A class Hotelling model for sequential auctions of close substitutes

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Abstract Against the background of supply chains, this paper constructs a class Hotelling model to describe and explore sequential auctions of close substitutes with slightly more general associated valuations. In this generalized model, both close substitutes and bidders are hypothetically distributed in the interval \([0, 1]\), types of bidders are continuous, and each bidder’s valuations for close substitutes are not independent. And with the aid of this model, equilibriums are explored and efficiencies of the auctions are analyzed under second-price sealed-bid auction formats. Further considering two typical information policies, we investigate some concrete bids and revenues of the efficient sequential auctions while bidders’ valuations are linear functions of distances between them and close substitutes. Results show that efficiencies of the sequential auctions are conditional, and influences of information policies on revenues of the auctions are related to both numbers of bidders and locations of items.

Keywords Sequential Auctions; Hotelling Model; Supply Chain; Associated Valuation; Information Policy

1. Introduction

In 1929, Hotelling developed a model of spatial competitions to demonstrate relationships between locations and pricing behaviors of firms through a line of fixed length, and predict an aggregation of two competing firms in the middle of the customers support interval \([1-2]\). The standard Hotelling model assumes that all consumers are identical (except for locations) and evenly dispersed along the line, both the firms and consumer respond to changes in demand and the economic environment. As a game model, it can also be used to describe some auction problems in supply chains: \(m\) suppliers sell their supply contracts sequentially or simultaneously to \(n\) agents with unit-demands via auctions, and both suppliers and agents are located in a same traffic line. Here, suppliers and agents correspond to firms and customers, respectively. And these contracts are deterministic and undifferentiated for agents, so differences among agents’ valuations for contracts mainly depend on their transportation costs.

The above problem can be abstracted as sequential auctions of close substitutes. Again because distances are associated between an agent and his or her suppliers, each agent’s valuations for the suppliers’ contracts are also associated, which is different from interdependent valuations among agents \([3]\). Hence with the help of the Hotelling model, our paper tries to focuses on sequential auctions of close substitutes with slightly more general associated valuations. Here, key functions of close substitutes are same or similar, but their configurations or external performances are slightly different in order to meet various needs of consumers, resulting in that private valuations of consumers are not exactly the same.

A key consideration of sequential auctions is a bidder’s expected surplus for follow-up auctions, which is usually concerned with future objects, bidder numbers, previous winning information and etc. Thus, information policy is always an important topic of auctions or other business activities, and theoretical and experimental studies show that revealing some information in advance will possibly affect overall efficiencies and revenues of auctions \([4-5]\). Presently, there are wide debates and correspondingly large literatures about sequential auctions and information policies. Usually while auctioned objects are heterogeneous and
bidders’ valuations are independent of each other, an expected revenues-maximizing auctioneer should fully and publicly reveal all information about auctioned objects [6-8]. But while auctioned objects are homogeneous and bidders’ valuations are not mutual independence, revealing future objects or other related information in advance would uncertainly affect overall efficiencies and revenues of sequential auctions [9-12]. Currently in these literatures, a bidder’s valuations for homogeneous objects are usually supposed identical or proportionable, which may be considered as special associated valuations and lead to learning behaviors [13]. Owing to difficulties of modeling general associated valuations and processing mathematical expectations while a bidder type is multi-dimensional and continuous, existing researches mostly focused on sequential auctions in which a bidder type is hypothetically discrete, namely H(High) or L(Low), and his or her valuations for objects are identical.

Recently, Zeithammer [14] studied sequential auctions of heterogeneous objects, discussed influences of revealing future objects on auction efficiencies, and proved existences of symmetric equilibriums and pure bidding strategies. Bayesian methods and models used to process a multi-dimensional continuous type in his paper inspired our studies. Accordingly combined with characteristics of supply chains, our paper explores a class Hotelling model for describing sequential auctions with some special assumptions under second-price sealed-bid auction mechanisms. In our model, both close substitutes and bidders are hypothetically distributed in the interval [0, 1], a bidder type is continuous and multi-dimensional (namely, each bidder has different valuation for each close substitute), and each bidder’s valuations for close substitutes are not independent. And this model can skillfully converts multi-dimensional types into one-dimensional types via interdependencies of distances between bidders and items in [0, 1].

The rest of this paper is organized as follows: Section 2 states formally the sequential auctions of close substitutes with associated valuations, and constructs a class Hotelling model. Section 3 proves existences of equilibrium bids based on the model under the second-price sealed-bid auction format, and explores some conditions of efficient auctions. Considering information policies, equilibrium bids and overall revenues of the sequential auctions of close substitutes with special associated valuations are specifically deduced and discussed in Section 4. Relevant conclusions are summarized in Section 5.

2. A Class Hotelling Model for Describing Sequential Auctions

It is supposed that two close substitutes, Item A and Item B, are auctioned sequentially to \( n \geq 3 \) bidders via second-price sealed-bid auctions. Auction rules and some assumptions are as follows:

1) Item A is sold in the first auction and Item B is sold in the second one. And in each auction, the bidder with the highest bid wins and pays the second highest bid. A tie is broken by rolling a fair coin.

2) Each bidder is risk-neutral and has a unit-demand. Namely the bidder getting Item A will exit from the second auction.

3) Both items and bidders are distributed in the interval [0, 1] and mutually stochastically independent, as Figure 1 shows. A location of each bidder is his or her private information.

4) Each bidder’s valuation for Item \( k \in \{A,B\} \) is \( V_k - \tau(d_k) \), where \( \tau(d_k) \geq 0 , \) \( \tau'(d_k) > 0, \) and \( d_k \) denotes the distance between the bidder and Item \( k \). Here, the value of Item \( k \) is \( \{A,B\} , V_k \geq 1, \) is common knowledge.

The last assumption is inspired by the standard Hotelling model [15]. When making trades in real-life markets, one important factor to consider is costs such as transportation costs or maintenance costs, which usually are increasing functions of transportation distances. In this
paper, \( V_k \) is the same to all bidders. But when taking into account the transportation cost \( \tau(d_k) \), each bidder’s true valuation for \( k \in \{A,B\} \) should be \( V_k - \tau(d_k) \), which is different from those of other bidders.

**Please insert Figure 1 here**

Figure 1 shows the class Hotelling Model. In Figure 1, the location of Item A and one of Item B are denoted by \( \alpha \in [0,1] \) and \( \beta \in [0,1] \), respectively. Without loss of generality, we assume \( \alpha \leq \beta \). The location of a bidder in the interval is denoted by \( t \in [0,1] \). To simplify the expressions, we use Bidder \( t \) to denote the bidder whose location in \([0,1]\) is \( t \). Thus, the distance between Bidder \( t \) and Item A is \( d_A(t) = |\alpha - t| \), and one between Bidder \( t \) and Item B is \( d_B(t) = |\beta - t| \). Accordingly, \( v_i(t) = V_k - \tau(d_k(t)) \) \((k \in \{A,B\})\) is Bidder \( t \)'s valuation for Item \( k \in \{A,B\} \). Furthermore, let \( b_k(t) \) represent Bidder \( t \)'s bid for \( k \in \{A,B\} \).

**Remark 1** In Figure 1, items and bidders correspond to firms and customers in the standard Hotelling model, respectively. While \( V_A = V_B \), \( \beta - \alpha = 0 \) means that the close substitutes (Item A and Item B) are identical for bidders, and \( \beta - \alpha = 1 \) means that maximum differences exist among them. And just like preferences in the standard Hotelling model, the smaller the distance \( d_k(t) \) \((k \in \{A,B\})\) is, the greater Bidder \( t \)'s valuation is for Item \( k \).

**Remark 2** In Figure 1, bidders' valuations for two items are their own private information and independent among bidders, but each bidder’s private valuations for two items \( v_A(t) = V_A - \tau(d_A(t)) \) and \( v_B(t) = V_B - \tau(d_B(t)) \) are correlated because either \( |d_A(t) - d_B(t)| = \beta - \alpha \) or \( d_A(t) + d_B(t) = \beta - \alpha \) while \( \alpha \) and \( \beta \) are given in advance. Thus, the model in Figure 1 can be used approximatively to describe sequential auctions of close substitutes with special assumptions of non-independences valuations, under which the probability of Bidder \( t \) winning Item B is intuitively relevant to one of his or her winning A. Concurrently with the help of \( t \), our model skillfully converts a two-dimensional type of a bidder into a one-dimensional one. Thus, \( t \) may be also regarded as the bidder type.

### 3. Equilibrium and Winners

Because the second stage of the sequential auctions in this paper is actually a private-value second-price sealed-bid auction, Bidder \( t \)'s dominant strategy for Item B is to bid his or her own valuation for Item B, namely \( b_B(t) = v_B(t) \). However, deciding the bid for Item A is more complicated than one for Item B, because Bidder \( t \) will consider his or her expected surplus in the second stage.

Let \( x = \max_{\theta \in \Theta} b_A(\theta) \) be the highest bid of Bidder \( t \)'s \( n-1 \) opponents in the first stage. If \( b_A(t) > x \), then Bidder \( t \) wins in the first auction and his or her surplus is \( v_A(t) - x \). If \( b_A(t) < x \), then Bidder \( t \) loses in the first auction and all his or her remaining opponents in the second auction belong to \( \Omega(x) = \{t' \in \Theta | b_A(t') \leq x\} \), which is a set of bidders whose bids for Item A are not more than \( x \). Further, let \( \Phi_{n-2/\tau}(z, \Omega(x)) = p\left( \max_{(\theta \in \Theta) \text{ and } \theta \neq t} v_B(\theta) \leq z \right) \) be the probability distribution function of the highest valuation of remaining bidders (except \( t \)) in the second auction. Then,
Bidder \( t \in \Omega(x) \) can win Item B with a probability of \( \Phi_{n-2-/1}(v_b(t), \Omega(x)) \) and his expected surplus in the second auction is

\[
\sigma(v_b(t), x) = \int_0^{v_b(t)} (v_b(t) - z) d\Phi_{n-2-/1}(z, \Omega(x)) \tag{1}
\]

Obviously, winning Item A is more beneficial to \( t \) if and only if \( v_A(t) - x > \sigma(v_b(t), x) \). And a logical and ideal bid \( b_A(t) \) should satisfy \( v_A(t) - b_A(t) = \sigma(v_b(t), b_A(t)) \) to make it undifferentiated for Bidder \( t \) to win Item A or to win Item B. In addition, because \( v_A(t) \leq \sigma(v_b(t), 0) \) implies that giving up Item A is always a good choice for Bidder \( t \) even though all other bidders bid 0 or do not submit their bids in the first auction, we assume that \( v_A(t) > \sigma(v_b(t), 0) \) holds for \( \forall t \in [0, 1] \) in this paper. Inspired by Zeithammer [14], Proposition 1 gives necessary and sufficient conditions of the equilibrium bid \( b_A(t) \).

**Proposition 1** Suppose \( v_A(t) > \sigma(v_b(t), 0) \) for \( \forall t \in [0, 1] \). If and only if \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \) for \( \forall t \in [0, 1] \), there is a unique equilibrium bid \( b_A(t) \) which satisfies the following equation

\[
b_A(t) = v_A(t) - \sigma\left(\begin{array}{c} v_b(t) \\ \frac{\partial \sigma(v_b(t), x)}{\partial x} \end{array}\right) \tag{2}
\]

**Proof.** Let \( S(x) = v_A(t) - x - \sigma(v_b(t), x) \). Proposition 1 need be proved from the following two aspects: 1) \( S(x) = 0 \) has a unique solution \( x_0 > 0 \) if and only if \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \) for \( \forall t \in [0, 1] \); 2) \( x_0 > 0 \) is Bidder \( t \)'s equilibrium bid for Item A.

1) If \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \), then \( S'(x) = -1 - \frac{\partial \sigma(v_b(t), x)}{\partial x} < 0 \). Therefore, \( S(x) \) strictly monotonically decreases with \( x \). According to \( v_A(t) > \sigma(v_b(t), 0) \) for \( \forall t \in [0, 1] \), we have \( S(0) > 0 \). Moreover, \( S(v_A(t)) = v_A(t) - v_A(t) - \sigma(v_b(t), v_A(t)) \leq 0 \). Because \( S(x) \) is continuous and strictly monotonic with \( x \), there must exist a unique solution \( x_0 \in (0, v_A(t)] \) satisfying \( S(x_0) = 0 \).

Suppose \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} \leq 0 \), then \( S'(x) = -1 - \frac{\partial \sigma(v_b(t), x)}{\partial x} \geq 0 \). Namely, \( S(x) \) monotonically increases with \( x \). Thus, for \( \forall x > 0 \), \( S(x) \geq S(0) = v_A(t) - \sigma(v_b(t), 0) > 0 \), which conflicts with the condition that \( S(x) = 0 \) has one positive solution \( x_0 > 0 \). Thus, \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \).

2) We assume \( b_A(t) < x_0 \). While \( b_A(t) < x_0 < x = \text{Max}(b_A(\theta)) \) or \( x < b_A(t) < x_0 \), Bidder \( t \) is indifferent to bid \( x_0 \) or \( b_A(t) \). While \( b_A(t) < x < x_0 \), Bidder \( t \) will lose Item A and his expected surplus in the second auction is \( \sigma(v_b(t), x) \). Notice that \( S(x) > S(x_0) = 0 \) because \( S(x) \) strictly monotonically decreases with \( x < x_0 \). Accordingly, \( \sigma(v_b(t), x) < \sigma(v_A(t) - x) \). Notice that \( v_A(t) - x \) is Bidder \( t \)'s surplus if he or she bids \( x_0 \) and wins Item A. Therefore, bidding \( b_A(t) < x_0 \) is worse than bidding \( x_0 \) for Bidder \( t \).

Similar proof shows that bidding \( b_A(t) > x_0 \) also makes Bidder \( t \) worse. Therefore, \( b_A(t) = x_0 \) is Bidder \( t \)'s equilibrium bid for Item A.
Remark 3 Essentially, \(1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0\) is the sufficient and necessary condition of that \(v_A(t) - \sigma(v_b(t), x)\) has a fixed point which just is Bidder \(i\)'s equilibrium bid for Item A. Here, \(v_A(t) - b_A(t)\) and \(\sigma(v_b(t), b_A(t))\) are Bidder \(i\)'s expected surplus in the first auction and one in the second auction, respectively. Accordingly, Proposition 1 implies that winning Item A or winning B should yield the same expected surplus for Bidder \(i\) while he or she determines the optimal bid for Item A. Intuitively, Proposition 1 is also appropriate for sequential auctions of more than two items while Item A is regarded as the current item and Item B is regarded as a sum of all follow-up (or future) items.

Based on Proposition 1, Proposition 2 and 3 identifies the winner in the first auction.

**Proposition 2** If \(\tau(d_k) (k \in \{A,B\})\) is a linear function of \(d_k\), \(\tau'(d_k) > 0\), and \(1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0\), then Bidder \(t = \arg \min_{t_i \leq \ldots \leq t_n} d_A(t_i) = \arg \min_{t_i \leq \ldots \leq t_n} |t_i - \alpha|\) will win Item A.

**Proof.** Proposition 1 indicates that if \(1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} > 0\), then the equilibrium bid \(b_A(t)\) for Item A is an implicit function satisfying Equation (2). According to \(v_A(t) = v_A - \tau(d_A(t))\) and \(v_b(t) = V_b - \tau(d_A(t))\), we have

\[
\frac{db_A(t)}{dd_A(t)} = \frac{dv_A(t)}{dd_A(t)} - \frac{\partial \sigma(v_b(t), b_A(t))}{\partial v_b(t)} \frac{dv_b(t)}{dd_A(t)} - \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} \frac{db_A(t)}{dd_A(t)}
\]

Namely,

\[
\frac{db_A(t)}{dd_A(t)} = \left[-\frac{d\tau(d_A(t))}{dd_A(t)} + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial v_b(t)} \frac{dv_b(t)}{dd_A(t)} \frac{dd_A(t)}{dd_A(t)} \frac{dd_A(t)}{dd_A(t)} \left[1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)}\right]^{-1}\right]
\]

\[
= w \left[-1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial v_b(t)} \frac{dv_b(t)}{dd_A(t)} \frac{dd_A(t)}{dd_A(t)} \left[1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)}\right]^{-1}\right]
\]

In Equation (3), \(w = \frac{d\tau(d_A(t))}{dd_A(t)} = \frac{d\tau(d_B(t))}{dd_B(t)} > 0\) because \(\tau(d_k) (k \in \{A,B\})\) is a linear function of \(d_k\) and \(\tau'(d_k) > 0\). And obviously, \(\frac{\partial \sigma(v_b(t), b_A(t))}{\partial v_b(t)} = \Phi_{n-2}^{-1}(v_b(t), \Omega(b_A(t))) \leq 1\) and \(1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} > 0\).

If \(\alpha < \beta \leq t\), then \(d_B(t) = t - \beta = (t - \alpha) - (\beta - \alpha) = d_A(t) - (\beta - \alpha)\); if \(\alpha \leq t < \beta\), then \(d_B(t) = \beta - t = (\beta - \alpha) - (t - \alpha) = (\beta - \alpha) - d_A(t)\); if \(t < \alpha < \beta\), then \(d_B(t) = \beta - t = (\beta - \alpha) + (\alpha - t) = (\beta - \alpha) + d_A(t)\). Accordingly, \(\left|\frac{dd_B(t)}{dd_A(t)}\right| = 1\). Thus, \(\frac{db_A(t)}{dd_A(t)} \leq 0\), that is, \(b_A(t)\) is a decreasing function of \(d_A(t)\) and an increasing function of \(v_A(t) = V_A - \tau(d_A(t))\). Therefore, Bidder \(t\) nearest to Item A (i.e., \(t = \arg \min_{t_i \leq \ldots \leq t_n} d_A(t_i) = \arg \min_{t_i \leq \ldots \leq t_n} |t_i - \alpha|\) will win Item A in first auction.

As a direct result of Proposition 2, the following corollary is obtained.

**Corollary 1** The auction for Item A is efficient while \(\tau(d_k) (k \in \{A,B\})\) is linear, \(\tau'(d_k) > 0\) and \(1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} > 0\).
While \( \tau(d_k) \) \((k \in \{A,B\})\) is a non-linear function of \( d_k \), it is contingent to predict the winner in the first auction in advance. For example, let \( \tau(d_k) = 1 - e^{-d_k} \) and \( \forall t \geq \beta > \alpha = 0 \), namely all bidders are on the right of Item B in Figure 1, then \( d_A(t) = t \), \( d_B(t) = t - \beta \) and

\[
\frac{db_A(t)}{dd_A(t)} = e^{-t} \left[ -1 + e^{-\beta} \Phi_{n-2-i} (v_b(t), \Omega(b_a(t))) \right] \left[ 1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} \right]^{-1}.
\]

If \( \beta \) is large enough, then \( \frac{db_A(t)}{dd_A(t)} \) or \( \frac{db_A(t)}{dt} \) may be more than zero, which means that \( b_A(t) \) may be an increasing function of \( d_A(t) \) or \( t \). However, \( v_A(t) = V_A - \tau(d_A(t)) = V_A - 1 + e^{-d_A(t)} = V_A - 1 + e^t \) is a decreasing function of \( d_A(t) \) or \( t \). Therefore, the auction for Item A may be inefficient.

**Proposition 3** If \( \tau(d_k) \geq 0, \tau''(d_k) \geq 0, k \in \{A,B\}, 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \) and \( t_i \geq \alpha \) \((i = 1, 2, \ldots, n)\), then Bidder \( t = \arg \min d_A(t) = \arg \min t_i - \alpha \) will win Item A.

**Proof.** Notice that \( t_i \geq \alpha \) \((i = 1, 2, \ldots, n)\) means all bidders on the right of Item A in [0, 1]. According to Proposition 1, Bidder \( t \)'s equilibrium bid \( b_A(t) = v_A(t) - \sigma(v_b(t), b_A(t)) \). Because \( v_A(t) = V_A - \tau(d_A(t)) \) and \( v_B(t) = V_B - \tau(d_B(t)) \), we have

\[
\frac{db_A(t)}{dd_A(t)} = \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} - \frac{1}{\partial \sigma(v_b(t), b_A(t))} \left[ \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} \right]^{-1} \frac{d\tau(d_A(t))}{dd_A(t)} + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} \frac{d\tau(d_B(t))}{dd_A(t)} \frac{dd_B(t)}{dd_A(t)} \left[ 1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} \right]^{-1}.
\]

Notice that \( \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} = \Phi_{n-2-i}(v_b(t), \Omega(b_A(t))) \leq 1 \) and \( 1 + \frac{\partial \sigma(v_b(t), b_A(t))}{\partial b_A(t)} > 0 \).

If \( t \geq \beta \), then \( d_B(t) = t - \beta = d_A(t) - (\beta - \alpha) \leq d_A(t) \) \((\therefore t \geq \alpha \) and \( \beta \geq \alpha \)). Accordingly,

\[
\frac{dd_B(t)}{dd_A(t)} = 1 \quad \text{and} \quad \frac{d\tau(d_A(t))}{dd_A(t)} \geq \frac{d\tau(d_B(t))}{dd_A(t)} \geq 0 \quad (\therefore \tau'(d_k) \geq 0, \tau''(d_k) \geq 0 \) and \( k \in \{A,B\} \). Hence,
\]

\[
\frac{db_A(t)}{dd_A(t)} \leq 0.
\]

If \( t < \beta \), then \( d_B(t) = \beta - t = (\beta - \alpha) - d_A(t) \geq 0 \) \((\therefore t \geq \alpha = 0 \) and \( \beta \geq \alpha \)). Accordingly,

\[
\frac{dd_B(t)}{dd_A(t)} = -1 \quad \text{and} \quad \frac{d\tau(d_A(t))}{dd_A(t)} \geq 0 \quad (\therefore \tau'(d_k) \geq 0, \ k \in \{A,B\}) \). Hence,
\]

\[
\frac{db_A(t)}{dd_A(t)} \leq 0.
\]

As a result, \( b_A(t) \) is a decreasing function of \( d_A(t) \) or \( t \). Therefore, Bidder \( t \) nearest to Item A \((i.e., t = \arg \min d_A(t) = \arg \min t_i - \alpha \) will submit the highest bid and win Item A.

**Proposition 3** actually gives nonlinear conditions of efficient auctions in the class Hotelling model. Concurrently, its proof implies the following corollary.

**Corollary 2** Let \( \alpha \leq \beta, \tau(d_k) \geq 0, k \in \{A,B\} \) and \( 1 + \frac{\partial \sigma(v_b(t), x)}{\partial x} > 0 \). The auction for Item A is efficient while any one of the following conditions is satisfied.

1. \( \tau''(d_k) \geq 0, \) and \( t_i \geq \alpha \) \((i = 1, 2, \ldots, n)\), namely all bidders are on the right of Item A.
2. \( \tau''(d_k) < 0, \) and \( t_i \leq \beta \) \((i = 1, 2, \ldots, n)\), namely all bidders are on the left of Item B.
3. All bidders are between A and B.
(4) $\tau(d_i)$ is linear, and Bidders’ locations are arbitrary in $[0, 1]$.

**Remark 4** Corollary 2 shows that efficiencies of the sequential auctions described by the class Hotelling model are conditional. More specifically, the auctions of supply contracts in Section 1 should take into consideration locations of suppliers and agents in a traffic line.

4. Bids and Information Policies about $\beta$

The key of determining the equilibrium bid $b_{\lambda}(t)=v_{\lambda}(t)-\sigma(v_{b}(t),b_{\lambda}(t))$ in Proposition 1 is the expected surplus $\sigma(v_{b}(t),b_{\lambda}(t))=\int_{0}^{t}(v_{b}(t)-z)d\Phi_{\alpha}(z)$, in which both $v_{b}(t)$ and $\Phi_{\alpha}(z)$ are closely related to the location of Item B. Thus, in order to obtain the concrete formula of $\sigma(v_{b}(t),b_{\lambda}(t))$, we need to consider information policies about $\beta \in [0,1]$, namely revealing Item B or hiding Item B before the auction of Item A ends. Here, revealing Item B means that all bidders know $\beta$, and hiding Item B means that all bidders do not know $\beta$ until the first auction ends. In this section, we will apply the class Hotelling model to deduce some concrete equilibrium bids of sequential auctions of close substitutes.

Intuitively, any bidder in auctions subjectively regards himself or herself as a marginal loser [12] or winner, because he or she will rationally give up auctions if he or she feels no opportunities to win. Here, a marginal loser or winner means the loser with the highest bid or the winner with the lowest bid. Specifically in our class Hotelling model with $\tau(d_i)=d_k$, the bidder nearest to Item A will win according to Proposition 2 or Proposition 3. Thus, Bidder $t$ regarding himself or herself as a marginal loser or winner of Item A always believes that his or her rest opponents are distributed on his right (namely in $[t, 1]$), except no more than one opponent in $[0, t)$. Such an idea of a preponderant rival exactly leads to the bidding equilibrium mentioned by Proposition 1. For convenience, Bidder $i$ is called a preponderant rival of Item A if and only if there is at most one opponent in $[0, t)$.

In order to facilitate discussions, we assume in this section

1) $V_{\lambda}=1$ and $V_{b}=1$.

2) $\alpha=0$ and $\tau(d_k)=d_k$ for $k \in \{A,B\}$.

The above assumptions imply that all bidders are on the right of Item A, namely $t \geq \alpha = 0$, and $\tau(d_k)$ is linear. Furthermore, let $f(x)$ $(F(x))$ and $g(x)$ $(G(x))$ denote the probability density (probability distribution) of $t$ and one of $\beta$, respectively.

With the help of Proposition 1 and Corollary 2, we will deduce the expected surplus $\sigma(v_{b}(t),b_{\lambda}(t))$ of Bidder $t$ as a preponderant rival, and verify and prove that $b_{\lambda}(t)=v_{\lambda}(t)-\sigma(v_{b}(t),b_{\lambda}(t))$ is exactly his or her equilibrium bid for Item A.

4.1 Equilibrium Bids when Revealing $\beta$

Each bidder knows Item A' location $\alpha=0$, Item B’s location $\beta \in [0,1]$, his or her own location (or type) $t \in [0,1]$ and bidders’ number $n$, and does not know other bidders’ locations (or types). But he or she regards other bidders’ locations as random variables distributed in $[0,1]$. First, all $n$ bidders submit their sealed bids for Item A. Then, the bidder with the highest bid wins Item A, pays the second highest bid, and exits from the second auction. Next, the remaining $n-1$ bidders submit their sealed bids for Item B. Finally, the bidder with the highest bid wins Item B and pays the second highest bid.
First, let \( \pi^y_i(v_b(t)) \) denote Bidder \( i \)'s expected surplus of winning Item B while he or she faces a group of opponents distributed in the interval \([x, y]\), where \( x, y \in [0,1] \) and \( x \leq y \). According to Proposition 2, a preponderant rival of Item A can only defeat the bidders who are far away from Item A. Obviously, the probability density of the location of an opponent in \([t,1]\) is \( \frac{f(s)}{1-F(t)} \) in which \( s \in [t,1] \). Thus, the probability and the probability density that all locations of \( n-2 \) opponents in \([t,1]\) are greater than \( s \in [t,1] \) should be

\[
\left( \frac{1-F(s)}{1-F(t)} \right)^{n-2} \text{ and } \frac{n-2}{1-F(t)} \left( \frac{1-F(s)}{1-F(t)} \right)^{n-3},
\]

respectively. Therefore, while Bidder \( t \) only considers opponents on his or her right, his or her expected surplus of winning Item B should be

\[
\pi^y_{\pi}(v_b(t)) = \int_{t}^{y} \left( \frac{n-2}{1-F(t)} \left( \frac{1-F(s)}{1-F(t)} \right)^{n-3} \right) (v_b(t) - v_b(s)) ds,
\]

where \( t \leq x \leq y \leq 1 \).

Then, in order to determine the expected surplus \( \sigma(v_b(t),b_i(t)) \) of Bidder \( t \) as a preponderant rival of Item A in Figure 2, the interval \([0, 1]\) is divided into three subintervals, namely \( W_1=[0, t] \), \( W_2=[t, (1+t)/2] \) and \( W_3=[(1+t)/2,1] \). Thus, three cases need to be considered for \( \beta \).

**Please insert Figure 2 here**

Figure 2 shows the case of \( W_2 \). In Figure 2, \( t^\prime \) denotes the symmetric location of \( t \) in \([0,1]\) with respect to \( \beta \), namely \( t^\prime = 2\beta - t \). Obviously, bids of opponents in \([t, t^\prime] \) for Item B should be greater than one of Bidder \( t \). Again because opponents are distributed in \([0,1] \), \( \sigma(v_b(t),b_i(t)) \) is only relevant to the opponents in \([0, t] \) and \([t^\prime,1] \). As a preponderant rival of Item A, Bidder \( t \) subjectively believes that his or her \( n-2 \) opponents in the second stage should be distributed on his or her right, namely in \([t,1] \). Therefore, \( \sigma(v_b(t),b_i(t)) = \pi^y_{\pi}(v_b(t)) \). And the other cases can be inferred similarly.

Finally, a concrete formula of \( \sigma(v_b(t),b_i(t)) \) is related to the specific probability distribution of \( t \) and one of \( \beta \). Further, we assume in Section 4: \( t \) and \( \beta \) is uniformly distributed in \([0,1]\).

Namely, \( f(x) = 1 \), \( g(x) = 1 \), \( F(x) = x \) and \( G(x) = x \) while \( x \in [0,1] \). In conclusions, the expected surplus \( \sigma(v_b(t),b_i(t)) \) in various cases can be shown as below.

1) \( \beta \in W_1 \):

\[
\sigma(v_b(t),b_i(t)) = \pi^y_{\pi}(v_b(t)) = \int_{t}^{1} \frac{n-2}{1-F(t)} \left( \frac{1-F(s)}{1-F(t)} \right)^{n-3} (s-t) ds = \frac{1-t}{n-1} \tag{5}
\]

2) \( \beta \in W_2 \):

\[
\sigma(v_b(t),b_i(t)) = \pi^y_{\pi,\beta}(v_b(t)) = \int_{2\beta-t}^{1} \frac{n-2}{1-F(t)} \left( \frac{1-F(s)}{1-F(t)} \right)^{n-3} (t+s-2\beta) ds = \frac{(1-2\beta+t)^{n-1}}{(n-1)(1-t)^{n-2}} \tag{6}
\]

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3) $\beta \in W3:
\sigma(v_b(t), b_\lambda(t)) = 0
(7)

**Proposition 4** If $\beta$ is revealed before auctioning Item A, then Bidder $i$’s equilibrium bid for Item A is $b_\lambda(t) = 1 - t - \sigma(v_b(t), b_\lambda(t))$, where $\sigma(v_b(t), b_\lambda(t))$ is shown as Equation (5), (6) and (7).

**Proof.** Let $x = b_\lambda(t) = \max_{\theta \in \Theta} b_\lambda(\theta)$ be the highest bid of Bidder $i$’s n-1 opponents in the first stage, and $h(x)$ denote the probability density of $x$. Usually, the equilibrium bid $b_\lambda(t)$ should maximize Bidder $i$’s expected surplus $\tau(b_\lambda(t))$ in the first auction, namely

$$
\max_{b_\lambda(t)} \tau(b_\lambda(t)) = \max_{b_\lambda(t)} \left( \int_{0}^{b_\lambda(t)} (V_\lambda(t) - x) h(x) dx + \int_{b_\lambda(t)}^{t} \sigma(v_b(t), x) h(x) dx \right)
$$

Let

$$
d\tau(b_\lambda(t)) = \frac{d\tau(b_\lambda(t))}{db_\lambda(t)} = \frac{(V_\lambda(t) - b_\lambda(t)) h(b_\lambda(t)) - \sigma(v_b(t), b_\lambda(t)) h(b_\lambda(t)) = 0, then}{\int_{b_\lambda(t)}^{t} \sigma(v_b(t), x) h(x) dx}
$$

$$
V_\lambda(t) - b_\lambda(t) - \sigma(v_b(t), b_\lambda(t)) = 0
$$

Hence, according to Proposition 1, we have $1 + \frac{d\sigma(v_b(t), x)}{dx} > 0$. Again on the base of Corollary 2, $x \geq b_\lambda(t)$ in Equation (8) means $t_i \leq t$. And $t_i \leq \theta \leq 1$ for $\forall \theta \in \Omega(x) = \{\theta | b_\lambda(\theta) \leq x\}$ while $x \geq b_\lambda(t)$.

$$
\therefore \Phi_{n-2-\theta}(z, \Omega(x)) = \mathbb{P}\left\{ \max_{\theta \in \Theta(x)} v_b(\theta) \leq z \right\}
$$

$$
= \left( \mathbb{P}\left\{ v_b(\theta) \leq z \text{ while } \theta \in \Omega(x) = \{\theta | b_\lambda(\theta) \leq x\} \text{ and } t_i \leq \theta \leq 1 \right\} \right)^{n-2}
$$

$$
= \left( \int_{b_\lambda(t)}^{t} \frac{f(\theta) d\theta}{1-t_i} \right)^{n-2}
$$

where $z \geq V_\lambda(1) = 1 - (1 - \beta) = \beta$ and $f(\theta) = 1$ in $[0, 1]$.

1) $\beta \in W1$: $\sigma(v_b(t), x) = \int_{0}^{a(t)} v_b(t) - z d\Phi_{n-2-\theta}(z, \Omega(x))$

$$
= \int_{0}^{a(t)} (1 + \beta - t - z) \frac{(n-2)}{1-t_i} \frac{(z-\beta)^{n-3}}{(n-1)(1-t_i)^{n-2}} dz = \frac{(1-t)^{n-1}}{(n-1)(1-t_i)^{n-2}}
$$

According to Corollary 2, $x = b_\lambda(t)$ means $t_i = t$. Hence, $\sigma(v_b(t), b_\lambda(t)) = \frac{1-t}{n-1}$.

2) $\beta \in W2$: $\sigma(v_b(t), x) = \int_{0}^{a(t)} v_b(t) - z d\Phi_{n-2-\theta}(z, \Omega(x))$

$$
= \int_{0}^{a(t)} (1 - \beta + t - z) \frac{(n-2)}{1-t_i} \frac{(z-\beta)^{n-3}}{(n-1)(1-t_i)^{n-2}} dz = \frac{(1-2\beta+t)^{n-1}}{(n-1)(1-t_i)^{n-2}}
$$

Hence, $\sigma(v_b(t), b_\lambda(t)) = \frac{(1-2\beta+t)^{n-1}}{(n-1)(1-t_i)^{n-2}}$.

3) $\beta \in W3$: $\beta \geq (1+t)/2$ means $v_b(t) = 1 - \beta + t \leq \beta$. Then, $\sigma(v_b(t), x) = \int_{0}^{a(t)} v_b(t) - z d\Phi_{n-2-\theta}(z, \Omega(x)) = 0$. Hence, $\sigma(v_b(t), b_\lambda(t)) = 0$.

Therefore, Bidder $i$’s equilibrium bid for Item A should be $b_\lambda(t) = 1 - t - \sigma(v_b(t), b_\lambda(t))$. 


where $\sigma(v_n(t), b_n(t))$ is shown as Equation (5), (6) and (7).

It is also easy to verify $\frac{db_n(t)}{dt} < 0$ by use of Equation (5), (6) and (7). Thus, Item A will be allocated to the bidder with the highest valuation, namely one closest to A.

**Remark 5** Equation (5), (6) and (7) imply that Bidder $t = 1$ or $t \leq 2\beta - 1$ always bids his or her true valuation in the first stage because he or she as a preponderant rival of Item A believes in advance that his or her expected surplus of winning Item B will be 0. However, Bidders $t = 0$ always hides his or her surplus $\frac{(1-2\text{Max}(|\beta, 1/2|))^{n-1}}{(n-1)}$ in the first stage by making use of the advantage of his or her location.

### 4.2 Equilibrium Bids when Hiding $\beta$

Item B (namely $\beta$) is hidden before the end of auctioning A. Each bidder knows $\alpha = 0$, his or her own location (or type) $t$ and bidders’ number $n$, and doesn’t know $\beta$ and other bidders’ locations (or types). But he or she regards other bidders’ locations as random variables distributed in $[0,1]$, and believes that Item B is also distributed in $[0,1]$. Similarly to Section 4.1, the key to determine the equilibrium bid is the expected surplus of winning Item B.

While $\beta$ is regarded as a random variable, the Bidder $t$’s expected surplus of winning B is $\sigma(\cdot, b_n(t)) = E_{\beta} \left[ \sigma(v_n(t), b_n(t)) \right]$, and the Bidder $t$’s equilibrium bid for Item A should be $b_n(t) = v_n(t) - \sigma(\cdot, b_n(t))$. Hence,

$$
\sigma(\cdot, b_n(t)) = E_{\beta} \left[ \sigma(v_n(t), b_n(t)) \right] = \int_0^1 \sigma(v_n(t), b_n(t)) g(\beta) d\beta 
$$

$$
= \int_0^{1-t} \left[ (1-2\beta + t) \right] \frac{(1-2\beta + t)^{n-1}}{(n-1)(1-t)^{n-1}} d\beta + \int_{1-t}^{1} \left[ (1-t) \right] \frac{(1-t)(1+(2n-1)t)}{2n(n-1)} d\beta
$$

**Proposition 5** If $\beta$ is hidden before the end of auctioning Item A, then Bidder $t$’s equilibrium bid for Item A is $b_n(t) = 1-t - \sigma(\cdot, b_n(t))$, where $\sigma(\cdot, b_n(t))$ is as Equation (9).

Proposition 5’s proof is similar to Proposition 4’s one. Also, $\frac{db_n(t)}{dt} = \frac{d}{dt} \left[ 1-t - \left( \frac{1+2(n-1)t-(2n-1)t^2}{2n(n-1)} \right) \right] = \frac{2-n}{n-1} < 0 (\because n \geq 3)$. Thus, $b_n(t)$ is monotonically decreasing with $t$, which is consistent with Proposition 2. Concurrently, $\frac{db_n(t)}{dt} < 0$ also indicates that Item A will be allocated to the bidder with the highest valuation, namely one closest to Item A.

**Remark 6** According to Equation (9), $\sigma(\cdot, b_n(0)) = \frac{1}{2n(n-1)}$, $\sigma(\cdot, b_n(0.5)) = \frac{2n+1}{8n(n-1)}$ and $\sigma(\cdot, b_n(1)) = 0$. It can easily be proved that $\sigma(\cdot, b_n(t))$ reaches the maximum $\frac{n}{2(n-1)(2n-1)}$ while $t = \frac{n-1}{2n-1}$, and $\sigma(\cdot, b_n(t))$ is monotonically increasing with $t$ in $\left[0, \frac{n-1}{2n-1}\right]$ and monotonically decreasing in $\left[\frac{n-1}{2n-1}, 1\right]$. Thus, bidders in $\left[0, \frac{n-1}{2n-1}\right]$ have advantages over ones.
4.3 Further Researches

Let $R_k(n)$ denote the auctioneer’s expected revenue from selling Item $k \in \{A,B\}$ while $\beta$ is revealed, and $R'_k(n)$ denote one while $\beta$ is hidden, where $n$ is the number of bidders. Then, $R(n)=R_A(n)+R_B(n)$ is the auctioneer’s total expected revenue in two auctions while $\beta$ is revealed, and $R'(n)=R'_A(n)+R'_B(n)$ is one while $\beta$ is hidden. In the second stage, dominant strategies of bidders are to bid their own valuations. Accordingly, expected revenues from selling Item B are the same under the above two information policies, namely $R_B(n)=R'_B(n)$. Thus, the difference between the total expected revenues $R(n)$ and $R'(n)$ are mainly determined by $R_A(n)$ and $R'_A(n)$.

Different from bidders, an auctioneer only knows $\alpha$, $\beta$, and $n$ in advance, and does not know locations of bidders, but believes that bidders are distributed in $[0, 1]$. Therefore, the expected revenue from selling Item A, $R_A(n)$ or $R'_A(n)$, is determined by the expectation of $\max_{t} b_A(t)$ or $\max_{t} b'_A(t)$. Thus while $F(t)=t$ or $f(t)=1$ in $[0,1]$,

$$R_A(n) = \int_{0}^{1} (1-t - \sigma(v_{bt}(t), b_A(t))) \cdot n(1-F(t))^{n-1} dt$$

$$= \int_{0}^{1} (1-t) \cdot n(1-t)^{n-1} dt - \int_{0}^{1} \sigma(b_{bt}(t), b_A(t)) \cdot n(1-t)^{n-1} dt + \int_{0}^{1} \frac{1-t}{n-1} \cdot n(1-t)^{n-1} dt$$

$$= \begin{cases} 
\frac{1}{n+1} - \frac{2}{n-1} (1-\beta)^{n+1}, & \text{for } \beta \in \left[0, \frac{1}{2}\right] \\
\frac{1}{n+1} - \frac{2}{n-1} (1-\beta)^{n+1}, & \text{for } \beta \in \left[\frac{1}{2}, 1\right] 
\end{cases}$$

(10)

$$R'_A(n) = \int_{0}^{1} (1-t - \sigma(v_{bt}(t), b'_A(t))) \cdot n(1-F(t))^{n-1} dt$$

$$= \int_{0}^{1} (1-t) \cdot \frac{2(n-1)r-(2n-1)r^2}{2n(n-1)} \cdot n(1-t)^{n-1} dt = 1 - \frac{(2n-1)(n+3)}{2(n-1)(n+1)(n+2)}$$

(11)

Curve clusters of $R_A(n)$ and $R'_A(n)$ with $n$ and $\beta$ are shown in Figure 3 and in Figure 4, respectively. They visually show the following propositions.

Proposition 6 $R_A(n)$ is monotonically increasing functions of $n$ and $\beta$; $R'_A(n)$ is monotonically increasing functions of $n$.

Proof. 1) While $\beta \in \left[\frac{1}{2}, 1\right]$, $R_A(n) = 1 - \frac{1}{n+1} - \frac{2}{n-1} (1-\beta)^{n+1}$. Obviously, $R_A(n)$ is monotonically increasing functions of $n$ and $\beta$.

While $\beta \in \left[0, \frac{1}{2}\right]$, $R'_A(n) = 1 - \frac{1}{n+1} - \frac{2}{n-1} (1-\beta)^{n+1} + \frac{(n+2-2\beta)(1-2\beta)^n}{(n+1)(n-1)}$. Then,

$$\frac{dR_A(n)}{d\beta} = \frac{2(n+1)}{n-1} (1-\beta)^{n-1} - \frac{2(1-2\beta)^n + 2n(n+2-2\beta)(1-2\beta)^{n-1}}{(n+1)(n-1)}$$

$$= \frac{2}{n-1} \left[ (n+1)(1-\beta)^n - n(1-2\beta)^{n-1} - (1-2\beta)^{n-1} \right]$$
\[
\frac{dR_A(n)}{d\beta} \geq 0, \text{ namely } R_A(n) \text{ is monotonically increasing functions of } \beta.
\]

Again according to Equation (9), let

\[
R_A(x) = \int_0^1 x(1-t)^x \, dt - \int_0^\beta \frac{x}{(x-1)} (1-2\beta + t)^{x-1} (1-t) \, dt \]

\[
+ \int_0^\beta \frac{x}{x-1} (1-t)^x \, dt = \frac{x}{x+1} - \left[\int_0^\beta h_1(x) \, dt + \int_0^\beta h_2(x) \, dt\right].
\]

Then,

\[
\frac{dh_1(x)}{dx} = \frac{d}{dx} \left(\frac{x}{(x-1)} (1-2\beta + t)^{x-1} (1-t)\right)
\]

\[
= \frac{(1-2\beta + t)^{x-1} (1-t)}{(x-1)^2} \left[(x-1) \ln(1-2\beta + t) - 1\right] \leq 0 \quad (\because x \geq 3 \text{ and } 0 \leq 1-2\beta + t \leq 1)
\]

\[
\frac{dh_2(x)}{dx} = \frac{d}{dx} \left(\frac{x}{x-1} (1-t)^x\right) = \frac{(1-t)^x}{(x-1)^2} \left[(x-1) \ln(1-t) - 1\right] \leq 0
\]

\[
\therefore \frac{\partial R_A(x)}{\partial x} = \frac{1}{(x+1)^2} - \left[\int_0^\beta \frac{dh_1(x)}{dx} \, dt + \int_0^\beta \frac{dh_2(x)}{dx} \, dt\right] > 0
\]

Hence, \( R_A(n) \) or \( R_A(x) \) is monotonically increasing functions of \( n \) or \( x \).

2) \( R_A(n) = 1 - \frac{(2n-1)(n+3)}{2(n-1)(n+1)(n+2)} = 1 - \frac{2n-2}{n^2 - 1} - \frac{5}{2(n-1)(n+2)} \). Obviously, \( R_A(n) \) is monotonically increasing functions of \( n \), owing to \( n \geq 3 \).

Equation (10) and (11) show complicated influences of \( \beta \) and \( n \) on the expected revenue. Here, both Proposition 6 and Figure 4 imply that if \( n \) is given, then these must exist \( \beta_0 \) satisfying \( R_A(n) \leq R_\lambda(n) \) while \( \beta \leq \beta_0 \), and \( R_\lambda(n) \geq R_A(n) \) while \( \beta \geq \beta_0 \). But it is difficult to fix the exact value of \( \beta_0 \) with \( n \). Next, Proposition 7 provides some sufficient conditions about comparisons of both \( R_A(n) \) and \( R_\lambda(n) \).

**Proposition 7** Let \( n \geq 3 \). 1) \( R_A(n) \leq R_\lambda(n) \) while \( \beta \leq 1 - \left(\frac{3n+1}{4(n+1)(n+2)}\right)^{1/n+1} \); 2) \( R_\lambda(n) > R_A(n) \) while \( \beta \geq 1 - \left(\frac{3n+1}{4(n+1)(n+2)}\right)^{1/n+1} \).

**Proof.** 1) While \( \beta \leq 1 - \left(\frac{3n+1}{2n(n+2)}\right)^{1/n+1} \), \( (1-\beta)^{n+1} \geq \frac{3n+1}{2n(n+2)} \).

\[
\therefore (1-\beta)^2 \geq 1-2\beta, \quad \frac{(n+2-2\beta)}{(n+1)(n-1)} (1-2\beta)^x \leq \frac{(n+2)}{(n+1)(n-1)} (1-\beta)^{n+1}.
\]

Hence,
\[
\left(1 - \frac{1}{n+1} \cdot \frac{2}{n-1} (1-\beta)_n^{\ast +1} + \frac{(n+2-2\beta)(1-2\beta)^n}{(n+1)(n-1)}\right) - \left(1 - \frac{(2n-1)(n+3)}{2(n-1)(n+1)(n+2)}\right) \leq 0 \quad (12)
\]

Inequation (12) means \( R_A(n) \leq R_A(n) \) for \( \beta \in [0, \frac{1}{2}] \), and also means \( R_A(n) \leq R_A(n) \) for \( \beta \in [\frac{1}{2}, 1] \).

Hence, according to Proposition 6, \( R_A(n) \leq R_A(n) \) for \( \beta \in [0,1] \).

2) While \( \beta \geq 1\): \(\frac{3n+1}{4(n+1)(n+2)} \), \( (1-\beta)_n^{\ast +1} \leq \frac{3n+1}{4(n+1)(n+2)} \). Then,

\[
\left(1 - \frac{1}{n+1} - \frac{2}{n-1} (1-\beta)_n^{\ast +1}\right) - \left(1 - \frac{(2n-1)(n+3)}{2(n-1)(n+1)(n+2)}\right) \geq 0 \quad (13)
\]

Inequation (13) means \( R_A(n) \geq R_A(n) \) for \( \beta \in [\frac{1}{2}, 1] \). \( \vdots \) \( \left(\frac{n+2-2\beta}{(n+1)(n-1)}\right)^n \geq 0 \).

Inequation (13) also means \( R_A(n) \geq R_A(n) \) for \( \beta \in [0, \frac{1}{2}] \).

Hence, according to Proposition 6, \( R_A(n) \geq R_A(n) \) for \( \beta \in [0,1] \).

Here, \( R_A(n) \leq R_A(n) \) \( \rightarrow \) \( R_A(n) > R_A(n) \) means that expected revenue for A while revealing \( \beta \) is less (more) than one while hiding \( \beta \). Proposition 7 implies that revealing latter items in advance would uncertainly affect overall efficiencies and revenues of sequential auctions, which is consistent with Cason [5], Jane [13], Mikusheva [9], Jackson [10], Kannan [11], Rao et al. [16, 17], and Colucci et al. [18]. Owing to \( R_B(n) = R_B(n) \), Proposition 7 is also sufficient to compare \( R(n) \) and \( R(n) \).

Please insert Figure 3 here

Please insert Figure 4 here

5. Conclusions

Focusing on sequential auctions of close substitutes with slightly more general associated valuations, this paper constructs a class Hotelling model and discusses equilibrium bids under second-price sealed-bid auction formats. Conclusions show that sequential auctions described by this model are efficient while bidders’ valuations satisfy conditions given by Corollary 2. Thus, the class Hotelling model can be used for supporting to solve some auctions in supply chains, and also are helpful for analysis and designs of some business mechanisms.

With the aid of this model, we specifically explore some sequential auctions while a bidder’s valuation is a linear function of a distance between him or her and an item. The equilibrium bid of a bidder as a preponderant rival is deduced and verified. And it depends on both numbers of bidders and locations of items whether the latter item (namely Item B) should be revealed or hidden. Generally speaking, revealing information would usually improve revenues of auctions with assumptions of independent valuations for multi-items. However, our conclusions are more complicated because each bidder’s valuations for Item A and Item B are not independent in our paper, but are correlated. In this paper, though the sequential auctions with only two items seem simple or far-fetched, a characterization in the class Hotelling model
for two items is an important step to achieving similar characterization for models with more than two items.

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References

Figure 1. The class Hotelling Model

Figure 2. The case of $W_2$

Figure 3 Curve clusters of $R_A(n)$ and $R_A'(n)$ as functions of $n$
Figure 4 Curve clusters of $R_{\lambda}(n)$ and $R'_{\lambda}(n)$ as functions of $\beta$

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