Bonferroni harmonic mean operators based on two-dimensional uncertain linguistic information and their applications in land utilization ratio evaluation

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Abstract. The Bonferroni Mean (BM) enjoys the advantage of capturing the interrelationship among the input arguments, and the harmonic mean is a conservative average lying between the max and min operators. The 2-dimensional uncertain linguistic variables add a subjective evaluation of the reliability of the evaluation results given by decision-makers; therefore, they can express fuzzy information better. In this paper, in order to combine their advantages, the 2-Dimensional Uncertain Linguistic Weighted Bonferroni Mean (2DULWBM) operator is first proposed. However, it cannot consider the case when the given arguments are too high or too low. Therefore, we further propose the 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Harmonic Mean (2DULIWBHM) operator, which combines the 2DULWBM with Harmonic Mean. Furthermore, some desirable properties and their special cases are studied. Further, a new method is developed to deal with some Multi-Attribute Group Decision Making (MAGDM) problems under 2-dimensional uncertain linguistic environment based on the proposed operators. Finally, an illustrative example is given to testify the validity of the developed method by comparing it with the other existing methods.

1. Introduction

The Multi-Attribute Decision Making (MADM) problems are common in real decision-making. Due to the multiple-attribute decision-making methods proposed by Churchman et al. [1], the classical (i.e., deterministic environment) MADM theory and methods have be developed. However, the attribute values of decision-making problems are not always expressed by crisp numbers because of the fuzziness, and some of them are more suitable for expressing by fuzzy information [2-6], such as fuzzy numbers, linguistic variable [7],

intuitionistic fuzzy numbers [8], etc. Particularly, for some qualitative information, we can easily conduct the subjective evaluation by linguistic information. For example, regarding some evaluation problems, such as site-choosing project and vehicle performance, their values can be easily expressed by linguistic terms, such as "very bad," "bad," "general," "good," and "very good". Zadeh [9-11] presented the concept of the linguistic variable firstly, which laid the foundation for the linguistic multiple-attribute decision-making methods; then, it was extended to many different types of fuzzy information, such as uncertain linguistic variables [12-16], 2-dimensional linguistic information [17], etc. Zhu et al. [18] proposed the 2-dimensional linguistic evaluation information to process the linguistic fuzzy decision problems, meaning that decision-makers use both Class I and Class II linguistic information to

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describe their judgment on an object. For example, if a vehicle performance is to be evaluated, Class I linguistic variable is used to perform personal assessment, and Class II linguistic variable is used to evaluate the reliability of assessment results. That is to say, decision-makers can use a 2-dimensional linguistic variable to describe their evaluation of an object. The first variable is used to describe the evaluation result of an attribute given by a decision-maker; the other is used to describe the subjective evaluation of decision-makers on the reliability of their given results. Liu and Zhang [19] further extended 2-dimensional linguistic information to 2-Dimensional Uncertain Linguistic Information (2DULI) and developed a method to deal with the Multiple Attribute Group Decision Making (MAGDM) problems in which the attribute weight is unknown and attribute values take the form of 2DULI. Then, Liu and Yu [20] extended Power Average (PA) to 2DULI and proposed a 2-dimensional uncertain linguistic power generalized weighted aggregation operator, which could detect the effects of unreasonable data from biased decision-makers. Obviously, 2-Dimensional Uncertain Linguistic Variable (2DULV) can more accurately reflect the evaluation of decision-makers on objects.

The aggregation operators [21-29] have become an important research topic for MADM problems, because they have more advantages than some traditional approaches such as TOPSIS [30], VIKOR [31], ELEC-TRE [32], TODIM [33], MULTIMOORA [34], and so on. In general, the basic function of the aggregation operators [8,35] is that they can aggregate a set of real values into one. Zhang et al. [36] proposed some power aggregation operators for Intuitionistic Fuzzy Numbers (IFNs), which can consider the influences of unreasonable attribute values based on power weights. Wang et al. [37] proposed some dependent aggregation operators for IFNs, which can eliminate the influences of unreasonable data by dependent weights. Meng et al. [38] proposed the Choquet aggregation operator for IFNs, considering the interaction among aggregating parameters. However, neither of them can consider the relationship between any two attributes. Bonferroni introduced the Bonferroni Mean (BM) operator [39], which can capture the correlations between the input arguments. Further, Wei et al. [40] extended the BM to the uncertain linguistic variables and proposed an Uncertain Linguistic Bonferroni Mean (ULBM) operator. Liu et al. [41] proposed some intuitionistic fuzzy Dombi Bonferroni mean operators. Ding and Wu [42] proposed some interval-valued hesitant fuzzy Bonferroni mean operators. However, BM cannot consider the case when the given arguments have a great difference; therefore, Sun and Sun [43] combined BM with harmonic operator and proposed Bonferroni Harmonic Mean (BHM), which can be more than

adequate to solve the MADM or MAGDM problems by considering the interrelationships and harmonicity.

To sum up the above discussion, the 2-Dimensional Uncertain Linguistic Variable (2DULV) can describe qualitative information better, and the BHM operator can consider the interrelationships and harmonicity; therefore, it is meaningful to extend the BHM to the 2DULV. The aim of this paper is to combine the 2DULVs with the BHM operator and to develop the Improved Weighted Bonferroni Harmonic Mean (IWBHM) operator to overcome the weaknesses of the existing operators. Thanks to the advantages of the newly proposed operator, it can capture the interrelationship among the input arguments that have the flexibility by Bonferroni mean parameters , and it can also consider the case when the given arguments have a great difference.

The rest of this paper is organized as follows. In Section 2, the concept and basic operations of 2DULVs, the Bonferroni Mean (BM) operator, 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Mean (2DULIWBM) operator, and Improved Weighted Bonferroni Harmonic Mean (IWBHM) operators are briefly reviewed. In Section 3, the 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Harmonic Mean (2DULI-WBHM) operator on the basis of the 2DULVs rules is proposed. In Section 4, a decision-making method based on the proposed operators for the MADM problems is proposed in which attribute values take the form of 2-dimensional uncertain linguistic information. In Section 5, an example is used to illustrate the effectiveness of the proposed new method, which is compared with the existing other methods. The conclusions are discussed in Section 6.

2. Preliminaries

In this section, some basic concepts including the concept of uncertain linguistic variables, 2DULVs, BM operator, 2-Dimensional Uncertain Linguistic Weighted Bonferroni Mean (2DULWBM) operator, the 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Mean (2DULIWBM) operator, and Improved Weighted Bonferroni Harmonic Mean (IWBM) operator are introduced.

2.1. The uncertain linguistic variables
Suppose that linguistic assessment set $[S = (s_\beta | \beta = 0, 1, \ldots, t - 1)$ is a finite and totally ordered discrete terms set, S_β represents the linguistic variable, and t is the odd number. In practice, t is set to 5, 7, 9, etc. For instance, when $t = 5$, it is represented as follows: $S = (S_1, S_2, S_3, S_4) =$ (very bad, bad, fair, good, very good).

In order to overcome the loss of information

during the operational process, discrete linguistic set $S = (s_\beta | \beta = 0, 1, \ldots, t-1)$ is extended to a continuous linguistic set $\tilde{S} = (s_\beta | \beta \in R)$ by Xu [44], where β is the large enough number, and if $s_\beta \in S$, then s_β is called an original linguistic term, or else s_β is called an extended linguistic term.

Definition 1 [12]. Suppose that $\tilde{s} = [s_a, s_b]$, $s_a, s_b \in \tilde{S}$, and $0 \leq a \leq b$; S_a and S_b are the lower and upper limits of $\tilde{s}$, respectively, and then $\tilde{s}$ is called an uncertain linguistic variable.

Definition 2 [12,44]. Let $\tilde{S}$ be a set of all uncertain linguistic variables, and $\tilde{s}_1 = [s_{a_1}, s_{b_1}]$, $\tilde{s}_2 = [s_{a_2}, s_{b_2}]$ be any two uncertain linguistic variables and $\lambda \geq 0$; the operation rules are defined as follows:

$$\tilde{s}_1 \oplus \tilde{s}_2 = [s_{a_1}, s_{b_1}] \oplus [s_{a_2}, s_{b_2}] = [s_{a_1+a_2}, s_{b_1+b_2}], \quad (1)$$

$$\tilde{s}_1 \otimes \tilde{s}_2 = [s_{a_1}, s_{b_1}] \otimes [s_{a_2}, s_{b_2}] = [s_{a_1 \times a_2}, s_{b_1 \times b_2}], \quad (2)$$

$$\lambda \tilde{s}_1 = \lambda [s_{a_1}, s_{b_1}] = [s_{\lambda a_1}, s_{\lambda b_1}], \quad (3)$$

$$(\tilde{s}_1)^\lambda = [s_{(a_1)^\lambda}, s_{(b_1)^\lambda}]. \quad (4)$$

However, the uncertain linguistic variable is just used to describe linguistic evaluation result of an attribute given by a decision-maker. It cannot describe the subjective evaluation of decision-makers on the reliability of their given results. To solve this problem, the 2DULV was proposed by Liu and Zhang [19].

2.2. The 2-Dimensional Uncertain Linguistic Variable (2DULV)

Definition 3 [20]. Let $\hat{s}_1 = ([\dot{s}_a, \dot{s}_b] [\ddot{s}_c, \ddot{s}_d])$, where $[\dot{s}_a, \dot{s}_b]$ is Class I uncertain linguistic variable, which represents decision-maker's judgment on an evaluated object, $\dot{s}_a$ and $\dot{s}_b$, are the elements from predefined linguistic evaluation set $S_I = (\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_{t-1})$, while $[\ddot{s}_c, \ddot{s}_d]$ is Class II uncertain linguistic variable which represents the reliability of the subjective evaluation of their given results, $\ddot{s}_c$ and $\ddot{s}_d$, are the elements from the predefined linguistic evaluation set $S_{II} = (\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_{L-1})$; then, $\hat{s}$ is called the 2-dimensional uncertain linguistic variable.

In order to reduce the loss of linguistic information as well as linguistic variable, the discrete linguistic assessment sets of 2-dimensional uncertain linguistic information are extended to continuous linguistic assessment sets, such that $\dot{s}_a, \dot{s}_b \in \tilde{S}_I = (\dot{s}_\beta | \beta \in [0, l])$ and $\ddot{s}_c, \ddot{s}_d \in \tilde{S}_{II} = \{\ddot{s}_\beta | \beta \in [0, l']\}$, where l and l' are the two sufficiently large numbers. At the same time, $\hat{S}$ is expressed as the set of all 2-dimensional uncertain linguistic variables.

2.3. The operational rules and characteristics of 2DULVs

Definition 4 [20]. Let $\hat{s}_1 = ([\dot{s}_{a_1}, \dot{s}_{b_1}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}])$ and

$\hat{s}_2 = ([\dot{s}_{a_2}, \dot{s}_{b_2}] [\ddot{s}_{c_2}, \ddot{s}_{d_2}])$ be any two 2DULVs and $\lambda \geq 0$. Then, the operational rules of 2DULVs are defined as follows:

$$\hat{s}_1 \oplus \hat{s}_2 = ([\dot{s}_{a_1+a_2}, \dot{s}_{b_1+b_2}] [\ddot{s}_{\min(c_1,c_2)}, \ddot{s}_{\min(d_1,d_2)}]), \quad (5)$$

$$\hat{s}_1 \otimes \hat{s}_2 = ([\dot{s}_{a_1 \times a_2}, \dot{s}_{b_1 \times b_2}] [\ddot{s}_{\min(c_1,c_2)}, \ddot{s}_{\min(d_1,d_2)}]), \quad (6)$$

$$\lambda \hat{s}_1 = ([\dot{s}_{\lambda \times a_1}, \dot{s}_{\lambda \times b_1}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}]) \lambda \geq 0, \quad (7)$$

$$(\hat{s}_1)^\lambda = ([\dot{s}_{(a_1)^\lambda}, \dot{s}_{(b_1)^\lambda}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}]). \quad (8)$$

When $\lambda < 0$, the operational rule (4) is defined as follows:

$$(\hat{s}_1)^\lambda = ([\dot{s}_{(b_1)^\lambda}, \dot{s}_{(a_1)^\lambda}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}]). \quad (9)$$

Theorem 1 [20]. Let $\hat{s}_1 = ([\dot{s}_{a_1}, \dot{s}_{b_1}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}])$, $\hat{s}_2 = ([\dot{s}_{a_2}, \dot{s}_{b_2}] [\ddot{s}_{c_2}, \ddot{s}_{d_2}])$, and $\hat{s}_3 = ([\dot{s}_{a_3}, \dot{s}_{b_3}] [\ddot{s}_{c_3}, \ddot{s}_{d_3}])$ be three 2DULVs, and λ, λ_1, and $\lambda_2 \geq 0$. Then:

$$\hat{s}_1 \oplus \hat{s}_2 = \hat{s}_2 \oplus \hat{s}_1, \quad (10)$$

$$\hat{s}_1 \otimes \hat{s}_2 = \hat{s}_2 \otimes \hat{s}_1, \quad (11)$$

$$\hat{s}_1 \oplus \hat{s}_2 \oplus \hat{s}_3 = \hat{s}_1 \oplus (\hat{s}_2 \oplus \hat{s}_3), \quad (12)$$

$$\hat{s}_1 \otimes \hat{s}_2 \otimes \hat{s}_3 = \hat{s}_1 \otimes (\hat{s}_2 \otimes \hat{s}_3), \quad (13)$$

$$\hat{s}_1 \otimes (\hat{s}_2 \oplus \hat{s}_3) = (\hat{s}_1 \otimes \hat{s}_2) \oplus (\hat{s}_1 \otimes \hat{s}_3), \quad (14)$$

$$\lambda (\hat{s}_1 \oplus \hat{s}_2) = (\lambda \hat{s}_1) \oplus (\lambda \hat{s}_2), \quad (15)$$

$$(\lambda_1 + \lambda_2) \hat{s}_1 = (\lambda_1 \hat{s}_1) \oplus (\lambda_2 \hat{s}_1). \quad (16)$$

Definition 5 [20]. Let $\hat{s}_1 = ([\dot{s}_{a_1}, \dot{s}_{b_1}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}])$ be a 2DULV. Then, the expected value $E(\hat{s}_1)$ of $\hat{s}_1$ is defined as follows:

$$E(\hat{s}_1) = \frac{a_1 + b_1}{2 \times (t-1)} \times \frac{c_1 + d_1}{2 \times (L-1)}, \quad (17)$$

where t and L can be defined in Definition 3.

Let $\hat{s}_1 = ([\dot{s}_{a_1}, \dot{s}_{b_1}] [\ddot{s}_{c_1}, \ddot{s}_{d_1}])$ and $\hat{s}_2 = ([\dot{s}_{a_2}, \dot{s}_{b_2}] [\ddot{s}_{c_2}, \ddot{s}_{d_2}])$ be any two 2DULVs. If $E(\hat{s}_1) \geq E(\hat{s}_2)$, then $\hat{s}_1 \geq \hat{s}_2$, or vice versa.

Example 1. If $t = 7$, $L = 5$, and $\hat{s}_1 = ([\dot{s}_4, \dot{s}_5] [\ddot{s}_2, \ddot{s}_3])$, $\hat{s}_2 = ([\dot{s}_3, \dot{s}_4] [\ddot{s}_1, \ddot{s}_2])$, then:

$$E(\hat{s}_1) = \frac{4+5}{2 \times 6} \times \frac{2+3}{2 \times 4} = \frac{15}{32},$$

$$E(\hat{s}_2) = \frac{3+4}{2 \times 6} \times \frac{1+2}{2 \times 4} = \frac{7}{32}.$$

Therefore, we have $\hat{s}_1 > \hat{s}_2$.

2.4. BM and 2DULIWBM operators

Definition 6 [39]. Let $p, q \geq 0$ and $\tilde{a}_i$ $(i = 1, 2, \cdots, n)$ be a collection of non-negative numbers. If:

$$BM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \tilde{a}_i^p \tilde{a}_j^q \right)^{\frac{1}{p+q}}, \quad (18)$$

then $BM^{p,q}$ is called the Bonferroni Mean (BM).

Obviously, the BM has the following properties:

1. Property 1 (Idempotency): $BM^{p,q}(\tilde{a}, \tilde{a}, \cdots, \tilde{a}) = \tilde{a}$.

2. Property 2 (Monotonicity): if $\tilde{a}_i \geq \tilde{b}_i$ $(i = 1, 2, ..., n)$ then $BM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \geq BM^{p,q}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n)$.

3. Property 3 (Boundedness): $\min_i\{\tilde{a}_i\} \leq BM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \leq \max_i\{\tilde{a}_i\}$.

The Bonferroni Mean (BM) operator can consider the interrelationships between the aggregated parameters. To consider the advantages of 2DULV and BM, the 2-Dimensional Uncertain Linguistic Weighted Bonferroni Mean (2DULWBM) is proposed.

Definition 7. Let $p, q \geq 0$, and:

$$\hat{s}_i = \left([\dot{s}_{a_i}, \dot{s}_{b_i}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}] \right) \quad (i = 1, 2, \ldots, n)$$

be a collection of 2DULVs with the weight vector:

$$w = (w_1, w_2, \cdots, w_n)^T$$

such that $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Then, the 2-dimensional uncertain linguistic weighted Bonferroni mean operator can be defined as follows:

$$2DULWBM^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((nw_i \hat{s}_i^p) \otimes (nw_j \hat{s}_j^q) \right) \right)^{\frac{1}{p+q}}. \quad (19)$$

It is obvious that Eq. (19) does not have the idempotency; thus, the improved 2-Dimensional Uncertain Linguistic Weighted Bonferroni Mean (2DULIWBM) operator is proposed as follows:

$$2DULIWBM^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \frac{1}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((w_i \hat{s}_i^p) \otimes (w_j \hat{s}_j^q) \right) \right)^{\frac{1}{p+q}}. \quad (20)$$

Based on the operational laws of the 2DULVs, the results are derived, as shown in Theorem 2.

Theorem 2. Let $p, q \geq 0$, and:

$$\hat{s}_i = \left([\dot{s}_{a_i}, \dot{s}_{b_i}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}] \right) \quad (i = 1, 2, \ldots, n),$$

be a collection of 2DULVs with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$; then, the result aggregated from Eq. (20) is still an 2DULV, and is calculated by Eq. (21) as shown in Box I.

Proof. On the basis of Definition 4, the aggregated value is also a 2DULV. Therefore, we can prove Eq. (21) by using a mathematical induction on n.

Firstly, we need to prove Eq. (22) shown in Box II.

According to the operations of 2DULVs defined in Eqs. (6)-(8), we have:

$$w_i \hat{s}_i^p = \left([\dot{s}_{w_i a_i p}, \dot{s}_{w_i b_i p}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}] \right),$$

$$w_j \hat{s}_j^q = \left([\dot{s}_{w_j a_j q}, \dot{s}_{w_j b_j q}] [\ddot{s}_{c_j}, \ddot{s}_{d_j}] \right), \quad (23)$$

$$2DULIWBM^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$$

$$= \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i a_i p)(w_j a_j q)) \right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i b_i p)(w_j b_j q)) \right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}}} \right] \left[\ddot{s}_{\min(c_i, c_j)}, \ddot{s}_{\min(d_i, d_j)} \right] \right). \quad (21)$$

<div align="center">Box I</div>

$$\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right) = \left(\left[s_{\left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\sum}} \left(\left(w_i a_i^{\ p} \right) \left(w_j a_j^{\ q} \right) \right) \right)}, s_{\left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\sum}} \left(\left(w_i b_i^{\ p} \right) \left(w_j b_j^{\ q} \right) \right) \right)} \right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)} \right] \right). \quad (22)$$

<div align="center">Box II</div>

$$\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right)$$

$$= \left(\left[\dot{s}_{(w_i a_i^{\ p})(w_j a_j^{\ q})}, \dot{s}_{(w_i b_i^{\ p})(w_j b_j^{\ q})} \right] \right.$$

$$\left. \ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)} \right). \quad (24)$$

When $n = 2$, by Eqs. (5) and (24), we can obtain:

$$\underset{\substack{i,j=1 \\ i \neq j}}{\overset{2}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right) = \left(w_1 \hat{s}_1^p \right) \otimes \left(w_2 \hat{s}_2^q \right)$$

$$+ \left(w_2 \hat{s}_2^p \right) \otimes \left(w_1 \hat{s}_1^q \right)$$

$$= \left(\left[s_{\left((w_1 a_1^{\ p})(w_2 a_2^{\ q}) + (w_2 a_2^{\ p})(w_1 a_1^{\ q}) \right)}, \right. \right.$$

$$\left. s_{\left((w_1 b_1^{\ p})(w_2 b_2^{\ q}) + (w_2 b_2^{\ p})(w_1 b_1^{\ q}) \right)} \right],$$

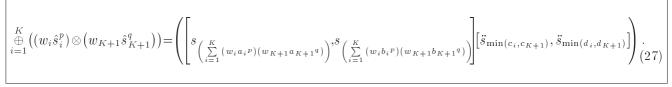$$\left. \left[\ddot{s}_{\min(c_1,c_2)}, \ddot{s}_{\min(d_1,d_2)} \right] \right),$$

that is, when $n = 2$, Eq. (22) is right.

If Eq. (22) holds for $n = K$, then we have Eq. (25) shown in Box III.

When $n = K + 1$, we have:

$$\underset{\substack{i,j=1 \\ i \neq j}}{\overset{K+1}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right) = \underset{\substack{i,j=1 \\ i \neq j}}{\overset{K}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \right.$$

$$\left. \otimes \left(w_j \hat{s}_j^q \right) \right) + \underset{i=1}{\overset{K}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \right.$$

$$\left. \otimes \left(w_{K+1} \hat{s}_{K+1}^q \right) \right) + \underset{j=1}{\overset{K}{\oplus}} \left(\left(w_{K+1} \hat{s}_{K+1}^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right). \quad (26)$$

Firstly, we prove Eq. (27) shown in Box IV. The mathematical induction on K is used as follows.
(a) When $K = 2$, we get:

$$\left(w_i \hat{s}_i^p \right) \otimes \left(w_3 \hat{s}_3^q \right)$$

$$= \left(\left[s_{\left((w_i a_i^{\ p})(w_3 a_3^{\ q}) \right)}, s_{\left((w_i b_i^{\ p})(w_3 b_3^{\ q}) \right)} \right], \right.$$

$$\left. \left[\ddot{s}_{\min(c_i,c_3)}, \ddot{s}_{\min(d_i,d_3)} \right] \right) \underset{i=1}{\overset{2}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \right.$$

$$\left. \otimes \left(w_{K+1} \hat{s}_{K+1}^q \right) \right) = \left(w_1 \hat{s}_1^p \right) \otimes \left(w_3 \hat{s}_3^q \right)$$

$$+ \left(w_2 \hat{s}_2^p \right) \otimes \left(w_3 \hat{s}_3^q \right)$$

$$= \left(\left[s_{\left((w_1 a_1^{\ p})(w_3 a_3^{\ q}) + (w_2 a_2^{\ p})(w_3 a_3^{\ q}) \right)}, \right. \right.$$

$$\underset{\substack{i,j=1 \\ i \neq j}}{\overset{K}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right) = \left(\left[s_{\left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{K}{\sum}} \left((w_i a_i^{\ p})(w_j a_j^{\ q}) \right) \right)}, s_{\left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{K}{\sum}} \left((w_i b_i^{\ p})(w_j b_j^{\ q}) \right) \right)} \right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)} \right] \right). \quad (25)$$

<div align="center">Box III</div>

$$\underset{i=1}{\overset{K}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_{K+1} \hat{s}_{K+1}^q \right) \right) = \left(\left[s_{\left(\underset{i=1}{\overset{K}{\sum}} (w_i a_i^{\ p})(w_{K+1} a_{K+1}^{\ q}) \right)}, s_{\left(\underset{i=1}{\overset{K}{\sum}} (w_i b_i^{\ p})(w_{K+1} b_{K+1}^{\ q}) \right)} \right] \left[\ddot{s}_{\min(c_i,c_{K+1})}, \ddot{s}_{\min(d_i,d_{K+1})} \right] \right). \quad (27)$$

<div align="center">Box IV</div>

$$\underset{i=1}{\overset{T+1}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_{T+2} \hat{s}_{T+2}^q \right) \right) = \underset{i=1}{\overset{T}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_{T+2} \hat{s}_{T+2}^q \right) \right) + \left(w_{T+1} \hat{s}_{T+1}^p \right) \otimes \left(w_{T+2} \hat{s}_{T+2}^q \right)$$

$$= \left(\left[s_{\left(\sum\limits_{i=1}^{T+1} (w_i a_i{}^p)(w_{T+2} a_{T+2}{}^q) \right)}, s_{\left(\sum\limits_{i=1}^{T+1} (w_i b_i{}^p)(w_{T+2} b_{T+2}{}^q) \right)} \right] \left[\ddot{s}_{\min(c_i, c_{T+2})}, \ddot{s}_{\min(d_i, d_{T+2})} \right] \right).$$

Box V

$$\underset{j=1}{\overset{K}{\oplus}} \left(\left(w_{K+1} \hat{s}_{K+1}^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right)$$

$$= \left(\left[s_{\left(\sum\limits_{j=1}^{K} (w_{K+1} a_{K+1}{}^p)(w_j a_j{}^q) \right)}, s_{\left(\sum\limits_{j=1}^{K} (w_{K+1} b_{K+1}{}^p)(w_j b_j{}^q) \right)} \right] \left[\ddot{s}_{\min(c_{K+1}, c_j)}, \ddot{s}_{\min(d_{K+1}, d_j)} \right] \right). \tag{28}$$

Box VI

$$s_{\left((w_1 b_1{}^p)(w_3 b_3{}^q) + \left(w_2 b_2{}^p \right)\left(w_3 b_3{}^q \right) \right)} \Big],$$

$$\left[\ddot{s}_{\min(c_i, c_3)}, \ddot{s}_{\min(d_i, d_3)} \right] \right).$$

In other words, when $n = 2$, Eq. (27) is right.

(b) If Eq. (27) holds for $K = T$, then:

$$\underset{i=1}{\overset{T}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_{T+1} \hat{s}_{T+1}^q \right) \right)$$

$$= \left(\left[s_{\left(\sum\limits_{i=1}^{T} (w_i a_i{}^p)(w_{T+1} a_{T+1}{}^q) \right)}, s_{\left(\sum\limits_{i=1}^{T} (w_i b_i{}^p)(w_{T+1} b_{T+1}{}^q) \right)} \right] \right.$$

$$\left. \left[\ddot{s}_{\min(c_i, c_{T+1})}, \ddot{s}_{\min(d_i, d_{T+1})} \right] \right).$$

When $K = T+1$, we have the relation shown in Box V. In other words, for $K = T+1$, Eq. (27) is also right. Similarly, we can obtain Eq. (28) as shown in Box VI.

Based on Eqs. (25), (27), and (28), we can obtain Eq. (26) as shown in Box VII. Therefore, when $n = K + 1$, Eq. (22) is also right, and it is right for all n.

Now, it is easy to prove that Eq. (21) is right. By Eq. (22), we obtain the relation shown in Box VIII. Hence, for all n, Eq. (21) is true.
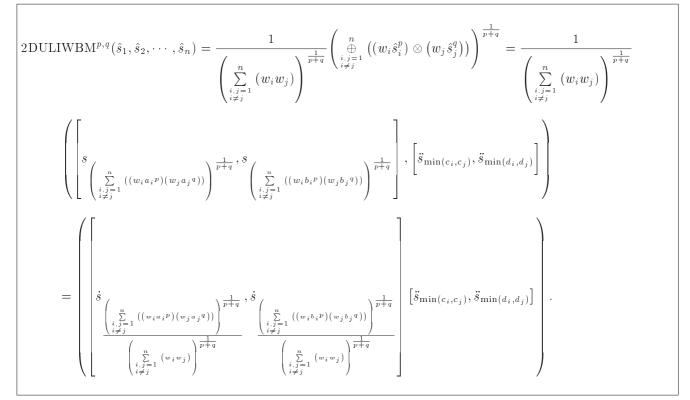
Obviously, Eq. (21) has also the properties similar to those of the $BM^{p,q}$: idempotency, monotonicity, commutativity, and boundedness.

Example 2. Let $\hat{s}_1 = ([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_3])$, $\hat{s}_2 = ([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$, and $\hat{s}_3 = ([\dot{s}_7, \dot{s}_9], [\ddot{s}_2, \ddot{s}_2])$ be three 2DULVs (suppose that Class I linguistic set is $S_I = (\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_9)$ and Class II linguistic set is $S_{II} = (\ddot{s}_0, \ddot{s}_1, \ddot{s}_2, \ddot{s}_3, \ddot{s}_4)$), and $w = (0.4, 0.25, 0.35)^T$ be the weight vector of $\hat{s}_i (i = 1, 2, 3)$. Then, the operator 2DULIWBM can be used to aggregate three 2DULVs as follows.

The comprehensive value $\hat{s} = ([\dot{s}_a, \dot{s}_b] [\ddot{s}_c, \ddot{s}_d])$ can be calculated by Eq. (21) (without loss of generality, we suppose that $p = q = 1$), and we obtain the relation shown in Box IX.

$$\underset{\substack{i,j=1 \\ i \neq j}}{\overset{K+1}{\oplus}} \left(\left(w_i \hat{s}_i^p \right) \otimes \left(w_j \hat{s}_j^q \right) \right) = \left(\left[s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{K+1} ((w_i a_i{}^p)(w_j a_j{}^q)) \right)}, s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{K+1} ((w_i b_i{}^p)(w_j b_j{}^q)) \right)} \right], \left[\ddot{s}_{\min(c_i, c_j)}, \ddot{s}_{\min(d_i, d_j)} \right] \right).$$

Box VII

$$2\text{DULIWBM}^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n) = \frac{1}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j)\right)^{\frac{1}{p+q}}} \left(\mathop{\oplus}\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left((w_i \hat{s}_i^p) \otimes (w_j \hat{s}_j^q)\right)\right)^{\frac{1}{p+q}} = \frac{1}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j)\right)^{\frac{1}{p+q}}}$$

$$\left(\left[s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i a_i^p)(w_j a_j^q))\right)^{\frac{1}{p+q}}}, s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i b_i^p)(w_j b_j^q))\right)^{\frac{1}{p+q}}}\right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right)$$

$$= \left(\left[\dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i a_i^p)(w_j a_j^q))\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i b_i^p)(w_j b_j^q))\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j)\right)^{\frac{1}{p+q}}}}\right] \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right).$$

Box VIII

$$2\text{DULIWBM}^{1,1}(\hat{s}_1, \hat{s}_2, \hat{s}_3)$$

$$= \left(\left[\dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} ((w_i a_i)(w_j a_j))\right)^{\frac{1}{1+1}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} (w_i w_j)\right)^{\frac{1}{1+1}}}}, \dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} ((w_i b_i)(w_j b_j))\right)^{\frac{1}{1+1}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{3} (w_i w_j)\right)^{\frac{1}{1+1}}}}\right] \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right) = \left(\left[\dot{s}_{3.664}, \dot{s}_{4.982}\right] \left[\ddot{s}_2, \ddot{s}_2\right]\right).$$

Box IX

However, BM operator cannot relieve the effects when the given arguments are too high or too low. Then, Sun and Sun [43] proposed Weighted Bonferroni Harmonic Mean (WBHM) operator to solve this problem.

2.5. Improved Weighted Bonferroni Harmonic Mean (IWBHM) operator

The Weighted Bonferroni Harmonic Mean (WBHM) operator was proposed by Sun and Sun [43], and it is

combined by Bonferroni Mean (BM) [39] and Harmonic Mean (HM) [45], which is defined as follows.

Definition 8 [43]. Let $p, q \geq 0$ and $\tilde{a}_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum\limits_{i=1}^{n} w_i = 1$. The WBHM is defined as follows:

$$WBHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \cfrac{1}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{\tilde{a}_i^p \tilde{a}_j^q} \right)^{\frac{1}{p+q}}}. \quad (29)$$

It is obvious that Eq. (29) does not have idempotency; thus, the Improved Weighted Bonferroni Harmonic Mean (IWBHM) operator is proposed as follows.

Definition 9. Let $p, q \geq 0$, and $\tilde{a}_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers with the weight vector, $w = (w_1, w_2, \cdots, w_n)^T$, such that $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum\limits_{i=1}^{n} w_i = 1$. The IWBHM operator is defined as follows:

$IWBHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$

$$= \cfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{w_i}{\tilde{a}_i^p} \right) \left(\frac{w_j}{\tilde{a}_j^q} \right) \right)^{\frac{1}{p+q}}}. \quad (30)$$

3. 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Harmonic Mean (2DULIWBHM) operator

In this section, based on the operational rules of 2DULV and the improved weighted Bonferroni harmonic mean, the 2-Dimensional Uncertain Linguistic Improved Weighted Bonferroni Harmonic Mean (2DULIWBHM) operator is proposed; then, some particular cases and properties will be investigated.

Definition 10. Let $\hat{s}_i = ([\dot{s}_{a_i}, \dot{s}_{b_i}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}])$ ($i = 1, 2, \ldots, n$) be a collection of 2DULVs with weight

vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} \omega_i = 1$; then, the 2-dimensional uncertain linguistic improved weighted Bonferroni harmonic mean operator is defined as follows:

$2DULIWBHM^{p,q}(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$

$$= \left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}$$

$$\left(\cfrac{1}{\bigoplus\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{w_i}{\hat{s}_i^p} \right) \otimes \left(\frac{w_j}{\hat{s}_j^q} \right) \right)} \right)^{\frac{1}{p+q}}. \quad (31)$$

Based on the operational laws of the 2DULVs, the aggregated result from Definition 10 is derived, as shown in Theorem 3.

Theorem 3. Let $p, q \geq 0$, and $\hat{s}_i = ([\dot{s}_{a_i}, \dot{s}_{b_i}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}])$ ($i = 1, 2, \ldots, n$) be a collection of 2DULVs with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_i \geq 0$ ($i = 1, 2, \cdots, n$); then, the result aggregated from Eq. (31) is still an 2DULV, and Eq. (32) shown in Box X is obtained.

Proof. Firstly, we can calculate $\hat{s}_i^p$ and $\hat{s}_j^q$ and obtain:

$\hat{s}_i^p = ([\dot{s}_{a_i^p}, \dot{s}_{b_i^p}] [\ddot{s}_{c_i}, \ddot{s}_{d_i}])$,

$\hat{s}_j^q = ([\dot{s}_{a_j^q}, \dot{s}_{b_j^q}] [\ddot{s}_{c_j}, \ddot{s}_{d_j}])$,

and they are also 2DULVs. Then, we obtain:

$\cfrac{w_i}{\hat{s}_i^p} = \left(\left[\dot{s}_{\frac{w_i}{b_i^p}}, \dot{s}_{\frac{w_i}{a_i^p}} \right] [\ddot{s}_{c_i}, \ddot{s}_{d_i}] \right)$,

$\cfrac{w_j}{\hat{s}_j^q} = \left(\left[\dot{s}_{\frac{w_j}{b_j^q}}, \dot{s}_{\frac{w_j}{a_j^q}} \right] [\ddot{s}_{c_j}, \ddot{s}_{d_j}] \right)$,

---

$2DULIWBHM^{p,q}(\hat{s}_1, \hat{s}_2, \ldots \hat{s}_n)$

$$= \left(\left[\dot{s}_{\cfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{w_i}{a_i^p} \right) \left(\frac{w_j}{a_j^q} \right) \right) \right)^{\frac{1}{p+q}}}}, \dot{s}_{\cfrac{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} (w_i w_j) \right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{w_i}{b_i^p} \right) \left(\frac{w_j}{b_j^q} \right) \right) \right)^{\frac{1}{p+q}}}} \right] , [\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}] \right). \quad (32)$$

Box X

and:

$$\left(\frac{w_i}{\dot{s}_i^p}\right) \otimes \left(\frac{w_j}{\dot{s}_j^q}\right) = \left(\left[\dot{s}_{\left(\frac{w_i}{b_i^p}\right)\left(\frac{w_j}{b_j^q}\right)}, \dot{s}_{\left(\frac{w_i}{a_i^p}\right)\left(\frac{w_j}{a_j^q}\right)}\right]\right.$$

$$\left.\left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right).$$

Then, we obtain the relation shown in Box XI. Therefore, we have the relation shown in Box XII. So, Theorem 3 is proved.

Example 3. If we solve Example 2 according to Theorem 3, then we obtain the relation shown in Box XIII.

Obviously, the result is smaller than Example 2, because our method can eliminate the effect when the given data are too high or too low. In this example, Class I uncertain linguistic variable of $\hat{s}_3$ is $[\dot{s}_7, \dot{s}_9]$, which is extremely bigger than the other two. The 2DULIWBM operator cannot consider this case; however, the proposed operator in this paper can tackle this shortcoming.

It is easy to prove that the 2DULIWBHM operator has the following properties.

Theorem 4 (Idempotency). If $\hat{s}_j = \left(\left[\dot{s}_{a_j}, \dot{s}_{b_j}\right]\left[\ddot{s}_{c_j}, \ddot{s}_{d_j}\right]\right)$ $j = 1,2\ldots,n$ all are equal, i.e., $\hat{s}_j = \hat{s} = \left(\left[\dot{s}_a, \dot{s}_b\right]\left[\ddot{s}_c, \ddot{s}_d\right]\right)$ for all j, then:

$$2DULIWBHM^{p,q}(\hat{s}, \hat{s}, ..., \hat{s}) = \hat{s}. \tag{33}$$

Proof. The relation shown in Box XIV completes the proof of Theorem 4.

Theorem 5 (commutativity). If $(\hat{s}'_1, \hat{s}'_2, ..., \hat{s}'_n)$ is any permutation of $(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$, then:

$$2DULIWBHM^{p,q}(\hat{s}'_1, \hat{s}'_2, ..., \hat{s}'_n)$$
$$= 2DULIWBHM^{p,q}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n). \tag{34}$$

Proof. Based on Eq. (32), we have the relation shown in Box XV. Since $(\hat{s}'_1, \hat{s}'_2, ..., \hat{s}'_n)$ is any permutation of $(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$, then we can get:
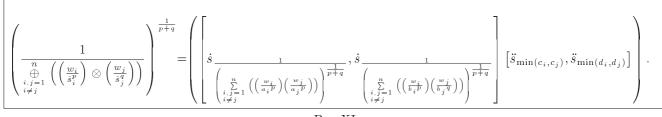$$2DULIWBHM^{p,q}(\hat{s}'_1, \hat{s}'_2, ..., \hat{s}'_n)$$
$$= 2DULIWBHM^{p,q}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n).$$

Theorem 6 (monotonicity). Let:
$$\hat{s}_\eta = \left(\left[\dot{s}_{a_\eta}, \dot{s}_{b_\eta}\right], \left[\ddot{s}_{c_\eta}, \ddot{s}_{d_\eta}\right]\right)$$
and:

$$\left(\frac{1}{\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\oplus}}\left(\left(\frac{w_i}{\dot{s}_i^p}\right)\otimes\left(\frac{w_j}{\dot{s}_j^q}\right)\right)}\right)^{\frac{1}{p+q}} = \left(\left[\dot{s}_{\frac{1}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i^p}\right)\left(\frac{w_j}{a_j^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{1}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i^p}\right)\left(\frac{w_j}{b_j^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right]\left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right).$$

Box XI

$$\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}\left(\frac{1}{\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\oplus}}\left(\left(\frac{w_i}{\dot{s}_i^p}\right)\otimes\left(\frac{w_j}{\dot{s}_j^q}\right)\right)}\right)^{\frac{1}{p+q}}$$

$$= \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i^p}\right)\left(\frac{w_j}{a_j^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i^p}\right)\left(\frac{w_j}{b_j^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right).$$

Box XII

$$
\begin{aligned}
\text{2DULIWBHM}^{1,1}\left(\hat{s}_1, \hat{s}_2, \hat{s}_3\right) &= \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{3} (w_i w_j)\right)^{\frac{1}{2}} \left(\frac{1}{\underset{\substack{i,j=1 \\ i \neq j}}{\overset{3}{\oplus}}\left(\left(\frac{w_i}{\hat{s}_i^{\frac{1}{2}}}\right) \otimes \left(\frac{w_j}{\hat{s}_j^{\frac{1}{2}}}\right)\right)}\right)^{\frac{1}{2}} \\
&= \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{3}(w_i w_j)\right)^{\frac{1}{2}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{3}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{2}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{3}(w_i w_j)\right)^{\frac{1}{2}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{3}\left(\left(\frac{w_i}{b_i{}^p}\right)\left(\frac{w_j}{b_j{}^q}\right)\right)\right)^{\frac{1}{2}}}}\right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right) = \left(\left[\dot{s}_{3.259}\right]\left[\dot{s}_{4.531}\right]\left[\ddot{s}_2, \ddot{s}_2\right]\right).
\end{aligned}
$$

<div align="center">Box XIII</div>

$$
\begin{aligned}
&\text{2DULIWBHM}^{p,q}\left(\hat{s}, \hat{s}, \ldots, \hat{s}\right) \\
&= \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(\left(\frac{w_i}{a^p}\right)\left(\frac{w_j}{a^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(\left(\frac{w_i}{b^p}\right)\left(\frac{w_j}{b^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right], \left[\ddot{s}_{\min(c,c)}, \ddot{s}_{\min(d,d)}\right]\right) \\
&= \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{a}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}(w_i w_j)\right)^{\frac{1}{b}}}}\right], \left[\ddot{s}_c, \ddot{s}_d\right]\right) = \left(\left[\dot{s}_a, \dot{s}_b\right], \left[\ddot{s}_c, \ddot{s}_d\right]\right).
\end{aligned}
$$

<div align="center">Box XIV</div>

$$
\hat{s}'_\eta = \left(\left[\dot{s}_{a'_\eta}, \dot{s}_{b'_\eta}\right], \left[\ddot{s}_{c'_\eta}, \ddot{s}_{d'_\eta}\right]\right)
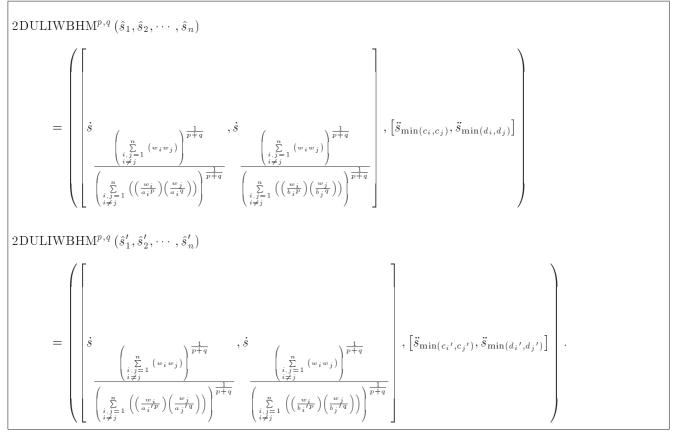$$

$$
(\eta = 1, 2, \ldots, n)
$$

be any two collections of 2DULVs. If $a_\eta \geq a'_\eta$, $b_\eta \geq b'_\eta$, $c_\eta \geq c'_\eta$, and $d_\eta \geq d'_\eta$ for all η, then:

$$
\text{2DULIWBHM}^{p,q}\left(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n\right)
$$

$$
\geq \text{2DULIWBHM}^{p,q}\left(\hat{s}'_1, \hat{s}'_2, \ldots, \hat{s}'_n\right). \tag{35}
$$

Proof. Since $a_\eta \geq a'_\eta$, $b_\eta \geq b'_\eta$, $c_\eta \geq c'_\eta$, $d_\eta \geq d'_\eta$, and $p, q > 0$, then based on Eq. (32), we have the relation shown in Box XVI.
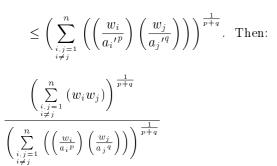
Since $a_\eta \geq a'_\eta$, we get:

$$
\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}
$$

$2\text{DULIWBHM}^{p,q}\left(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n\right)$

$$= \left(\left[\dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i{}^p}\right)\left(\frac{w_j}{b_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right)$$

$2\text{DULIWBHM}^{p,q}\left(\hat{s}_1', \hat{s}_2', \cdots, \hat{s}_n'\right)$

$$= \left(\left[\dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i'^p}\right)\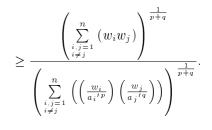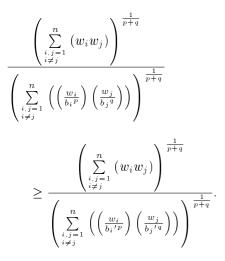left(\frac{w_j}{a_j'^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i'^p}\right)\left(\frac{w_j}{b_j'^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right], \left[\ddot{s}_{\min(c_i',c_j')}, \ddot{s}_{\min(d_i',d_j')}\right]\right).$$

<div style="text-align:center">Box XV</div>

$2\text{DULIWBHM}^{p,q}\left(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n\right)$

$$= \left(\left[\dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i{}^p}\right)\left(\frac{w_j}{b_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}}}\right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)}\right]\right).$$

<div style="text-align:center">Box XVI</div>

$$\leq \left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i'^p}\right)\left(\frac{w_j}{a_j'^q}\right)\right)\right)^{\frac{1}{p+q}}. \text{ Then:}$$

$$\dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{p+q}}}$$

$$\geq \dfrac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p+q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i'^p}\right)\left(\frac{w_j}{a_j'^q}\right)\right)\right)^{\frac{1}{p+q}}}.$$

Similarly, we also have:

$$\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i^{\,p}}\right)\left(\frac{w_j}{b_j^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}$$

$$\geq \frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i'^{\,p}}\right)\left(\frac{w_j}{b_j'^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}.$$

Further, we have:

$$\min\left(\ddot{s}_{c_1},\ddot{s}_{c_2},\ldots,\ddot{s}_{c_n}\right) \geq \min\left(\ddot{s}'_{c_1},\ddot{s}'_{c_2},\ldots,\ddot{s}'_{c_n}\right),$$

$$\min\left(\ddot{s}_{d_1},\ddot{s}_{d_2},\ldots,\ddot{s}_{d_n}\right) \geq \min\left(\ddot{s}'_{d_1},\ddot{s}'_{d_2},\ldots,\ddot{s}'_{d_n}\right).$$

Finally, we obtain the relation shown in Box XVII.

That is, $2\text{DULIWBHM}^{p,q}(\hat{s}_1,\hat{s}_2,....,\hat{s}_n) \geq 2\text{DULIWBHM}^{p,q}(\hat{s}'_1,\hat{s}'_2,....,\hat{s}'_n)$.

Thus, the proof of Theorem 6 is completed.

Theorem 7 (boundedness). Let:

$$\hat{s}_j = \left(\left[\dot{s}_{a_j},\dot{s}_{b_j}\right],\left[\ddot{s}_{c_j},\ddot{s}_{d_j}\right]\right) \quad (j=1,2,\ldots,n)$$

be a collection of 2DULVs, and:

$$\hat{s}^- = \left(\left[\dot{s}_{\min_j(a_j)},\dot{s}_{\min_j(b_j)}\right],\left[\ddot{s}_{\min_j(c_j)},\ddot{s}_{\min_j(d_j)}\right]\right)$$

$$\hat{s}^+ = \left(\left[\dot{s}_{\max_j(a_j)},\dot{s}_{\max_j(b_j)}\right],\left[\ddot{s}_{\min_j(c_j)},\ddot{s}_{\min_j(d_j)}\right]\right).$$

Then:

$$\hat{s}^- \leq 2\text{DULIWBHM}^{p,q}(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n) \leq \hat{s}^+. \qquad (36)$$

Proof. Since $\hat{s}_j \geq \hat{s}^-$, based on Theorems 4 and 6, we have:

$$2\text{DULIWBHM}^{p,q}(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n)$$

$$\geq 2\text{DULIWBHM}^{p,q}(\hat{s}^-,\hat{s}^-,\ldots,\hat{s}^-) = \hat{s}^-.$$

Similarly, we also have:

$$2\text{DULIWBHM}^{p,q}(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n)$$

$$\leq 2\text{DULIWBHM}^{p,q}(\hat{s}^+,\hat{s}^+,\ldots,\hat{s}^+) = \hat{s}^+.$$
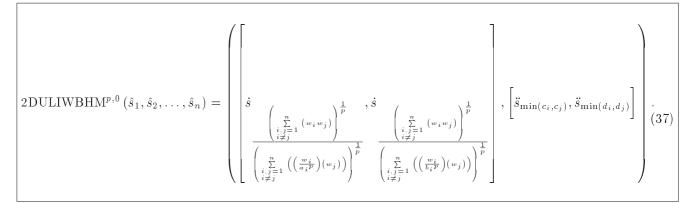
Then, we obtain:

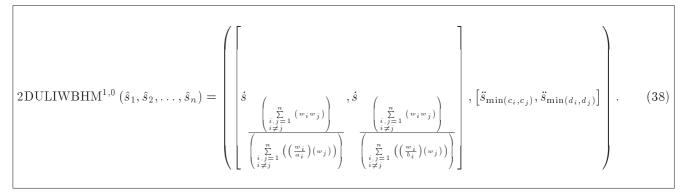$$\hat{s}^- \leq 2\text{DULIWBHM}^{p,q}(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n) \leq \hat{s}^+,$$

which completes the proof of Theorem 7.

In the following, we will study some particular cases of the $2\text{DULIWBHM}^{p,q}$ operator with regard to parameters p and q.

$$\left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i^{\,p}}\right)\left(\frac{w_j}{a_j^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i^{\,p}}\right)\left(\frac{w_j}{b_j^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}}\right],\left[\ddot{s}_{\min(c_i,c_j)},\ddot{s}_{\min(d_i,d_j)}\right]\right)$$

$$\geq \left(\left[\dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a'_i^{\,p}}\right)\left(\frac{w_j}{a'_j^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}}, \dot{s}_{\frac{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{p+q}}}{\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b'_i^{\,p}}\right)\left(\frac{w_j}{b'_j^{\,q}}\right)\right)\right)^{\frac{1}{p+q}}}}\right],\left[\ddot{s}_{\min(c'_i,c'_j)},\ddot{s}_{\min(d'_i,d'_j)}\right]\right).$$

Box XVII

$$
\text{2DULIWBHM}^{p,0}\left(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n\right) = \left(\left[\dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i{}^p}\right)\left(w_j\right)\right)\right)^{\frac{1}{p}}}}, \dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i{}^p}\right)\left(w_j\right)\right)\right)^{\frac{1}{p}}}} \right], \left[\dddot{s}_{\min(c_i,c_j)},\dddot{s}_{\min(d_i,d_j)}\right] \right). \tag{37}
$$

<div align="center">Box XVIII</div>

$$
\text{2DULIWBHM}^{1,0}\left(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n\right) = \left(\left[\dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i}\right)\left(w_j\right)\right)\right)}}, \dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i}\right)\left(w_j\right)\right)\right)}} \right], \left[\dddot{s}_{\min(c_i,c_j)},\dddot{s}_{\min(d_i,d_j)}\right] \right). \tag{38}
$$

<div align="center">Box XIX</div>

(i)   When $q \to 0$, we can obtain Eq. (37) shown in Box XVIII;

(ii) When $p = 1$, $q \to 0$, we can obtain Eq. (38) shown in Box XIX;

(iii) When $p \to 0$, we can obtain Eq. (39) shown in Box XX;

(iv) When $p = q = 1$, we can obtain Eq. (40) shown in Box XXI;

4. A group decision-making approach based on the 2DULIWBHM operator

In this section, we will apply the 2DULIWBHM operator to solve the MAGDM problems.

A MAGDM problem with 2DULVs is described as follows. Suppose that $\{A_1, A_2, \cdots, A_m\}$ is a set of alternatives, and $\{C_1, C_2, \cdots, C_n\}$ is a set of attributes whose weight vector is $w = (w_1, w_2, \cdots, w_n)^T$ with $w_j \geq 0$, $j = 1, 2, \cdots, n$, and $\sum\limits_{j=1}^{n} w_j = 1$. Further,

$$
\text{2DULIWBHM}^{0,q}\left(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n\right) = \left(\left[\dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{q}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(w_i\right)\left(\frac{w_j}{a_j{}^q}\right)\right)\right)^{\frac{1}{q}}}}, \dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(w_i w_j\right)\right)^{\frac{1}{p}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(w_i\right)\left(\frac{w_j}{b_j{}^q}\right)\right)\right)^{\frac{1}{q}}}} \right], \left[\dddot{s}_{\min(c_i,c_j)},\dddot{s}_{\min(d_i,d_j)}\right] \right). \tag{39}
$$

<div align="center">Box XX</div>

$$\text{2DULIWBHM}^{1,1}\left(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n\right) = \left(\left[\dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{2}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{a_i}\right)\left(\frac{w_j}{a_j}\right)\right)\right)^{\frac{1}{2}}}}, \dot{s}_{\frac{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}(w_i w_j)\right)^{\frac{1}{2}}}{\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(\frac{w_i}{b_i}\right)\left(\frac{w_j}{b_j}\right)\right)\right)^{\frac{1}{2}}}} \right], \left[\ddot{s}_{\min(c_i,c_j)}, \ddot{s}_{\min(d_i,d_j)} \right] \right). \tag{40}$$

<div align="center">Box XXI</div>

suppose that $D = \{D_1, D_2, \cdots, D_\mu\}$ is a set of decision-makers and $\omega = (\omega_1, \omega_2, \cdots, \omega_\mu)$ is the weight vector of them with $\omega_k \geq 0$ $(k = 1, 2, \cdots, \mu)$, and $\sum\limits_{k=1}^{\mu} \omega_k = 1$. Let $\hat{S}^k = \left[\hat{s}_{ij}^k\right]_{m \times n}$ $(k = 1, 2, \ldots, \mu)$ be the decision matrix of MAGDM problems, where:

$$\hat{s}_{ij}^k = \left([\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]\right),$$

is the evaluation information expressed by the 2DULVs with respect to alternative A_i for attribute C_j given by the decision maker D_k, $\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k} \in S_I$, $S_I = (\dot{s}_0, \dot{s}_1, \ldots, s_{t-1})$, $\dot{s}_{c_{ij}^k}, \dot{s}_{d_{ij}^k} \in S_{II}$, and $S_{II} = (\dot{s}_0, \dot{s}_1, \ldots, s_{L-1})$. Then, the goal of this decision problem is to rank alternatives.

Based on the 2DULIWBHM operator proposed in Section 3, we will give its application in the MAGDM problems and establish the detailed decision-making process, as shown in the following:

Step 1. Normalize the attribute values. In real decision making, the attribute values have two types, i.e., cost attribute and benefit attribute. In order to eliminate the differences between types, we need to convert them to the same type. In general, we need convert cost type to benefit type. If the attribute value $\hat{s}_{ij}^k$ $(k = 1, 2, \ldots, \mu)$ is cost type, it can be transformed to benefit one as shown below (suppose that the transformed attribute value is still expressed by: $\hat{s}_{ij}^k = \left([\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]\right)$):

$$\hat{s}_{ij}^k = \left([\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]\right)$$

$$= \begin{cases} \left([\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]\right) \\ \quad \text{for benefit attribute } C_j \\ \left([\dot{s}_{t-1-b_{ij}^k}, \dot{s}_{t-1-a_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]\right) \\ \quad \text{for cost attribute } C_j \end{cases} \tag{41}$$

Step 2. Aggregate the evaluation information of an individual decision-maker with collective information by 2DULIWBHM operator as shown below:

$$\hat{s}_{ij} = \text{2DULIWBHM}\left(\hat{s}_{ij}^1, \hat{s}_{ij}^2, \ldots, \hat{s}_{ij}^\mu\right)$$

$$= \left([\dot{s}_{a_{ij}}, \dot{s}_{b_{ij}}][\ddot{s}_{c_{ij}}, \ddot{s}_{d_{ij}}]\right). \tag{42}$$

Step 3. Aggregate the evaluation information of each attribute with the comprehensive evaluation value of each alternative by 2DULIWBHM operator as follows:

$$\hat{s}_i = \text{2DULIWBHM}\left(\hat{s}_{i1}, \hat{s}_{i2}, \ldots, \hat{s}_{in}\right)$$

$$= \left([\dot{s}_{a_i}, \dot{s}_{b_i}][\ddot{s}_{c_i}, \ddot{s}_{d_i}]\right). \tag{43}$$

Step 4. Calculate the expectations $E(\hat{s}_i)$ $(i = 1, 2, \cdots, m)$ of the collective overall values $\hat{s}_i$ $(i = 1, 2, \cdots, m)$ by Eq. (17).

Step 5. Rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$.

Step 6. End.

5. An illustrative example

Example 4. In order to show the application of the proposed method, an example about the land utilization ratio evaluation of four cities $\{A_1, A_2, A_3, A_4\}$ is given. There are three decision-makers, $D_k = \{k = 1, 2, 3\}$, who are invited to evaluate the land utilization ratio of cities according to four attributes shown as follows: The proportion of building covering C_1, the ratio of vegetation covering C_2, the proportion of traffic covering C_3, and the ratio of the land C_4 that should be developed. Suppose that weight vector of three decision-makers is $\omega = (0.4, 0.3, 0.3)^T$, and the attribute weight vector is $w = (0.25, 0.25, 0.25, 0.25)^T$. Decision maker D_K gives the evaluation value, $\hat{s}_{ij}^k$, of city A_i with respect to attribute C_j by uncertain linguistic variable $[\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}]$, and then he/she must also evaluate the credibility of the result $[\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}]$,

which is also expressed by uncertain linguistic variable $[\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}]$ in another dimension. Then, we can achieve the final evaluation result of city A_i with respect to attribute C_j by decision-maker D_k by the 2DULV $([\dot{s}_{a_{ij}^k}, \dot{s}_{b_{ij}^k}][\ddot{s}_{c_{ij}^k}, \ddot{s}_{d_{ij}^k}])$ (suppose that Class I linguistic set is $S_I = (\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4, \dot{s}_5, \dot{s}_6)$, and Class II linguistic set is $S_{II} = (\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4)$. For example, decision-maker D_1 gives the evaluation value $[\dot{s}_5, \dot{s}_5]$ for index C_1 of city A_1; then, he/she gives the reliability $[\ddot{s}_2, \ddot{s}_3]$ of the evaluation value $[\dot{s}_5, \dot{s}_5]$; then, the 2DULV $([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$ is produced. To evaluate all cities with respect to all indexes by all decision-makers, three decision matrices, $\hat{S}^k = [\hat{s}_{ij}^k]_{4\times4}$ $(k = 1, 2, 3)$, should be constructed, which are listed in Tables 1, 2, and 3.

Then, the goal is to rank four cities according to the land utilization ratio.

5.1. The decision-making steps
To rank the cities, the steps are given as follows:

Step 1. Normalize the attribute values. All the measured values are of benefit type; thus, they do not need to do the standardization.

Step 2. Aggregate the evaluation information of an individual decision-maker (as shown in Box XXII) with collective information by 2DULIWBHM operator expressed in Eq. (42) (without loss of generality, we suppose that $p = q = 1$).

Step 3. Aggregate the evaluation information of each attribute with the comprehensive evaluation value of each alternative by 2DULIWBHM operator expressed in Eq. (43) (without loss of generality, we let $p = q = 1$):

$$\hat{s}_1 = \left([\dot{s}_{3.591}, \dot{s}_{4.266}] [\ddot{s}_2, \ddot{s}_3]\right),$$

$$\hat{s}_2 = \left([\dot{s}_{2.893}, \dot{s}_{3.623}] [\ddot{s}_2, \ddot{s}_2]\right),$$

$$\hat{s}_3 = \left([\dot{s}_{2.734}, \dot{s}_{3.378}] [\ddot{s}_2, \ddot{s}_3]\right),$$

$$\hat{S} = \begin{pmatrix} ([\dot{s}_{4.644}, \dot{s}_{4.644}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{2.569}, \dot{s}_{3.271}][\ddot{s}_2, \ddot{s}_2])([\dot{s}_{3.873}, \dot{s}_{4.644}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{3.830}, \dot{s}_{4.855}][\ddot{s}_1, \ddot{s}_2]) \\ ([\dot{s}_{3.589}, \dot{s}_{4.282}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{3.359}, \dot{s}_{4.190}][\ddot{s}_2, \ddot{s}_2])([\dot{s}_{2.133}, \dot{s}_{2.986}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{2.872}, \dot{s}_{3.317}][\ddot{s}_1, \ddot{s}_2]) \\ ([\dot{s}_{2.569}, \dot{s}_{3.589}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{3.830}, \dot{s}_{4.282}][\ddot{s}_2, \ddot{s}_2])([\dot{s}_{1.773}, \dot{s}_{2.171}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{3.633}, \dot{s}_{4.327}][\ddot{s}_1, \ddot{s}_2]) \\ ([\dot{s}_{3.576}, \dot{s}_{4.450}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{1.805}, \dot{s}_{2.947}][\ddot{s}_2, \ddot{s}_2])([\dot{s}_{1.840}, \dot{s}_{2.987}][\ddot{s}_2, \ddot{s}_3])([\dot{s}_{3.589}, \dot{s}_{4.282}][\ddot{s}_1, \ddot{s}_2]) \end{pmatrix}.$$

Box XXII

Table 1. Decision matrix $\hat{S}^1$.

	C_1	C_2	C_3	C_4
A_1	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_4, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$
A_2	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_3], [\ddot{s}_4, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$
A_3	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_4, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$
A_4	$([\dot{s}_5, \dot{s}_6], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_1, \dot{s}_2], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_4, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$

Table 2. Decision matrix $\hat{S}^2$.

	C_1	C_2	C_3	C_4
A_1	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_6], [\ddot{s}_3, \ddot{s}_4])$
A_2	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_4])$
A_3	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$
A_4	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_1, \dot{s}_2], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$

Table 3. Decision matrix $\hat{S}^3$.

	C_1	C_2	C_3	C_4
A_1	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_3], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_1, \ddot{s}_2])$
A_2	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_1, \dot{s}_2], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_3], [\ddot{s}_1, \ddot{s}_2])$
A_3	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_1, \dot{s}_1], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_1, \ddot{s}_2])$
A_4	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_1, \ddot{s}_2])$

$\hat{s}_4 = \left([\dot{s}_{2.460}, \dot{s}_{3.554}] \, [\ddot{s}_1, \ddot{s}_2] \right).$

Step 4. Calculate the expectations $E(\hat{s}_i)$ $(i = 1, 2, 3, 4)$ of the collective overall values $\hat{s}_i$ $(i = 1, 2, 3, 4)$:

$E(\hat{s}_1) = 0.409, \qquad E(\hat{s}_2) = 0.271,$

$E(\hat{s}_3) = 0.318, \qquad E(\hat{s}_4) = 0.188.$

Step 5. Rank the alternatives. Based on Definition 5, we can rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows:

$A_1 \succ A_3 \succ A_2 \succ A_4.$

Therefore, the best alternative is A_1.

Step 6. End.

5.2. The influence of the parameters p and q on decision making result of this example

In order to demonstrate the influence of parameters p and q on decision-making in this example, different values of p and q in Steps 2 and 3 are used to rank the alternatives. The ranking results are shown in Table 4.

The ranking results with different parameters p and q are slightly different; however, the best choice is always A_1 and the worst is A_4, indicating that the 2DULIWBHM operator can reflect the decision-maker's subjective preferences. When $p = q$, we can find that the ranking of the alternatives is all the same. In addition, because the score functions of alternatives become smaller and smaller as the parameter p and q increase, we can regard parameters p and q as decision-makers' risk attitude. When decision-

Table 4. Ordering of the alternatives by utilizing different p and q.

Parameters	Expectations $E(\hat{s}_i)$	Ranking
$p = q = \frac{1}{2}$	$E(\hat{s}_1) = 0.413, E(\hat{s}_2) = 0.276,$ $E(\hat{s}_3) = 0.326, E(\hat{s}_4) = 0.193$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = q = 1$	$E(\hat{s}_1) = 0.409, E(\hat{s}_2) = 0.271,$ $E(\hat{s}_3) = 0.318, E(\hat{s}_4) = 0.188$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 2, q = 1$	$E(\hat{s}_1) = 0.402, E(\hat{s}_2) = 0.262,$ $E(\hat{s}_3) = 0.301, E(\hat{s}_4) = 0.178$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 1, q = 2$	$E(\hat{s}_1) = 0.402, E(\hat{s}_2) = 0.262,$ $E(\hat{s}_3) = 0.301, E(\hat{s}_4) = 0.178$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 2, q = 2$	$E(\hat{s}_1) = 0.402, E(\hat{s}_2) = 0.263,$ $E(\hat{s}_3) = 0.304, E(\hat{s}_4) = 0.178$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 1, q = 5$	$E(\hat{s}_1) = 0.371, E(\hat{s}_2) = 0.218,$ $E(\hat{s}_3) = 0.217, E(\hat{s}_4) = 0.145$	$A_1 \succ A_2 \succ A_3 \succ A_4$
$p = 5, q = 1$	$E(\hat{s}_1) = 0.371, E(\hat{s}_2) = 0.218,$ $E(\hat{s}_3) = 0.217, E(\hat{s}_4) = 0.145$	$A_1 \succ A_2 \succ A_3 \succ A_4$
$p = 5, q = 5$	$E(\hat{s}_1) = 0.386, E(\hat{s}_2) = 0.244,$ $E(\hat{s}_3) = 0.274, E(\hat{s}_4) = 0.155$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$p = 1, q = 10$	$E(\hat{s}_1) = 0.329, E(\hat{s}_2) = 0.177,$ $E(\hat{s}_3) = 0.159, E(\hat{s}_4) = 0.121$	$A_1 \succ A_2 \succ A_3 \succ A_4$
$p = 10, q = 1$	$E(\hat{s}_1) = 0.329, E(\hat{s}_2) = 0.177,$ $E(\hat{s}_3) = 0.159, E(\hat{s}_4) = 0.121$	$A_1 \succ A_2 \succ A_3 \succ A_4$
$p = 10, q = 10$	$E(\hat{s}_1) = 0.369, E(\hat{s}_2) = 0.227,$ $E(\hat{s}_3) = 0.253, E(\hat{s}_4) = 0.139$	$A_1 \succ A_3 \succ A_2 \succ A_4$

makers are optimistic, the small values of parameters p and q can be adopted, and when decision-makers are pessimistic, the big values of parameters p and q can be used. Therefore, in real practical decision-making situations, decision-makers can choose an appropriate value in accordance with their risk preferences. When $p \geq 5$ or $q \geq 5$ and $p \neq q$, the ranking changes to $A_1 \succ A_2 \succ A_3 \succ A_4$.

5.3. The verification of the validity

To prove the effectiveness of the improved method in this paper, the same illustrative example is solved using the two existing MAGDM methods: the 2-Dimensional Uncertain Linguistic Weighted Geometric Heronian Mean (2DULWGHM) operator proposed by Liu and Chu [46]; 2-Dimensional Uncertain Linguistic Power Generalized Weighted Aggregation (2DULPGWA) operator proposed by Liu and Yu [20].

For convenience, we let $p = q = 1$; then, the final ranking orders of the alternatives obtained by these methods are described in Table 5.

From Table 5, we can find that there are the same ranking results using four methods. Therefore, the method in this paper is effective and feasible.

5.4. Further discussion on the proposed methods

According to the above subsections, the validity of our proposed method has been confirmed. However, it has been found that they have the same ranking results; hence, it is difficult to illustrate the advantages of our method and the drawbacks of the existing methods in some situations. Therefore, two examples are given to show the advantages of our proposed method based on the 2DULIWBHM operator. These examples are particular cases of Example 4 given as follows.

Example 5. For easily calculation, the evaluation information of decision-maker D_2 who is invited to evaluate the land utilization ratio of four cities $\{A_1, A_2, A_3, A_4\}$ is adopted according to the four attributes, i.e., the proportion of building covering C_1, the ratio of vegetation covering C_2, the proportion of traffic covering C_3, and the ratio of land C_4 that should be developed. Suppose that weight vector of the attribute is $w = (0.25, 0.25, 0.25, 0.25)^T$. Decision-maker D_2 gives the evaluation value $\hat{s}_{ij}$ of city A_i with respect to attribute C_j, which is expressed by the 2-dimensional uncertain linguistic variables (suppose that Class I linguistic set is $S_I = (\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4, \dot{s}_5, \dot{s}_6)$, and Class II linguistic set is $S_{II} = (\ddot{s}_0, \ddot{s}_1, \ddot{s}_2, \ddot{s}_3, \ddot{s}_4)$. In addition, decision matrices $\hat{S} = [\hat{s}_{ij}]_{4 \times 4}$ are listed in Table 6, where $\hat{s}_{ij}$ can be expressed as $\left([\dot{s}_{a_{ij}}, \dot{s}_{b_{ij}}][\ddot{s}_{c_{ij}}, \ddot{s}_{d_{ij}}] \right)$. Then, the goal is to rank four cities according to the land utilization ratio.

The ranking results are shown in Table 7, from which, we can find that when $p = q = 1$, the best alternative to the proposed method is A_3 and other is A_1. We can explain that the ranking result produced by the proposed method is more reasonable than that produced by Liu and Chu [46], because the proposed method considers the effects of unreasonable data from biased decision-makers, such as too high or too low arguments. In this example, because Class I uncertain linguistic variable of $\hat{s}_{12}$ and $\hat{s}_{14}$ is $[\dot{s}_5, \dot{s}_6]$, which are quite larger than the others. The proposed method can

Table 5. Ranking results by different methods.

Method	Aggregation operator	Score values $S(Z_i)$	Ranking
Liu and Chu [46]	2DULWGHM	$E(\hat{s}_1) = 0.429, E(\hat{s}_2) = 0.302,$ $E(\hat{s}_3) = 0.361, E(\hat{s}_4) = 0.221$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Liu and Yu [20]	2DULPGWA	$E(\hat{s}_1) = 0.328, E(\hat{s}_2) = 0.177,$ $E(\hat{s}_3) = 0.159, E(\hat{s}_4) = 0.121$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Proposed method	2DULIWBHM	$E(\hat{s}_1) = 0.427, E(\hat{s}_2) = 0.298,$ $E(\hat{s}_3) = 0.356, E(\hat{s}_4) = 0.216$	$A_1 \succ A_3 \succ A_2 \succ A_4$

Table 6. Decision matrix $\hat{S}$ of Example 5.

	C_1	C_2	C_3	C_4
A_1	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_6], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_5, \dot{s}_6], [\ddot{s}_1, \ddot{s}_2])$
A_2	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_1, \ddot{s}_2])$
A_3	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_1, \ddot{s}_2])$
A_4	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_3, \ddot{s}_4])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_1, \ddot{s}_2])$

Table 7. Ranking results by different methods for Example 5.

Method by	Aggregation operator	Expected values $E(\hat{s}_i)$	Ranking
Liu and Chu [46]	2DULWGHM ($p=q=1$)	$E(\hat{s}_1)=0.4837$, $E(\hat{s}_2)=0.2941$, $E(\hat{s}_3)=0.4748$, $E(\hat{s}_4)=0.2662$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Proposed method	2DULIWBHM ($p=q=1$)	$E(\hat{s}_1)=0.4653$, $E(\hat{s}_2)=0.2711$, $E(\hat{s}_3)=0.4657$, $E(\hat{s}_4)=0.2608$	$A_3 \succ A_1 \succ A_2 \succ A_4$

better process this type of decision problem and relieve the effects of unreasonable data, while the method of Liu and Chu [46] cannot eliminate the effect of extreme cases; thus, the ranking result produced by the proposed method is more reasonable than the result produced by the method of Liu and Chu [46].

Example 6. Similar to Example 5, the evaluation results of decision-maker D_3 who is invited to evaluate the land utilization ratio of four cities $\{A_1, A_2, A_3, A_4\}$ are adopted according to the four attributes: the proportion of building covering C_1, the ratio of vegetation covering C_2, the proportion of traffic covering C_3, and the ratio of the land C_4 that should be developed. Suppose that weight vector of the attribute is $w = (0.25, 0.25, 0.25, 0.25)^T$. Decision-maker D_3 gives evaluation value, $\hat{s}_{ij}$, of city A_i with respect to attribute C_j, which is expressed by the 2-dimensional uncertain linguistic variables. Suppose that Class I linguistic set is $S_I = (\dot{s}_0, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4, \dot{s}_5, \dot{s}_6)$ and Class II linguistic set is $S_{II} = (\ddot{s}_0, \ddot{s}_1, \ddot{s}_2, \ddot{s}_3, \ddot{s}_4)$. In addition, decision matrices $\hat{S} = [\hat{s}_{ij}]_{4\times4}$ are listed in Table 8, where $\hat{s}_{ij}$ can be expressed as $\left([\dot{s}_{a_{ij}}, \dot{s}_{b_{ij}}][\ddot{s}_{c_{ij}}, \ddot{s}_{d_{ij}}]\right)$. Then, the goal is to rank the four cities according to the land utilization ratio.

The ranking results are shown in Table 9, from

which we can find that the best alternative of the proposed method is A_1 and the other is A_3. Obviously, these two methods can relieve the effect of unreasonable data, such as too high or too low data given by biased decision-makers; however, the only difference between them is that the proposed method can also consider the relationship of attributes. In real decision-making problems, it is common that there is the relationship among attributes. Therefore, the proposed method in this paper is more reasonable than that proposed by Liu et al. [20]. In this example, there is the interrelationship among four attributes; therefore, the ranking result produced by the proposed method is more reasonable than the result produced by the method of Liu and Yu [20].

According to the above analysis, the comparison of the proposed method based on 2DULIWBHM operator with the other methods can be described as follows (see Table 10):

1. Compared with the method in [46] based on the 2DULWGHM operator, we can find that these two methods adopt the same operational rules which are more accurate in operations. At the same time, these two methods all consider the interrelationship for input arguments. However, the method in [46] cannot consider the case when the given arguments

Table 8. Decision matrix $\hat{S}$ of Example 6.

	C_1	C_2	C_3	C_4
A_1	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_1, \ddot{s}_2])$
A_2	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_1, \dot{s}_2], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_3, \dot{s}_3], [\ddot{s}_1, \ddot{s}_2])$
A_3	$([\dot{s}_3, \dot{s}_4], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_5, \dot{s}_5], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_4], [\ddot{s}_1, \ddot{s}_2])$
A_4	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_3])$	$([\dot{s}_2, \dot{s}_3], [\ddot{s}_2, \ddot{s}_2])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_3, \ddot{s}_3])$	$([\dot{s}_4, \dot{s}_5], [\ddot{s}_1, \ddot{s}_2])$

Table 9. Ranking results by different methods for Example 6.

Method by	Aggregation operator	Expected values $E(\hat{s}_i)$	Ranking
Liu and Yu [20]	2DULPGWA	$E(\hat{s}_1)=0.4676$, $E(\hat{s}_2)=0.2790$, $E(\hat{s}_3)=0.4678$, $E(\hat{s}_4)=0.2167$	$A_3 \succ A_1 \succ A_2 \succ A_4$
Proposed method	2DULIWBHM ($p=q=1$)	$E(\hat{s}_1)=0.4632$, $E(\hat{s}_2)=0.2317$, $E(\hat{s}_3)=0.4424$, $E(\hat{s}_4)=0.2033$	$A_1 \succ A_3 \succ A_2 \succ A_4$

Table 10. The comparisons of different methods.

Methods by	Whether considers interrelationship of two attributes	Whether considers the effect from too high or too low arguments
Liu and Chu [46]	Yes	No
Liu and Yu [20]	No	Yes
2DULIWBHM	Yes	Yes

have a great difference; in other words, it cannot eliminate the effects of unreasonable data from biased decision-makers. For instance, if the given data are too high or too low, it cannot achieve the reasonable result. Obviously, the proposed method is more flexible and general to solve the MAGDM problems than the method proposed by Liu and Chu [46].

2. Compared with the method in [20] based on the 2DULPGWA operator, the two methods both can deal with the problems about unreasonable data given by biased decision-makers. However, the attributes are independent in the method proposed in [20], and the interrelationship between them is not considered. In many times, there exists much relevance between the given arguments. The proposed method in this paper can easily solve this problem with different parameters p and q using BM; thus, our method is more flexible and reasonable than the method in [20].

According to the comparisons and analyses above, our method can combine the advantages of the two methods efficiently. In other words, the proposed method based on the 2DULIWBHM operator in this paper is better than the other existing methods for aggregating the 2DULVs. Of course, there is a shortcoming in the proposed method, i.e., it is more complex in the calculation process than the method in [20].

6. Conclusion

Because 2DULVs can express fuzzy information better, this study extended the Bonferroni Mean (BM) operator to 2DULVs and proposed the 2-Dimensional Uncertain Linguistic Weighted Bonferroni Mean (2DULWBM). However, it cannot consider the case when the given arguments are too high or too low. Therefore, the 2DULWBM was combined with harmonic mean to solve this problem. Then, some of their desirable characteristics, such as idempotency, monotonicity, boundedness, and commutativity, were discussed. Further, some particular cases of these operators were analyzed, and a method was proposed for the MAGDM based on the 2DULIWBHM operators. Compared with the existing methods, the proposed method is more general

than some existing methods. The significant advantages of the proposed method include (1) capturing the interrelationship among the input arguments that have the flexibility with Bonferroni mean parameters p, q and (2) considering the case when the given arguments are too high or too low. For future studies, it is necessary and significant to apply these operators to solve the real decision-making problems such as fuzzy cluster analysis, uncertain programming, etc.

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