An Analytical Approach to Estimate Optimum River Channel Dimensions

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Abstract:

Extremal hypotheses without bank stability constraint typically over-predict and under-predict channel width in large rivers and natural streams, respectively. In general, results obtained from unconstrained extremal hypotheses are indicative of inappropriate agreement between predicted and observed dimensions of the rivers. One of the important factors in disparity of the data may be lack of appropriate relationships to assess bank vegetation of the rivers. For this reason, a modified analytical model has been developed to reduce the effect of bias by considering bank stability and vegetation. The model takes into account channel shape factor, a wide range of bed load equations in the form excess shear stress and vegetation quantification is able to predict optimal channel geometry dimensions. Finally, developed model was calibrated using the field data of the United Kingdom and Iran. In addition to indicating the effect of bank stability and vegetation on estimation of the geometric characteristics of the channel, obtained results also confirmed the efficiency of the constrained model in comparison to the unconstrained model. This study also provides support for the use of the concepts of maximum sediment transporting capacity and minimum stream power for understanding the operation of alluvial rivers.

Keywords: extremal hypotheses, bank vegetation, analytical model.
1. Introduction

In the advent of new millennium, a great deal of advancement in engineering science has been achieved. However, the problem of alluvial river response to natural and man induced environmental changes is still waiting for a simple, reasonable and understanding solution. In general, there are two principle approaches of analytical and empirical to determine stable channel geometry in alluvial rivers among which the application could have advantages and disadvantages [1]. Among analytical models in the last decades, extremal hypotheses have been used as objective functions to estimate optimum channel dimensions. By this means, sediment transport and flow resistance equations are applied, together with, a third equation to predict stable channel dimensions. This equation plays the role of an objective function for optimization by which it is expressed in terms of stream power, energy dissipation rate, sediment transport, etc. [2]. In early 1960, researchers like Leopold and Langbein [3] have applied minimum variance theory to design stable channel dimensions. Pickup [4] and Kirkby [5] suggested maximum sediment transporting capacity (MSTC). Yang [6] stated minimum energy dissipation rate (MEDR) as a general rule in hydraulic. Although MEDR hypothesis has made remarkable progress to determine velocity and sediment concentration, Yang et al. [7] indicated that MEDR hypothesis results in a width-depth ratio of 2 that is hardly encountered in natural rivers. Subsequently, Song and Yang [8] suggested minimum unit stream power hypothesis (MUSP) on the basis of which a self-formed stable channel may adjust its
cross shape and dimensions in such a way that transport of water and sediment takes place at optimum condition. Huang and Nanson [9] also emphasize that the stable condition of adjustable alluvial rivers is equivalent to the condition at which the maximum sediment load is transported. In their opinion, due to insufficient basic flow relationships (i.e. continuity, resistance and sediment) to express equilibrium state, application of extremal hypothesis is suggested. Based on their research, since the number of unknown are more than the number of equations, this may be resolved by defining channel shape factor (width/depth). They calculated optimum shape factor by assuming rectangular cross-section and solving basic flow equations, together with, extremal hypothesis simultaneously. They derived hydraulic geometry relationships analytically similar to others. They also defined maximum flow efficiency (MFE) as the flow at which maximum sediment transporting capacity occurs per unit available stream power.h

Further, on, Eaton and Millar [10] applied MTC hypothesis with bank stability constraint by considering trapezoidal shape for the cross-section under Parker’s bed load [11] and Manning’s flow resistance equations [12]. They showed that channel geometry depends upon bank material and vegetation. Investigations state that results obtained by applying extremal hypotheses without bank stability constraint (unconstrained model) are indicative of a relative disagreement between observed and predicted channel geometry, particularly, in large rivers [13]. For this reason, an analytical model is developed to accomplish unconstrained model in which effect of bank stability and vegetation has been used to estimate channel dimensions of gravel bed rivers. The model is able to predict optimum channel dimensions under static and dynamic equilibrium by considering shape factor in trapezoidal form, together with, a wide range of flow resistance and sediment transport equations as a function of excess
bed shear stress (According to Knighton and Lane [14,15], when the ratio of river design discharge to the threshold discharge of the sediment motion is about 1 or smaller, or in other words, the entrainment of sediments from the river bed and banks is zero (flow can carry sediments but cannot erode the river boundaries), the river is considered in the static stability condition. However, when the ratio is larger than 1 (sediments are transported from bed and banks, but the erosion and sedimentation rates are almost the same), the river is considered in the dynamic stability condition). In general, this model is meant to follow Huang and Nanson [9] and couples up with Eaton and Millar [10]. The aim of the presented model in this study was to design dimensions of a stable section carrying a dominant discharge equal to the discharge in the actual river section with the same parameters such as the Manning’s roughness coefficient, longitudinal slope, bank vegetation and bed and bank materials. In other words, the model is meant to be a practical tool for river restoration and engineering applications, e.g. river dredging. Attempts have been made to design a section that would: 1) have stable characteristics, and 2) pass a discharge equal to the river dominant discharge.

Material and methods

Selection of dependent and independent variables is one of the most important problems suggested in analytical models. The suggested model may be capable for both constant and variable slopes. For constant slope, model input data are dominant discharge, longitudinal slope, bed and bank material size, friction angle of bank material and roughness coefficient. Whereas in variable slope case, slope is replaced by sediment transport rate in input data to work out optimum slope. River cross section is considered trapezoidal with constant side slope of 1 (horizontal): z (vertical).
In Fig. 1, \( P_{\text{bed}} \), \( P_{\text{bank}} \), \( W \), \( D \), \( z \) and \( \phi' \) are bed perimeter, bank perimeter, channel surface width, maximum channel depth, bank slope (\( \theta \) is bank angle) and bank friction angle, respectively. According to Eaton and Millar and Lane, in natural rivers, the angle between banks of the channel and floodplain surface can be considered as \( \phi' \) [10,15]. Bed and banks may be easily distinguishable by Parker's definition, which is followed by proceeding from the floodplain margins to the center of the river channel and getting to the first point where the depth of 0.99\( D_{\text{max}} \), the river bank area is determined [17].

1.1. Model assumptions

- Flow is considered steady and uniform;
- Model is applicable in straight reach of the river through which sediment is mainly transported as bedload;
- Bed and bank are mainly composed of coarse grain sized materials, hence resistance due to form roughness is negligible;
- River cross-section is in trapezoidal form;
- Stability is achieved by satisfying extremal hypotheses;
- Dominant discharge is considered to be estimated by bankfull discharge.
1.2. Governing flow equations

There are six equations to develop channel geometry as continuity, resistance, sediment transport, mean bank shear stress, mean bed shear stress and bank stability. Whereas, seven dependent variables are available as bed perimeter ($P_{\text{bed}}$), maximum channel depth ($D$), longitudinal slope ($S$) (in constant slope state) or bed load discharge ($Q_s$) (in variable slope state), mean velocity ($V$), mean bank shear stress ($\tau_{\text{bank}}$), mean bed shear stress ($\tau_{\text{bed}}$) and bank angle ($\theta$). To estimate these variables, it is required to have seven equations as well. This has made the author to use extremal hypothesis as the seventh equation. The equations are defined as follows:

Flow continuity to be maintained in alluvial channels is:

$$Q = A \times V$$  \hspace{1cm} (1)

In which $A$, $V$ and $Q$ are cross-sectional area, average flow velocity, and dominant or bankfull flow discharge, respectively.

In the model, a generalized form of flow resistance based on Huang and Nanson [9] with some modifications may be expressed as:

$$V = c_r R^x S^y D^\alpha$$  \hspace{1cm} (2)

In which $c_r$ is a coefficient determined by sediment size, $x$, $y$ and $\alpha$ are functions of channel bed forms or flow regimes and $R$ is hydraulic radius. Many resistance equations, such as Manning [12], Lacey [18], Brownlie [19] etc., may be the same as above.

To avoid a particular equation of bed load in the model, a general form of the bed load equation, in the form of excess bed shear stress, was initially suggested by Huang and Nanson [9]. Although there are bed load equations with no sediment threshold of motion [20], hydraulic engineers emphasize on the determination of sediment threshold of motion. As this is the state at which sediment particles are in equilibrium, therefore, formulation of this will help solving problems such
as non-erodible stable channel design, riprap size design to protect bed and banks of the channels and calculation of sediment transport in rivers [21].

$$q_s = c_s \tau_{\text{bed}}^m (\tau_{\text{bed}} - \tau_c)^j$$

(3)

Where \(q_s\), \(c_s\), \(\tau_{\text{bed}}\), and \(\tau_c\) are bed load discharge per unit channel width, a constant relating to sediment characteristics, mean bed shear stress and critical shear stress for the incipient motion of sediments, respectively. Exponents \(m\) and \(j\) vary widely, as shown in many bed load transport models. Huang and Nanson [9] indicated that the above equation can easily be in the form of Meyer-Peter and Muller [22], Duboy [23], Parker [24] etc.

In this paper Flintham and Carling [25] relations for trapezoidal section, which were initially derived by Knight [26] and Knight et al. [27], are used to estimate boundary shear stress distribution. The proportion of the shear force acting on the bank (\(SF_{\text{bank}}\)), and the mean bank and bed shear stress values (\(\tau_{\text{bank}}\) and \(\tau_{\text{bed}}\), respectively) are estimated using the following equations:

$$\log \%SF_{\text{bank}} = -1.4026 \log(\frac{P_{\text{bed}}}{P_{\text{bank}}} + 1.5) + 2.247$$

(4)

$$\frac{\tau_{\text{bank}}}{\gamma DS} = 0.01\%SF_{\text{bank}} \left[ \frac{(W + P_{\text{bed}})\sin \theta}{4D} \right]$$

(5)

$$\frac{\tau_{\text{bed}}}{\gamma DS} = (1 - 0.01\%SF_{\text{bank}}) \left[ \frac{W}{2P_{\text{bed}}} + 0.5 \right]$$

(6)

Where \(\gamma\) is unit weight of water. The rest of the parameters are the same as before. It is necessary to notice that in trapezoidal channel \(P_{\text{bank}} = 2D\sqrt{1 + z^2}\).

The stability of the bank is assessed by comparing the value \(\tau_{\text{bank}}\) calculated from Equation 5 with a modified USBR bank stability criterion (after Eaton and Millar [10]), based on the bank sediment calibre (\(D_{50\text{bank}}\)) and bank friction angle (\(\phi'\)):
The coefficient \( c \) is dependent upon the properties of the unconsolidated, non-cohesive sediment where bank strength is unmodified by bank vegetation. The coefficient is defined as [28]:

\[
c = c^*_c / \tan \phi
\]  

(8)

In which \( c^*_c \) is the critical dimensionless shear stress for bed material of the same caliber, and \( \phi \) is the angle of repose.

The value of \( \phi \) varies with grain size and shape [28], ranging from a minimum of about 25º for fine sand to about 40º for sub-rounded gravel. Setting \( c^*_c \approx 0.04 \), \( c \) will thus vary between about 0.086–0.048. In this paper, we use \( c = 0.048 \) for natural gravel rivers. The value of \( \phi' \) ranges from a lower bound of \( \phi' = \phi \), where the bank sediment is unaffected by bank vegetation or interstitial cohesive sediment, up to a maximum value approaching 90º, which corresponds to a non-erodible bank [10].

Here in this research, MSTC and MSP hypotheses were combined as the seventh equation in accordance with an alluvial river adjusts its slope and geometry to maximize sediment transport.

### 1.3. Theory

As it was mentioned, the model is meant to follow Huang and Nanson [9] and couples up with Eaton and Millar [10].

At first, a non-dimensional channel shape factor \( \zeta \) is defined as:

\[
\zeta = P_{bed} / D
\]

(9)

By which trapezoidal cross-section geometric parameters are expressed as:

\[
A = D^2 (\zeta + z)
\]

(10)
\[ P = P_{\text{bed}} + P_{\text{bank}} = D(\zeta + 2\sqrt{1+z^2}) \]  
\[ R = \frac{A}{P} = \frac{(\zeta + z)}{(\zeta + 2\sqrt{1+z^2})}D \]  
\[ W = P_{\text{bed}} + 2zD = D(\zeta + 2z) \]  

To determine a relationship between depth with \( \zeta \), \( Q \), \( z \) and \( S \), it is sufficient to substitute \( Q/A \) from Equation 1 for velocity in Equation 2 and incorporate into Equations 10 and 11:

\[ D = \frac{(\zeta + 2\sqrt{1+z^2})^{(1/x + 2\alpha)}(Q/c_j)^{(1/x + 2\alpha)}}{(\zeta + z)^{(x+1)/(x + 2\alpha)}S^{1/(x + 2\alpha)}} \]  

By incorporating Equations 14 with Equations 5, 6 and 9-12, relationships are derived to estimate \( P_{\text{bed}} \), \( V \), \( \tau_{\text{bed}} \) and \( \tau_{\text{bank}} \) in terms of \( \zeta \), \( Q \), \( z \), \( S \) and \( c_j \):

\[ P_{\text{bed}} = \zeta D = \frac{\zeta(\zeta + 2\sqrt{1+z^2})^{(1/x + 2\alpha)}(Q/c_j)^{(1/x + 2\alpha)}}{(\zeta + z)^{(x+1)/(x + 2\alpha)}S^{1/(x + 2\alpha)}} \]  
\[ V = Q/A = \frac{(S^2)Q^{(1-xa)}c_j^{(1-xa)}(\zeta + z)^{(x-a)/(x + 2\alpha)}}{(\zeta + 2\sqrt{1+z^2})^{(2\alpha)/(x + 2\alpha)}} \]  
\[ \tau_{\text{bed}} = \gamma DS (1 - 0.01\%SF_{\text{bank}})(\frac{\zeta + z}{\zeta}) \]  
\[ \tau_{\text{bank}} = \gamma DS \times 0.01\%SF_{\text{bank}} \frac{(\zeta + z)\sin\theta}{2} \]  

Value of \( D \) can be replaced in Equations 17 and 18 from Equation 14. \( \%SF_{\text{bank}} \) could be estimated in terms of \( \zeta \) and \( z \) as:

\[ \%SF_{\text{bank}} = 10^{2.247} \left( \frac{\zeta}{2\sqrt{1+z^2}} + 1.5 \right)^{-1.4026} \]
Huang and Nanson [9] suggested that channel slope $S$ is regarded as a function of channel shape factor $\zeta$, therefore, $P_{\text{bed}} = P_{\text{bed}}(\zeta, S(\zeta))$, $D = D(\zeta, S(\zeta))$, $\tau_{\text{bed}} = \tau_{\text{bed}}(\zeta, S(\zeta))$. Hence, this leads to:

$$\frac{dP_{\text{bed}}}{d\zeta} = \frac{\partial P_{\text{bed}}}{\partial \zeta} + \frac{\partial P_{\text{bed}}}{\partial S} \frac{dS}{d\zeta}$$  \hspace{1cm} (20a)$$

$$\frac{d\tau_{\text{bed}}}{d\zeta} = \frac{\partial \tau_{\text{bed}}}{\partial \zeta} + \frac{\partial \tau_{\text{bed}}}{\partial S} \frac{dS}{d\zeta}$$  \hspace{1cm} (20b)

Where Equations 15 and 17 give:

$$\frac{\partial P_{\text{bed}}}{\partial \zeta} = \left[ \frac{(1+x+\alpha)\zeta + 2(1+\alpha)\sqrt{1+z^2}}{(\zeta + 2\sqrt{1+z^2})(\zeta + z)(2+x+\alpha)} + \frac{(2+2\alpha)z + (4+2\alpha)z\sqrt{1+z^2}}{(\zeta + 2\sqrt{1+z^2})(\zeta + z)(2+x+\alpha)} \right] P_{\text{bed}}$$  \hspace{1cm} (21a)

$$\frac{\partial P_{\text{bed}}}{\partial S} = \frac{-y}{(2+x+\alpha)S} P_{\text{bed}}$$

$$\frac{\partial \tau_{\text{bed}}}{\partial \zeta} = \left[ \frac{-\zeta - 2(1+x)\sqrt{1+z^2} - (2+\alpha)\zeta - 2(2+x+\alpha)\sqrt{1+z^2}}{(\zeta + 2\sqrt{1+z^2})(\zeta + z)(2+x+\alpha)} + \frac{0.01 \times 10^{2.37} \times 1.4026(1-0.01\%SF_{\text{bank}})^{-1}}{2\sqrt{1+z^2}(\frac{S}{2\sqrt{1+z^2}} + 1.5)^{4.026}} \right] \tau_{\text{bed}}$$  \hspace{1cm} (21b)

$$\frac{\partial \tau_{\text{bed}}}{\partial S} = \frac{(2+x+\alpha - y)}{(2+x+\alpha)S} \tau_{\text{bed}}$$

To have an in-depth understanding of bed width variations, depth and mean bed shear stress with shape factor, it is required to determine $S = S(\zeta)$. The following section states that obtaining an explicit relationship $S = S(\zeta)$ is still impossible with selection of a bed load equation owing to the lack of an extra flow equation.

Letting $Q_s$ be sediment discharge on total channel width and because $Q_s = P_{\text{bed}} q_s$, $P_{\text{bed}} = P_{\text{bed}}(\zeta, S(\zeta))$, $\tau_{\text{bed}} = \tau_{\text{bed}}(\zeta, S(\zeta))$ and thus $Q_s = Q_s(\zeta, S(\zeta))$, the following relationship is maintained:

$$\frac{dQ_s}{d\zeta} = \frac{\partial Q_s}{\partial \zeta} + \frac{\partial Q_s}{\partial S} \frac{dS}{d\zeta}$$  \hspace{1cm} (22)
Where the selected bed load transport relationship of \( q_s = c_s \bar{Q}_{bed}^m (\bar{Q}_{bed} - \tau_c)^j \) and \( Q_s = P_{bed} q_s \) give:

\[
\frac{\partial Q_s}{\partial \tau} = \frac{\partial Q_s}{\partial \tau_0} + \frac{\partial Q_s}{\partial P_{bed}} \quad \text{and} \quad Q_s = Q_s \left( \frac{m}{\bar{Q}_{bed}} + \frac{j}{(\bar{Q}_{bed} - \tau_c)} \right) + \frac{1}{P_{bed}} \frac{\partial P_{bed}}{\partial \tau} \tag{23a}
\]

\[
\frac{\partial Q_s}{\partial S} = \frac{\partial Q_s}{\partial \tau_0} + \frac{\partial Q_s}{\partial P_{bed}} \quad \text{and} \quad \frac{\partial Q_s}{\partial S} = Q_s \left( \frac{m}{\bar{Q}_{bed}} + \frac{j}{(\bar{Q}_{bed} - \tau_c)} \right) + \frac{1}{P_{bed}} \frac{\partial P_{bed}}{\partial S} \tag{23b}
\]

Incorporating the expressions of \( \frac{\partial P_{bed}}{\partial \zeta} \), \( \frac{\partial P_{bed}}{\partial S} \), \( \frac{\partial \bar{Q}_{bed}}{\partial \zeta} \) and \( \frac{\partial \bar{Q}_{bed}}{\partial S} \) in Equation 21 (a,b) into Equation 23 (a,b) produces:

\[
\frac{\partial Q_s}{\partial \zeta} = \frac{x(1 + \alpha)z + 2(1 + \alpha)z^2 + (2 + 2x + \alpha)z}{(2 + 2x + \alpha)z + \sqrt{1 + z^2}} - \zeta \quad \text{and} \quad \frac{\partial Q_s}{\partial S} = \zeta \left( \frac{m}{\bar{Q}_{bed}} + \frac{j}{(\bar{Q}_{bed} - \tau_c)} \right) + \frac{1}{P_{bed}} \frac{\partial P_{bed}}{\partial S} \tag{24a}
\]

Where:

\[
K = -\left[ \frac{2\sqrt{1 + z^2} \left( \frac{z}{2\sqrt{1 + z^2}} + 1.5 \right)^{4.026} [-z - (2 + x)z - 2(1 + x)z\sqrt{1 + z^2} - 2(2 + x + \alpha)z\sqrt{1 + z^2}]}\right] + \frac{0.01 \times 10^{2.245} \times 1.4026 (1 - 0.01% SF_{bed})^{-1} (2 + 2x + \alpha)z + \sqrt{1 + z^2} \zeta}{2\sqrt{1 + z^2} \left( \frac{z}{2\sqrt{1 + z^2}} + 1.5 \right)^{4.026}}
\]

\[
\frac{\partial Q_s}{\partial S} = \left[ \frac{(x + 2 + \alpha - y)(m + j) - y}{S(2 + x + \alpha)(\tau_0 - \tau_c)} \right] + \frac{y - m(x + 2 + \alpha - y)}{S(2 + x + \alpha)(\tau_0 - \tau_c)} \tag{24b}
\]

Now to determine optimum cross-section characteristics, \( Q_s \) is differentiated with respect to \( \zeta \) and made equal naught, under which from Equation 22, it is resulted that:

\[
\frac{dS}{d\zeta} = \frac{\partial Q_s}{\partial \zeta} \cdot \frac{\partial Q_s}{\partial \zeta} \tag{25}
\]

From Equations 24 (a,b) and 25, it is clear that variations of \( dS/d\zeta \) depend on variations of \( \bar{Q}_{bed}/\tau_c \) and \( \zeta \). For given values of the ratio \( \bar{Q}_{bed}/\tau_c \), \( dS/d\zeta \) varies from negative to positive values by an
increase in $\zeta$ value. This means that there is an upward concavity in $S-\zeta$ plot and $S$ reaches its least value where intersects horizontal axis.

### 1.4. Optimum dimensions of channel cross section ($Q_s = Q_{\text{max}}$)

Now, with regards to Equation 25, to minimize slope ($dS/d\zeta = 0$), $\partial Q_s/\partial \zeta$ should be equal to zero:

\[
\tau_{\text{bed}} = \tau_c \frac{(x + 1 + \alpha)\zeta_m + 2(1 + \alpha)\sqrt{1 + \zeta^2} + (2 + 2x + \alpha)z + \frac{(4 + 2x + 2\alpha)z \sqrt{1 + \zeta^2}}{\zeta_m} - m K}{(x + 1 + \alpha)\zeta_m + 2(1 + \alpha)\sqrt{1 + \zeta^2} + (2 + 2x + \alpha)z + \frac{(4 + 2x + 2\alpha)z \sqrt{1 + \zeta^2}}{\zeta_m} - (m + j) K}
\]  

(26)

In which $K$ is calculated from Equation 24(a). In the above equation, for given values of $Q$, $S$, $c_r$ and $D_{50}$ (i.e. constant slope), shape factor ($\zeta$) and bank slope ($z$) are unknowns, therefore, another equation is required. This may be obtained by coupling USBR modified bank stability equation (Equation 7) and mean bank shear stress (Equation 18).

\[
D S \times 0.01\% S F_{\text{bank}} \left[ \frac{(\zeta + z) \sin \theta}{2} \right] = 0.048(G_s - 1)D_{50\text{bank}} \tan \phi' \sqrt{1 - \sin^2 \theta \over \sin^2 \phi'}
\]  

(27)

In which $D$ is calculated from Equation 14. By solving Equations 26 and 27 simultaneously, optimum values of shape factor ($\zeta$) and bank slope ($z$) are computed and by substituting these values in Equation 14 to 17, $D_m$, $(P_{\text{bed}})_m$, $V_m$ and $(\tau_{\text{bed}})_m$ are computed likewise for other parameters. Optimum channel slope may also be computed by applying Equation 17 and 26.

\[
S_m = \frac{(\zeta_n)(\zeta_m)^{(x + 2 + \alpha)}}{Q} \left[ \frac{(1 - 0.01\% S F_{\text{bank}})^{-1} \zeta_m (2 + 2x \alpha)z^2}{(\zeta_m + z)^{(1 + 2x \alpha)}(1 + (x + 2\alpha)z)} \right] \left[ \frac{(4 + 2x + 2\alpha)z \sqrt{1 + \zeta^2}}{\zeta_m} - m K \right]^{(1 + 2\alpha)}
\]

(28)

By substituting $S_m$ from Equations 28 into Equations 14 and 15, relationships of bed width ($(P_{\text{bed}})_m$) and maximum depth ($D_m$) are derived (variable slope):
In which $K$ is calculated from Equation 24(a). As in stable channels, $Q_{\text{max}} = Q$, therefore, for given $Q$, optimum shape factor ($\alpha$) and stable bank slope ($\beta$) can be computed by solving Equations 27 and 29 into Equations 26 and 28 into Equations 25 and 16 into Equations 15 and 14 into Equations 13 and 12 into Equations 11 and 10 into Equations 9 and 8 into Equations 7 and 6 into Equations 5 and 4 into Equations 3 and 2 into Equations 1.

To determine optimum surface width $w_{\text{opt}}$ for a channel of given bank slope, the relationship of optimum surface width is produced:

$$w_{\text{opt}} = \frac{Q}{g^2} \left[ \frac{x + 1}{2} + 2 \alpha N + x + \left( x + 1 \right) \beta \right]$$

$$w_{\text{opt}} = \frac{Q}{g^2} \left[ \frac{x + 1}{2} + 2 \alpha N + x + \left( x + 1 \right) \beta \right]$$

Incorporating the expressions of $Q_{\text{max}}$ and $F_{\text{opt}}$ into Equations 26 and 29 into Equations 25 and 16 into Equations 24 and 23 into Equations 22 and 21 into Equations 20 and 19 into Equations 18 and 17 into Equations 16 and 15 into Equations 14 and 13 into Equations 12 and 11 into Equations 10 and 9 into Equations 8 and 7 into Equations 6 and 5 into Equations 4 and 3 into Equations 2 into Equations 1.

Substituting Equation 30 into Equation 13, the relationship of optimum surface width $w_{\text{opt}}$ is produced:

$$w_{\text{opt}} = \frac{Q}{g^2} \left[ \frac{x + 1}{2} + 2 \alpha N + x + \left( x + 1 \right) \beta \right]$$

$$w_{\text{opt}} = \frac{Q}{g^2} \left[ \frac{x + 1}{2} + 2 \alpha N + x + \left( x + 1 \right) \beta \right]$$

Therefore, $w_{\text{opt}}$ is maintained.
\[ Q_s = P_{bed} \cdot c_s \cdot \bar{\tau}_{bed} \cdot (\bar{\tau}_{bed} - \tau_c) \cdot (\tau_c) \cdot \bar{\tau}_{bed} \] is considered to be equal to \( \tau_c \) for stable channel design, in which \( Q_s = 0 \). In otherwords, channel dimensions are designed for bed sediment threshold of motion which may be interpreted as static stability of the channel. Under these circumstances, Equation 26 may be applied as:

\[
K \cdot j = 0 \Rightarrow K = 0 \Rightarrow \\
0.01 \times 10^{247} \times 14.026 \left( 1 - 0.01\% S F_{\text{bank}} \right)^{-1} (\zeta_m + 2\sqrt{1+z^2})(\zeta_m + z)(x + 2 + \alpha) = \\
-2\sqrt{1+z^2} \left( \frac{\zeta_m}{2\sqrt{1+z^2}} + 1.5 \right)^{2.4026} \left[ -\zeta_m - (2 + \alpha)z - 2(x + 1)\sqrt{1+z^2} - (2 + 2 + \alpha) \frac{z\sqrt{1+z^2}}{\zeta_m} \right] \quad (33)
\]

With regards to Equation 26, if \( K = 0 \) (i.e. Equation 33), \( Q_s = 0 \) which is in conformation with sediment threshold of motion. Likewise, for \( \frac{\partial \tau_{bed}}{\partial \zeta} = 0 \), from Equation 21(b), Equation 33 is resulted and bed shear stress reaches its maximum value which is again a confirmation of sediment threshold. This Equation (i.e. Equation 33) could be applied for static equilibrium in channel where flow can transport sediment without erosion in the channel [14,15].

2. Results and Discussion

2.1. Sensitivity Analysis

In this section, effect of modified bank friction angle (\( \phi' \)) on river geometry has been investigated by applying Manning's roughness [12] and MPM [22] and Parker [24] bed load equations, where other input variables have been kept constant (Fig. 2). It is noticed that by an increase in \( \phi' \), surface width (\( W \)) decreases while water depth (\( D \)) increases. This could be justified as \( \phi' \) increases (i.e. bank vegetation increases), bank resistance also increases and, therefore, bed sediments are more exposed to erosion than those of the banks, which may lead to channel deepening. This could also be confirmed by velocity reduction in the vicinity of the banks which provides the situation where deposition occurs faster. This figure also states that \( \phi' \) could be effective on river geometry in such a way that its variations from 40° to 60°, the most variations occur in width and depth while under
its variations from 60° to 90°, variations of width and depth decrease. In general, it is obvious that the above bed load equations have the same functions with respect to $\phi'$. Furthermore, sensitivity analysis was accomplished by keeping $Q$, $S$, $D_{50}$ and $c_s$ constants for four different values of $\phi'$ (i.e. 50°, 60°, 70° and 90°) to compute variations of bed load transport, using MPM [22] equation, versus bed width ($P_{\text{bed}}$). It is worth mentioning that the above analysis was also conducted for unconstrained model under two extremal hypotheses MSTC and MSP (Fig. 3). This illustrates that with an increase in $\phi'$, optimum bed width decreases and maximum sediment transport increases.

Sensitivity analysis was also conducted on variations of $D_{50\text{bank}}$ using MPM [22] and Parker [24] equations (Fig. 4). It shows that with an increase in $D_{50\text{bank}}$, surface width decreases while increasing depth. The results also state that channel geometry are most sensitive to $D_{50\text{bank}}$ for its values up to 0.05 (m) while for $D_{50\text{bank}}>0.05$m, no changes are observed in the geometry. In other words, in this case, with an increase in $D_{50\text{bank}}$, bank stability increases as a result of which channel widening decreases. When bed erosion occurs, it may reach the state at which armouring take place where channel deepening ceases.

### 2.2. Model Calibration

In this study, two case studies have been used to calibrate the model as follows:

#### Part I: In this section, Hey and Thorne [29] data set which was then updated by Darby [30] were used to calibrate the model. The data set consists of 62 river stations at which river geometry have been measured in low flow conditions, as characterized stable single thread with mobile bed and four bank vegetation density classifications: Type 1 represents grassy banks with no trees or bushes; Type 2, 1-5% tree/shrub cover; Type 3, 5-50% tree/shrub cover, Type 4, greater than 50% tree/shrub cover or incised into flood plain.

In this study, $D_{50\text{bank}}$ suggested by Darby [30] did not give good results. However, when
$D_{50\text{bank}}=D_{50\text{surface}}$ was considered [10], less discrepancy, was observed in the results. This assumption may be in agreement with the case when banks are exposed to erosion and their composition deposits on the river bed.

### 2.2.1. $\phi'$ Estimation

As it was shown in sensitivity analysis section, $\phi'$ values could be key control parameter on bank stability, which vary by bank vegetation density. It is worth mentioning that these values are not given by Hey and Thorne [29]. $\phi'$ estimation was made for 48 stations among the data. This was undertaken for a particular value of $\phi'$ for each station and the value was changed to make the difference between observed and predicted widths to be approximately $\pm 1\%$ (Table 1). As it is shown average $\phi'$ value for each vegetation type, in agreement with vegetation cover, varies. This confirms impact of vegetation cover on channel geometry can be considered by $\phi'$.

### 2.2.2. Comparison of model outputs with field data

In this section, predicted results were plotted against observed data where predicted results were obtained by Manning’s roughness equation [12] and Parker’s bed load equation [24] (Fig. 5).

### 2.3. Effect of Bank vegetation

The aim of this section is to investigate the effect of vegetation in estimation of stable width and comparison of the results for constrained and unconstrained models by considering a combination of MSTC and MSP extremal hypotheses. In order to have deep understanding of effect of vegetation cover, this was initially considered in two phases of sparse vegetation (type 1 and 2) and dense vegetation (type 3 and 4), respectively, (Fig. 6). Therefore, the process was repeated for all types 1, 2, 3 and 4 in details (Table 2). As it is shown in Fig. 6(a), channels with observed width
less than 30 m and sparse bank vegetation are more scatter about best-fit line while for those with
dense vegetation less scatter about the best-fit line is observed. For channel width of greater than 30
m, data variation from best-fit line increases which states that the model behavior is questionable.
Likewise, in Table 2, observed width values are often greater than the predicted values. $W_{\text{pred}}/W_{\text{obs}}$ in
unconstrained model of combination of MSTC and MSP vary 0.44-0.75 for bank vegetation type 1
to 4, respectively. In other words, with regards to an increase in vegetation type from 1 to 4, bank
resistance is also increased to meet the assumption of bank inedibility in unconstrained model.
Hence, it is expected to have predicted width values much closer to the observed ones in types 3
and 4 than those of the types 1 and 2. With regards to Table 2 and Fig. 6, it is obvious that by
inserting vegetation cover in the model, in addition to improving of data disparity about best fit line,
$W_{\text{pred}}/W_{\text{obs}}$ is also closer to 1 for all vegetation types. This confirms the improvement in the accuracy
of the model in width prediction by applying bank vegetation.
Following investigation of vegetation impact, it has been suggested by many researchers [29, 31]
that vegetation type does not cause variation in the power of width hydraulic geometry
relationships. However, this may influence the coefficients in them. Hence, based on conducted
researchers, the width of hydraulic geometry, was considered as $W = aQ^{0.5}$. The model was applied
with Manning’s equation as resistance and Parker [24] as bed load equation to compute width for all
types of vegetation. Therefore, values of ‘a’ were calibrated for each type by fitting $W = aQ^{0.5}$ to the
data on log-log scale to obtain the followings (Fig. 7):

\[
W = 4.59Q^{0.5}, m \text{VegetationType1} \quad (34a)
\]
\[
W = 3.20Q^{0.5}, m \text{VegetationType2} \quad (34b)
\]
\[
W = 2.69Q^{0.5}, m \text{VegetationType3} \quad (34c)
\]
\[
W = 1.82Q^{0.5}, m \text{VegetationType4} \quad (34d)
\]
These equations indicate that for a given discharge, there is a decrease in channel width with an increase in vegetation density. Values of observed and computed ‘a’ were compared for each vegetation type and relative errors are presented in Table 3. The results state a reasonable conformation between computed and observed values of ‘a’.

Part II: In this section, the data from the four river reaches of Iran (Khuzestan province) were used. The rivers were described as stable channels with movable gravel beds. The studied stations are located in the northern and eastern parts of Khuzestan as mountainous regions on the margin of the Zagros mountains. The measured data including cross section measurements, river channel slope and Manning’s roughness coefficient were made available at each hydrometric station by Khuzestan Water and Power Organization. The average size of bed particles and bankfull discharge were determined by Mahmoudi et al. [32]. It should be noted that bank angle (θ) was not found in the reference data and was, therefore, determined at each station using the AutoCAD software and cross-section shape for the two riverbanks and the mean values were used as the input data to the unconstrained model. In addition, because of the lack of bed load (Qs) measurements in the data set, the constant slope state of the model was used for the calibration. This section was meant to evaluate the efficiency of the presented model to work out river bank vegetation type by estimating φ’ values. Therefore, the method presented in section (3.2.1) for estimation of φ’ angle was used by applying the developed model in this study and combining the Manning and Parker equations [12, 24], the calibrated values of φ’ were obtained for the four studied river reaches. The results are presented in Table 4. Then, considering the estimated φ’ values in comparison to the range of the modified bank friction angle (Table 1) as well as the mean defined φ’, the values with the minimum differences used to estimate the riverbank vegetation. It should be noted that the estimated φ’ values were reasonable considering the location of study river reaches and field surveys.
In the next stage, the model was used considering the Manning’s flow resistance equation [12] and combining it with the Meyer-Peter and Muller [22], Parker [24], and modified Meyer-Peter and Muller by Huang [33] relations. In addition, mean $\phi'$ values corresponding to different vegetation types according to Table 1 and other input data were considered with and without bank constraint. The model error (in percent) in estimation of the bankfull width and depth is presented in Table 5. The results clearly showed a better efficiency of the constrained model compared to the unconstrained model in estimation of the optimum channel dimensions in the study river reach. The results also showed the better efficiency of the Meyer-Peter and Muller bed load function compared to others used in the study reach.

3. Conclusion

The developed model in this study has a great deal of flexibility with regards to resistance and bed load equations. The model has a broad scope of applicability based on available data. Data analyses suggest that predicted width values by MSTC and MSP under unconstrained condition are only valid in alluvial channels with highly resistant banks, so it may not give correct results for erodible banks with sparse vegetation cover. Therefore, consideration of bank vegetation (constrained condition) may increase the accuracy of channel dimensions estimation to some extent, particularly, in wide rivers with sparse vegetation cover. Sensitivity analysis indicates that bed and surface widths could decrease by increasing bank vegetation or resistance while depth increases. The model may also be applicable for the circumstances where $\phi'$ values are required to be calibrated to assess bank stability at a cross section of a river.

Acknowledgements
$Q$  Discharge ($m^3/s$)  

$A$  Cross-sectional area ($m^2$)  

$V$  Mean velocity ($m/s$)  

$c_r$  Coefficient  

$R$  Hydraulic radius ($m$)  

$S$  Longitudinal slope  

$D$  Maximum channel depth ($m$)  

$x, y, \alpha$  Exponents  

$q_s$  Bed load discharge per unit channel width  

$c_s$  Coefficient  

$\tau_{bed}$  Mean bed shear stress ($N/m^2$)  

$\tau_c$  Critical shear stress ($N/m^2$)  

$m, j$  Exponents  

$\%SF_{bank}$  Percentage of the shear force acting on banks  

$P_{bed}$  Bed perimeter  

$P_{bank}$  Bank perimeter  

$\tau_{bank}$  Mean bank shear stress ($N/m^2$)  

$\gamma$  Unit weight of water ($N/m^3$)  

$W$  Channel surface width ($m$)  

$\theta$  Bank angle  

$z$  Bank slope  

$\tau_{cb}^*$  Critical dimensionless shear stress for bank sediment  

$\gamma_s$  Unit weight of sediment ($N/m^3$)  

$D_{50_{bank}}$  Bank material size  

$\phi'$  Bank friction angle  

$\tau_c^*$  Critical dimensionless shear stress  

$\phi$  Angle of repose  

$\zeta$  Non-dimensional channel shape factor  

$P$  Channel perimeter ($m$)  

$Q_s$  Bed load discharge  

$K$  Coefficient  

$D_{50_{bed}}, D_{50}$  Bed material size
Reference


Table 1. Calibrated $\phi'$ values for different bank vegetation types for 48 stations of Hey and Thorne [29]

<table>
<thead>
<tr>
<th>Vegetation type</th>
<th>$\phi'(\degree)$</th>
<th>Minimum</th>
<th>Mean</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30.6</td>
<td>43.5</td>
<td>54.2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>32.6</td>
<td>47.6</td>
<td>59.7</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>35.0</td>
<td>50.0</td>
<td>72.0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>44.1</td>
<td>64.3</td>
<td>87.0</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Predicted and observed widths for different vegetation types

<table>
<thead>
<tr>
<th>Vegetation type</th>
<th>Observed $W_{\text{obs}}$ (m)</th>
<th>Bank strength unconstrained model</th>
<th>Bank strength constrained model</th>
<th>$\phi'(\degree)$</th>
<th>$W_{\text{Pred}}/W_{\text{obs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32.3</td>
<td>14.40</td>
<td>0.44</td>
<td>43.5</td>
<td>36.46</td>
</tr>
<tr>
<td>2</td>
<td>22.4</td>
<td>11.50</td>
<td>0.54</td>
<td>47.6</td>
<td>22.04</td>
</tr>
<tr>
<td>3</td>
<td>27.0</td>
<td>16.48</td>
<td>0.64</td>
<td>50.0</td>
<td>29.08</td>
</tr>
<tr>
<td>4</td>
<td>20.2</td>
<td>14.03</td>
<td>0.75</td>
<td>64.3</td>
<td>16.72</td>
</tr>
</tbody>
</table>
Table 3. Mean relative errors of ‘a’ values for $W = aQ^{0.5}$

<table>
<thead>
<tr>
<th>Bank vegetation type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean relative error</td>
<td>6.00</td>
<td>(3.90)</td>
<td>(1.47)</td>
<td>(22.22)</td>
</tr>
</tbody>
</table>

() indicates negative values (predicted values are less than observed values).

Table 4. Calibrated $\phi'$ values using Iranian data [32]

<table>
<thead>
<tr>
<th>River</th>
<th>Station</th>
<th>$\phi'$ (%)</th>
<th>Predicted bank vegetation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Karoun</td>
<td>Sousan</td>
<td>61.5</td>
<td>4</td>
</tr>
<tr>
<td>Aab Shirin</td>
<td>Kheir Abad</td>
<td>59.8</td>
<td>3</td>
</tr>
<tr>
<td>Maroun</td>
<td>Cham Nezam</td>
<td>65.6</td>
<td>4</td>
</tr>
<tr>
<td>Zal</td>
<td>Pol-e-Zal</td>
<td>47.9</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5. Mean relative errors

<table>
<thead>
<tr>
<th>Bed load equation</th>
<th>unconstrained model</th>
<th>constrained model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bankfull width (%)</td>
<td>Bankfull depth (%)</td>
</tr>
<tr>
<td>Meyer-Peter &amp; Muller [22]</td>
<td>(25.57)</td>
<td>23.65</td>
</tr>
<tr>
<td>MPM-H [33]</td>
<td>(29.13)</td>
<td>28.72</td>
</tr>
<tr>
<td>Parker [24]</td>
<td>(29.37)</td>
<td>29.57</td>
</tr>
</tbody>
</table>

() indicates negative values (predicted values are less than observed values).
Fig. 1. River cross section [16]
Fig. 2. Sensitivity analysis of $\phi'$ to water surface width and depth

Fig. 3. Variations of computed bed load versus bed width for different values of $\phi'$

Fig. 4. Sensitivity analysis of $D_{50\text{bank}}$ to water surface width and depth
Fig. 5. Comparison of calculated versus observed bankfull width and depth

Fig. 6. Widths predicted using Parker’s equation [24] by (A) an unconstrained model and (B) a constrained model are plotted against the observed widths for gravel bed channels.
Fig. 7. Relation between discharge and width for different types of vegetation.

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