_9}^2
\]

In order to get the optimal values of $$q_i$$'s, we differentiate eq. (25) with respect to $$q_i$$, $$i = 9, 10, 11$$ and then equate them to zero. So, we get

\[
q_{9(\text{opt})} = \frac{2B_1}{\lambda B_2}
\]

\[
q_{10(\text{opt})} = \frac{\bar{Y}}{X} \left[ V_1 A_2 B_2 - 4V_2 A_2 B_1 - V_6 (A_2 B_2 - A_6 B_1) \right]
\]

\[
q_{11(\text{opt})} = \frac{\bar{Y}}{Z} \left[ A_6 B_2 - A_6 B_1 \right]
\]

We obtained the minimum variance at optimal values of $$q_i, i = 9, 10, 11$$ by inserting them in (25)

\[
\text{Var}_{\text{min}} \left( \bar{y}^{(i)}_p \right) \equiv \frac{1}{C_{x_9} V_9^2 A_2 B_2} \bar{Y}^2 \psi \left[ C_{x_9} V_9^2 A_2 \left\{ C_{x_9}^2 B_2^2 - 2V_1 B_1 B_2 + 4V_1 B_1^2 \right\} \\
+ C_1 \left\{ C_{x_9} V_9 B_1 - 2C_{x_9} V_9 A_2 B_2 + 2C_{x_9} V_9^2 A_2 B_1 \right\} \\
+ C_{x_9}^2 V_9^2 \left\{ C_{x_9}^2 V_9 C_2 - 2C_{x_9} A_2 B_2 + 2V_6 A_2 B_1 \right\} + 2C_{x_9} V_9 V_9 C_1 C_2 \right]
\]  

(26)

where

\[
A_1 = C_{x_9} V_9 - C_{x_9} V_5, \quad A_2 = C_{x_9} V_9 V_6 - C_{x_9}^2 C_{x_9} V_5, \quad A_3 = C_{x_9} V_9 - C_{x_9} C_{x_9} V_5
\]

\[
A_4 = C_{x_9} V_9 V_6 - C_{x_9} V_9 V_5, \quad A_5 = C_{x_9} \left( 4V_2 V_9 - V_5^2 \right), \quad A_6 = 4C_{x_9}^2 V_9 - C_{x_9} V_9 V_6
\]

\[
B_1 = A_4 A_2 - A_3 A_4, \quad B_2 = A_2 A_3 - A_4 A_6
\]

\[
C_1 = V_1 A_2 B_2 - 4V_2 A_2 B_1 - V_6 (A_2 B_2 - A_6 B_1), \quad C_2 = (A_1 B_2 - A_6 B_1)
\]
4. Empirical Illustration

In this section, three natural populations are used to present the empirical performance of suggested almost unbiased estimators as compared to other estimators. Table 3 contains helpful information regarding data sets. We calculated percent relative efficiency (PRE) through following expression.

\[
PRE = \frac{MSE\left(\bar{y}_0^{(u)}\right)}{MSE\left(\bullet\right)} \times 100
\]

Where \(\bullet = \bar{y}_0^{(u)} , \bar{y}_{HR}^{(u)} , \bar{y}_{GK}^{(u)} , \bar{y}_{S1}^{(u)} , \bar{y}_{S2}^{(u)} , \bar{y}_{CK1}^{(u)} , \bar{y}_{CK2}^{(u)} , \bar{y}_{MU,0}^{(u)} , \bar{y}_{MU,1}^{(u)} , \bar{y}_{SG}^{(u)} , \bar{y}_P^{(u)}\)

We calculated the PRE’s of all the estimators taken from the literature as well as for the proposed family of estimators with respect to \(\bar{y}_0^{(u)}\). Empirical findings are reported in Tables 4, 5 and 6 for populations 1, 2 and 3 respectively. From Tables 4, 5 and 6 following observations are made:

- A comparison between all unbiased/Hartley-Ross type unbiased estimators i.e., \(\bar{y}_0^{(u)} , \bar{y}_{HR}^{(u)} , \bar{y}_{GK}^{(u)} , \bar{y}_{S1}^{(u)} , \bar{y}_{S2}^{(u)} , \bar{y}_{CK1}^{(u)} , \bar{y}_{CK2}^{(u)}\) and \(\bar{y}_P^{(u)}\) reveals that \(\bar{y}_P^{(u)}\) provide maximum gain in PREs as compared to others. This gain in PREs based on the fact that proposed estimators used the information of two auxiliary variables while the others use the information of only one auxiliary variable.
- Proposed family of estimators also provide greater PREs as compared to well-known Grover and Kaur [23] class of biased estimators i.e., \(\bar{y}_{GK}\). This class considers the information of only one auxiliary variable.
- When we contrast the proposed estimators \(\bar{y}_P^{(u)}\) against the biased estimators using the auxiliary information of two variables i.e., \(\bar{y}_{MU,0}, \bar{y}_{MU,1}, \bar{y}_{SG}\). We again observe the gain in PREs of the suggested estimators.
- Finally, it is quite obvious that proposed family gives maximum PREs as compared to all other estimators under study.

Hence, the suggested family outperforms and provides almost unbiased and efficient estimators for estimating population mean in case of SRSWOR.

5. Concluding Remarks

In this article, a new family of Hartley-Ross type unbiased estimators for estimating population mean under SRSWOR is suggested. In this family, we utilized the information of two auxiliary variables to generate new estimators. We derived the mathematical expressions for the Bias and the minimum variance of new family up to first order of approximation. To evaluate the potentiality of new family three natural populations are used. Numerical findings confirm the efficiency of new estimators as compared to the i) unbiased/ Hartley-Ross type unbiased estimators for instance, traditional estimator, Hartley and Ross [12], Singh et al. [19] and Cekim and Kadilar [20] ii) biased estimators using the information of one or two auxiliary variable(s)
like Grover and Kaur [23], Muneer et al. [10] and Shabbir and Gupta [11]. On the basis of these findings, it can be recommended that new family may be used for future applications.

Acknowledgments

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References


**Table Captions**

Table 1 Measures associated with study variable (y) and auxiliary variables (x & z)
Table 2 Measures used in relative error terms
Table 3 Data Statistics
Table 4 PRE’s of different estimators for Population 1
Table 5 PRE’s of different estimators for Population 2
Table 6 PRE’s of different estimators for Population 3
Table 1

<table>
<thead>
<tr>
<th>Measure</th>
<th>Study Variable y</th>
<th>Auxiliary Variable x</th>
<th>Auxiliary Variable z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population Mean</td>
<td>( \bar{Y} = N^{-1} \sum_{i=1}^{N} y_i )</td>
<td>( \bar{X} = N^{-1} \sum_{i=1}^{N} x_i )</td>
<td>( \bar{Z} = N^{-1} \sum_{i=1}^{N} z_i )</td>
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<tr>
<td>Sample Mean</td>
<td>( \bar{y} = n^{-1} \sum_{i=1}^{n} y_i )</td>
<td>( \bar{x} = n^{-1} \sum_{i=1}^{n} x_i )</td>
<td>( \bar{z} = n^{-1} \sum_{i=1}^{n} z_i )</td>
</tr>
<tr>
<td>Population Variance</td>
<td>( S_y^2 = \frac{\sum_{i=1}^{N} (y_i - \bar{Y})^2}{(N-1)} )</td>
<td>( S_x^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{X})^2}{(N-1)} )</td>
<td>( S_z^2 = \frac{\sum_{i=1}^{N} (z_i - \bar{Z})^2}{(N-1)} )</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
<td>( C_y = \bar{Y}^{-1} S_y )</td>
<td>( C_x = \bar{X}^{-1} S_x )</td>
<td>( C_z = \bar{Z}^{-1} S_z )</td>
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Table 2

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<thead>
<tr>
<th>Description</th>
<th>Sample</th>
<th>Population</th>
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<tr>
<td>Covariance between ( y ) and ( x )</td>
<td>( s_{yx} = (n-1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) )</td>
<td>( S_{yx} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) )</td>
</tr>
<tr>
<td>Mean of transformed ( x )</td>
<td>( \bar{x}^{(1)} = C_x \bar{x} + \rho_{yx} )</td>
<td>( \bar{x}^{(1)} = C_x \bar{X} + \rho_{yx} )</td>
</tr>
<tr>
<td></td>
<td>( \bar{x}^{(2)} = C_x \bar{x} + \beta_{2(x)} )</td>
<td>( \bar{x}^{(2)} = C_x \bar{X} + \beta_{2(x)} )</td>
</tr>
<tr>
<td></td>
<td>( \bar{x}^{(3)} = \alpha \bar{x} + \beta )</td>
<td>( \bar{x}^{(3)} = \alpha \bar{X} + \beta )</td>
</tr>
<tr>
<td>Ratio of ( y ) to the transformed ( x )</td>
<td>( r_i^{(0)} = \frac{y_i}{x_i} ), ( i = 1, 2, \ldots, n )</td>
<td>( R_i^{(0)} = \frac{y_i}{x_i} ), ( i = 1, 2, \ldots, N )</td>
</tr>
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<td>( r_i^{(1)} = \frac{y_i}{C_x x_i + \rho_{yx}} ), ( i = 1, 2, \ldots, n )</td>
<td>( R_i^{(1)} = \frac{y_i}{C_x x_i + \rho_{yx}} ), ( i = 1, 2, \ldots, N )</td>
</tr>
<tr>
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<td>( r_i^{(2)} = \frac{y_i}{C_x x_i + \beta_{2(x)}} ), ( i = 1, 2, \ldots, n )</td>
<td>( R_i^{(2)} = \frac{y_i}{C_x x_i + \beta_{2(x)}} ), ( i = 1, 2, \ldots, N )</td>
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<td>( r_i^{(3)} = \frac{y_i}{\alpha x_i + \beta} ), ( i = 1, 2, \ldots, n )</td>
<td>( R_i^{(3)} = \frac{y_i}{\alpha x_i + \beta} ), ( i = 1, 2, \ldots, N )</td>
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<tr>
<td>Mean of the ratio of ( y ) to transformed ( x )</td>
<td>( \bar{r}^{(k)} = n^{-1} \sum_{i=1}^{n} r_i^{(k)} ) for ( k = 0, 1, 2, 3 )</td>
<td>( \bar{R}^{(k)} = N^{-1} \sum_{i=1}^{N} r_i^{(k)} ) for ( k = 0, 1, 2, 3 )</td>
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Table 3

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<tr>
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<th>Marmara Region</th>
<th>Singh and Mangat [22], Page 369</th>
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</thead>
<tbody>
<tr>
<td>Study variable $y$</td>
<td>Number of Teachers</td>
<td>Number of Teachers</td>
<td>Number of Tube wells</td>
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<tr>
<td>Auxiliary variable $x$</td>
<td>Number of classes</td>
<td>Number of classes</td>
<td>Number of tractors</td>
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<td>Auxiliary variable $z$</td>
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<td>Net irrigated area</td>
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Bold values indicate maximum PREs
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<th>Estimator</th>
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<tr>
<td>( \bar{y}_0^{(a)} )</td>
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<td>( \alpha )</td>
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Bold values indicate maximum PREs
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<th>Families of Estimators</th>
</tr>
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<tr>
<td>(\tilde{y}_0^{(a)})</td>
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<td>(\alpha) (\beta) (\tilde{y}<em>{CK1}) (\tilde{y}</em>{CK2}) (\tilde{y}_{GK}) (\tilde{y}_P^{(a)})</td>
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<tr>
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</tr>
</tbody>
</table>

Bold values indicate maximum PREs
Authors’ Biographies

Maria Javed is currently a Ph.D. scholar in Statistics at Zhejiang University, Hangzhou, China. She received her M.Phil. degree in Statistics from Government College University, Faisalabad, Pakistan. She has been working as Lecturer in the Department of Statistics, Government College University, Faisalabad, Pakistan since 2004. Her research interests include Sampling Theory and Probability Distributions.

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