Optimal Lot Size in a Manufacturing System with Imperfect Raw Materials and Defective Finished Products

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Abstract
In real-world manufacturing systems, encountering with imperfect raw materials and generation of defective finished products are inevitable. In order to cope with these practical problems, this paper studies a manufacturer which orders raw materials from external source (supplier), and then produces a finished product. The raw materials contain imperfect quality items and, in addition, the production system is defective. The imperfect raw materials are sold after screening process, while the defective finished products go under a further rework process. It is also assumed that defective rate of machine is a random variable, resulting three possible cases regarding occurrence of backordering shortage. The aim is to determine economic order/production lot sizes for each case in such a way that the total cost of system is minimized. The optimal closed form solution is derived for each case separately. Moreover the applicability of the proposed manufacturing model is illustrated via a numerical example.
**Keywords:** Manufacturing Systems; Manufacturing Planning; Imperfect Raw Material; Defective Finished Product; Reworking Process

1. **Introduction**

One of the basic and useful production-inventory planning models is economic order quantity (EOQ). The aim of EOQ is to find economic lot size of materials to order from external sources so as to minimize total cost of system like holding and ordering costs. The classic EOQ was customized for manufacturing settings through economic production/manufacturing quantity (EPQ/EMQ) model. The traditional models of EOQ/EPQ are based on some simple assumptions such as: (i) demand rate for an item is pre-known and constant, (ii) all quantity of an order are received instantaneously for EOQ, and gradually for EPQ, (iii) items are entirely consumed when the next order is received, (iv) The shortage is not permitted and no safety stock is allowed, (v) there is no quantity discount, (vi) ordering/setup cost is fixed per order/production, (vii) all parameters are deterministic and (viii) all received/produced items are of perfect quality. Since the holding and ordering costs behave inversely in basic EOQ/EPQ models, the total cost function is convex and then an intermediate amount of lot size is optimal. Based on the above assumptions, closed-form lot sizing can be simply calculated for the basic EOQ/EPQ models. The above assumptions are far from real conditions to justify the use of the basic EOQ/EPQ models in practice. Therefore, a high amount of literature has been allocated to relaxing these assumptions.
One of the assumptions in the manufacturing planning models like EPQ which is unrealistic is that all items (raw materials received from external sources and finished products by own manufacturer) are of perfect quality and conform to all required characteristics perfectly. However, in reality, this assumption is not necessarily true and then it is a crucial weakness of traditional manufacturing models. For this reason, the problem of imperfect quality items has received attention of researchers during recent years. Porteus [1] analyzed production lines in out of control state when the products are of imperfect quality and rate of defective items is dependent to lot size. Moreover, Rosenblatt and Lee [2] studied the EPQ model with an imperfect process. Salameh and Jaber [3] did a great improvement in imperfect quality context via proposing an EOQ model considering a 100% inspection process upon receiving the products to identify defective products. The research of Salameh and Jaber [3] was modified by Cárdenas-Barrón [4] and Maddah and Jaber [5]. Rezaei [6] developed inventory model with imperfect quality of items and fully backordered shortage. In addition, Yu et al. [7] extended the inventory model with imperfect with mixed partial backordering shortage and lost sales. After, Wee et al. [8] studied an imperfect quality on EPQ model with deterioration and partial backordering shortage. Additionally, Papachristos and Konstantaras [9] proposed an inventory model with a constraint to avoid shortages. Moreover, Wee et al. [10] proposed an inventory model with fully backordering shortage. Khan et al. [11] presented a review on imperfect quality inventory models. At same time, Wahab et al. [12] derived a coordinated level supply with shortages and environmental effects considering imperfect quality of items. Afterward, Konstantaras et al. [13] studied the impact of learning effect on the quantity of imperfect products and lot size. Liu and Zheng [14] suggested a fuzzy model with inspection errors on imperfect products. Moreover, Hsu and Hsu [15] proposed a general model on imperfect products with inspection errors, backordering
shortage, and sales returns. In addition, Rad et al. [16] derived a price-dependent demand model for an integrated supply chain with imperfect process and allowed shortage. Skouri et al. [17] developed an EOQ inventory model with imperfect product in the received lot, and rejection of imperfect items to supplier. In another work, Paul et al. [18] presented a joint replenishment problem to determine the lot size for products with defective items. Hlioui et al. [19] investigated a supply chain model with defective items with 100% screening. Sharifi et al. [20] studied the effect of inspection errors on inventory models with imperfect items and partial backordering shortage. Alamri et al. [21] evaluated the impact of learning on inventory model with imperfect items. Chang et al. [22] developed an extension to the inventory model with imperfect items considering permissible delays in payments and inspection errors. Rezaei [23] proposed using sample inspection instead of full inspection in inventory models with imperfect items. Yu and Hsu [24] propose an unequal sized shipments for a production-inventory problem with a 100% inspection and item return. Sarkar and Saren [25] investigated warranty cost in EPQ model considering defective items with inspection error. Ongkunaruk et al. [26] proposed considering some constraints such as shipment, budget, and transportation capacity constraints in an inventory model with defective items. Taleizadeh et al. [27] proposed an imperfect EPQ manufacturing model with backordering allowed and trade credits. Cheikhrouhou et al., [28] proposed joint optimization of sample size and order size considering lot inspection policy and withdrawn of defective batches. Taleizadeh and Zamani Dehkordi, [29] considered an economic order quantity model with partial backordering and sampling inspection. Mokhtari and Rezvan, [30] discussed a production-inventory system under VMI condition and partial backordering. Jaber et al., [31] extended the work of Salameh and Jaber [3] by proposing economic order quantity models considering imperfect items and buy and repair options when encountering with a distant supplier. In

In this paper, we propose a production-inventory model where a manufacturer orders raw materials from external source (supplier), and produces a finished product via a finite production rate. The raw materials contain imperfect quality items and hence a 100% screening process is done upon receive of lot size. On the other hand, the production system is also defective and a fraction of finished products are imperfect. The imperfect raw materials are sold after screening process, while the defective finished products go under a further rework process on same machine. A percent of defective products are reworkable and have potential to become perfect after reworking process, while others are scrapped items and sold at a lower price. It is also assumed that defective rate of machine is a random variable, resulting three possible cases regarding occurrence of backordering shortage. We design two scenarios for each case, resulting six total states. The aim is to determine economic order/production lot sizes for each case in such a way that the total cost of system is minimized.

The rest of this paper is structured as follows. Section 2 presents overall description of problem and introduce proposed manufacturing models via
three possible cases. Then, Section 3 discussed numerical experiments. Finally, Section 4 presents conclusions.

2. Manufacturing System Modeling

2.1. Overall Description and Notations

We propose a new manufacturing model where a manufacturer produces a product to satisfy an external demand $D$. The demand is assumed to be constant over time horizon. The manufacturer produces the product via a finite production rate $P_1$ under EPQ setting. In addition, the shortage is not allowed and purchase cost is fixed. In contrast to standard models, it is assumed that the production system is defective and produces a percentage of imperfect items $\beta$. The imperfect items are also under rework process to become perfect and return to consumption cycle. In general, an inventory cycle is composed of production, reworking and depletion periods. After production period, a percentage of defective items $\beta$ which are reworkable go under rework process with rework rate $P_2$. In real world situations, many environmental features there exist which affect the production system, and cause a fluctuation in the quality of produced items. Hence, as a realistic assumption, it is considered that the percentage of defective items $\beta$ is a random variable. At the end of rework period, the stored inventory is consumed till reaches to zero during depletion period. The production is carried out via production rate $P_1$ within production period $t_p$. Once the production ends, there exist imperfect items $\beta Q$. Among them, the reworkable items $\alpha \beta Q$ go under reworking process and the scrapped items $(1-\alpha)\beta Q$ are disposed from system. During rework period $t_r$, all the reworkable items $\alpha \beta Q$ become perfect with the rate of $P_2$ and return to the system. At the end of rework period, the stored inventory are consumed during depletion period $t_d$ till reach to zero. The next cycles repeat this
process continuously. The aim is to determine optimal/economic production quantity $Q$, in such a way that the total profit is maximized. The total profit per cycle $TP$ is obtained as total revenue per cycle $TR$ minus total cost per cycle $TC$.

Here we analyze three possible cases (I, II, III) regarding occurrence of shortage in the proposed system. The case I considers a case in which shortage does not occur, so we consider initial condition as $I_{max} - \beta Q \geq 0$.

Since $I_{max} = Q (1 - D/P_i)$, this condition simplifies to:

$$0 \leq \beta \leq 1 - \frac{D}{P_i}$$  \hspace{1cm} (1)

While the case II considers a case where shortage occurs but is backordered fully. In this case, we encounter a shortage due to $(1 - D/P_i) - \beta < 0$, but it is backordered because the amount of reworked items minus demand during rework period is greater than the amount of shortage, i.e., $(1 - D/P_i) - \beta + \alpha \beta (1 - D/P_2) > 0$. These conditions are summarized to:

$$1 - \frac{D}{P_i} < \beta < \frac{1 - D/P_i}{1 - \alpha (1 - D/P_2)}$$ \hspace{1cm} (2)

Finally, case III considers a case in which not only shortage occurs but also the amount of reworked items is not sufficient to cover all shortages occurred. In this case, there will be an amount of unsatisfied demand. Due to avoid lost sale, we use a special order at the end of rework period (will be discussed later). The condition for this case is as follows.

$$\frac{1 - D/P_i}{1 - \alpha (1 - D/P_2)} \leq \beta \leq 1$$ \hspace{1cm} (3)

A famous condition in an EPQ model is that production rate $P$ should be greater than the demand rate $D$ in a classic EPQ model ($P > D$). This condition ensures feasibility of model and avoid ultimate shortage in all planning horizon. The conditions I-III play such a role in our model and should be checked before starting to solve the problem. Of course, the
expected value of defective rat should be considered in these constraints, since it is a random variable.

Before formulating the problem, we summarize the notations used throughout the paper as follows:

- \( D \) The demand rate of finished product
- \( P_1 \) The production rate of finished product
- \( P_2 \) The rework rate of finished product
- \( A_1 \) The fixed order cost of raw material
- \( A_2 \) The fixed setup cost of finished product
- \( h_1 \) The holding cost of raw material per item per unit time
- \( h_2 \) The holding cost of finished product per item per unit time
- \( C_1 \) The purchase cost of raw material per item
- \( C_2 \) The production cost of finished product per item
- \( d_1 \) The screening cost of raw material per item
- \( d_2 \) The screening cost of finished product per item
- \( r \) The rework cost of finished product per item
- \( p \) The selling price of imperfect raw material per item
- \( v \) The selling price of perfect finished product per item
- \( s \) The selling price of scrapped finished product per item
- \( x \) The screening rate of raw material
- \( \beta \) The defective rate of finished product
- \( \alpha \) The reworkable rate of defective finished product
- \( q \) The imperfect rate of raw material
- \( t_s \) The duration of screening period of raw material
- \( t_p \) The duration of production period of finished product
- \( t_r \) The duration of rework period of finished product
- \( t_D \) The duration of depletion period of finished product
- \( E[\cdot] \) The expected value of random variable
- \( Y \) The order quantity of raw material
- \( Q \) The production quantity of finished product
- \( b \) The backorder quantity of finished product
2.2. Case I: When shortage does not occur

Figure 1 shows one cycle of the proposed manufacturing system in case I. To encounter the complexity of modelling procedure gradually, we consider two scenarios of this case in the sequel. In first scenario, we just consider the finished product cycle, and do not consider the raw material cycle, while in second scenario, we consider both raw material and finished product cycles, simultaneously.

>> Please insert Figure 1 here <<

In first scenario, the production quantity $Q$ is assumed to be decision variable, independently. Hence, the total revenue per cycle $TR_i$ involves sales of perfect and scrapped items which is given as

$$TR_i = v \{ Q - \beta Q + \alpha \beta Q \} + s \{ (1-\alpha) \beta Q \}$$

where $v$ and $s$ represent the unit selling price of perfect and scrapped finished products respectively. Note that the unit selling price of perfects is greater than that of scrapped items ($v > s$). The total cost in this case involves production, setup, holding, screening and reworking costs. The production cost per cycle $PC_i$ is calculated as $PC_i = C \cdot Q$. In addition, the setup cost $SC_i$ is incurred per production cycle by $SC_i = A_z$. Moreover, in order to formulate the holding cost, we first calculate the area under inventory level in three periods, i.e., production period $S_1$, rework period $S_2$ and depletion period $S_3$. The first area is calculated as $S_1 = I_{max} \cdot t_p / 2$. Since $I_{max} = Q \left( 1 - D / P_i \right)$ and $t_p = Q / P_i$, the $S_1$ is re-written as $S_1 = \frac{Q^2}{2P_i} \left( 1 - D / P_i \right)$. To calculate $S_2$, we should first formulate the inventory level at the start of rework period $I_2$ and the inventory level at the end of rework period $I_3$, as $I_2 = Q \left( 1 - D / P_i \right) - \beta Q$ and
During rework period \( t_r \) the reworkable items \( \alpha \beta Q \) are under reworking process. Hence, we can calculate \( t_r \) in terms of model parameters as \( t_r = \alpha \beta Q / P_2 \). So the inventory level \( I_3 = Q \left( 1 - D / P_1 \right) - \beta Q + \alpha \beta Q \left( 1 - D / P_2 \right) \). Therefore, the area under inventory level in rework period is calculated as \( S_2 = t_r \left( I_2 + I_3 \right) / 2 \) which can be re-written as \( S_2 = \frac{\alpha \beta Q}{2P_2} \left\{ 2Q \left( 1 - D / P_1 \right) - 2\beta Q + \alpha \beta Q \left( 1 - D / P_2 \right) \right\} \). Moreover, the area under inventory level of depletion period is formulated as \( S_3 = I_3 / 2D \) where depletion period is \( t_d = I_3 / D \), hence, we have \( S_3 = I_3^2 / 2D \) which is re-written as \( S_3 = \frac{1}{2D} \left\{ Q \left( 1 - D / P_1 \right) - \beta Q + \alpha \beta Q \left( 1 - D / P_2 \right) \right\}^2 \). Utilizing \( S_1 \), \( S_2 \) and \( S_3 \), the holding cost is formulated by \( h_2 \{ S_1 + S_2 + S_3 \} \), as follows.

\[
H_{Ci} = h_2 Q^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha \beta}{2P_2} \left\{ 1 - \frac{D}{P_1} \right\} - \beta + G \right\} \tag{4}
\]

in which \( h_2 \) is holding cost per item per unit time and \( G = \left( 1 - D / P_1 \right) - \beta + \alpha \beta \left( 1 - D / P_2 \right) \). The screening cost per cycle \( W_{Ci} \) is computed as \( W_{Ci} = d_2 Q \) in which \( d_2 \) represents the screening cost of finished product per item. Moreover, the reworking cost per cycle \( R_{Ci} \) is obtained \( R_{Ci} = r \alpha \beta Q \) where \( r \) denotes the rework cost per item.

Therefore the total cost per cycle is obtained by \( P_{Ci} + S_{Ci} + H_{Ci} + W_{Ci} + R_{Ci} \) as follows.

\[
T_{Ci} = C_2 Q + A_2 + h_2 Q^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha \beta}{2P_2} \left\{ 1 - \frac{D}{P_1} \right\} - \beta + G \right\} + d_2 Q + r \alpha \beta Q \tag{5}
\]

Here, the total profit per cycle in first scenario of case I, \( T_{Pi} \), can be calculated by \( T_{Pi} - T_{Ci} \) as follows.
\[ TP_i = v \{Q - \beta Q + \alpha \beta Q\} + s \{(1 - \alpha) \beta Q\} \]
\[-C_2 Q - A_2 - h_2 Q^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha \beta}{2P_2} \left\{ \left( 1 - \frac{D}{P_1} \right) - \beta + G \right\} \right\} \]
\[-d_2 Q - r \alpha \beta Q \]

Since \( \beta \) is random variable, it should be replaced with expected value \( E[\beta] \) in \( TP_i \) to calculate expected total profit \( E[TP_i] \) as follows.

\[ E[TP_i] = v \{Q - E[\beta] Q + \alpha E[\beta] Q\} + s \{(1 - \alpha) E[\beta] Q\} \]
\[-C_2 Q - A_2 - h_2 Q^2 \left\{ \frac{E[G^2]}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha E[\beta]}{2P_2} \left\{ \left( 1 - \frac{D}{P_1} \right) - E[\beta] + E[G] \right\} \right\} \]
\[-d_2 Q - r \alpha E[\beta] Q \]

where \( E[G] = (1 - D / P_1) - E[\beta] + \alpha E[\beta](1 - D / P_2) \). Moreover, the cycle time \( T_i \) is also a random variable, which can be obtained by \( T_i = t_p + t_r + t_d \) as \( T_i = \frac{Q}{D} (\beta (\alpha - 1) + 1) \) whose expected value is calculated as follows.

\[ E[T_i] = \frac{Q}{D} (E[\beta] (\alpha - 1) + 1) \]

Hence the expected total profit per unit time is given as follows:

\[ E[TPU_i] = \frac{E[TP_i]}{E[T_i]} \]

By simplifying the expressions in Eq. 8, we reach to:

\[ E[TPU_i] = \frac{D}{E[\beta] (\alpha - 1) + 1} \{ v \{1 - E[\beta] + \alpha E[\beta]\} + s \{(1 - \alpha) E[\beta]\} \}
\[-C_2 Q - A_2 - h_2 Q^2 \left\{ \frac{E[G^2]}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha E[\beta]}{2P_2} \left\{ \left( 1 - \frac{D}{P_1} \right) - E[\beta] + E[G] \right\} \right\} \]
\[-d_2 Q - r \alpha E[\beta] Q \]

The above expected total profit per unit time \( E[TPU_i] \) is concave, since:
\[
\frac{\partial^3 E[TPU_j]}{\partial Q^2} = \frac{-2A_jD}{Q^3 \{E[\beta](\alpha-1)+1\}} \leq 0
\]  

(10)

So we can set the first derivative of \( E[TPU_j] \) to zero so as to reach the economic lot size, as follows.

\[
Q_i^* = \left[ \frac{2A_jD}{h_2 \left\{ E[G] \right\}^2 + \frac{D}{P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha DE[\beta]}{P_2} \left[ \left( 1 - \frac{D}{P_1} \right) - E[\beta] + E[G] \right] \} \frac{1}{2} \right]
\]  

(11)

Here, we consider second scenario with both raw material and finished product cycles. In this scenario, order quantity of raw material \( Y \) is decision variable, and production quantity of finished product \( Q \) is dependent variable. Since a fraction of raw material \( q \) is imperfect, then the relation between order and production quantities is given as \( Q = (1-q)Y \). After receiving lot of raw material from external source, a 100% screening process is started, and simultaneously, the perfect raw materials go to production system under rate of \( P_i \). This process is continued till all of raw materials are inspected. Since the screening rate is constant \( x \), duration of screening period is calculated \( t_s = Y / x \). Moreover, quantity of produced items during screening period is \( t_s P_i \). In addition, the inventory level after disposal (selling) of imperfect raw material is obtained \( I_1 = Y - t_s P_i - qY \), which simplifies, by substituting \( t_s \), to \( I_1 = (1-q-P_i / x)Y \). The production period \( t_p \) is the time interval in which production quantity \( Q = (1-q)Y \) is processed. Therefore it is computed as \( t_p = Q / P_i \) or equivalently \( t_p = (1-q)Y / P_i \). In this scenario, the total revenue per cycle \( TR_i \) involves sales of perfect and scrapped finished products and imperfect raw materials which is given as

\[
TR_i = v \left\{ (1-q)Y - \beta (1-q)Y + \alpha \beta (1-q)Y \right\} + s \left\{ (1-\alpha) \beta (1-q)Y \right\} + pqY
\]

The total cost, in this scenario, is associated with two cycles, i.e., finished product
cycle and raw material cycles. The total cost of finished product $TC_{fp}$, similar to that of first scenario, involves production, setup, holding, screening and reworking costs. Hence, it can be simply obtained by substituting $Q = (1-q)Y$ to total cost of first scenario as follows.

$$ TC_{fp} = C_2 (1-q)Y + A_2 + h_2 (1-q)^2 Y^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha \beta}{2P_2} \left\{ \frac{1}{P_1} - \beta + G \right\} \right\}$$

(12)

$$ + d_2 (1-q)Y + r \alpha \beta (1-q)Y $$

The total cost of raw material cycle $TC_{rm}$ involves purchasing, ordering, holding and screening costs. The purchasing cost of raw material per cycle is calculated as $CY$. In addition, the ordering cost is incurred per cycle as $A_1$. Moreover, in order to formulate the holding cost, we first calculate the area under inventory level in two periods, i.e., screening period $S_1$ and after screening period (till end of production period) $S_2$. The first area is calculated as $S_1 = \{ Y + (qY + I_1) \} t_s / 2$. By substituting $I_1$ and $t_s$ into $S_1$, it simplifies to $Y^2 (2-P_1/x) / (2x)$. In addition, the second area $S_2$ is formulated as $S_2 = I_2^2 / (2P_1)$ which simplifies to $S_2 = Y^2 (1-q-P_1/x)^2 / (2P_1)$. Hence, we can calculate total area as $S_1 + S_2 = Y^2 \{ (1-q)^2 / (2P_1) + q / x \}$. Therefore, the holding cost of raw material is formulated as follows.

$$ HC_{rm} = hY^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} $$

(13)

Now, the screening cost of raw material is calculated by $dY$. So the total cost of raw material cycle is obtained as

$$ TC_{rm} = CY + A_1 + hY^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} + dY $$

(14)
Therefore the total cost per cycle of second scenario is obtained by $TC_{fp} + TC_{rm}$ as follows.

$$TC_f = \{C_1 + C_2 (1-q)\}Y + A_1 + A_2 + h_2 Y^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\}$$

$$+ h_2 (1-q)^2 Y^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left(1 - \frac{D}{P_1}\right) + \frac{\alpha \beta}{2P_2} \left\{ \left(1 - \frac{D}{P_1}\right) - \beta + G \right\} \right\}$$

$$+ d_Y + d_2 (1-q)Y + r \alpha \beta (1-q)Y$$

(15)

Then, the total profit per cycle in second scenario of case I can be calculated by $TR_f - TC_f$ as follows.

$$TP_f = v \{ (1-q)Y - \beta (1-q)Y + \alpha \beta (1-q)Y \} + s \{ (1-\alpha) \beta (1-q)Y \} + pqY$$

$$- \{C_1 + C_2 (1-q)\}Y - A_1 - A_2 - h_2 Y^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\}$$

$$- h_2 (1-q)^2 Y^2 \left\{ \frac{G^2}{2D} + \frac{1}{2P_1} \left(1 - \frac{D}{P_1}\right) + \frac{\alpha \beta}{2P_2} \left\{ \left(1 - \frac{D}{P_1}\right) - \beta + G \right\} \right\}$$

$$- d_Y - d_2 (1-q)Y - r \alpha \beta (1-q)Y$$

(16)

Since $\beta$ is random variable, it should be replaced with expected value $E[\beta]$ in $TP_f$ to calculate expected total profit $E[TP_f]$ as follows.

$$TP_f = v \{ (1-q)Y - E[\beta] (1-q)Y + \alpha E[\beta] (1-q)Y \} + s \{ (1-\alpha) E[\beta] (1-q)Y \} + pqY$$

$$- \{C_1 + C_2 (1-q)\}Y - A_1 - A_2 - h_2 Y^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\}$$

$$- h_2 (1-q)^2 Y^2 \left\{ \frac{E[G^2]}{2D} + \frac{1}{2P_1} \left(1 - \frac{D}{P_1}\right) + \frac{\alpha E[\beta]}{2P_2} \left\{ \left(1 - \frac{D}{P_1}\right) - E[\beta] + E[G] \right\} \right\}$$

$$- d_Y - d_2 (1-q)Y - r \alpha E[\beta] (1-q)Y$$

(17)

Moreover, the cycle time $T_i$, similar to that of first scenario, is

$$T_i = \frac{Q}{D} (E[\beta] (\alpha - 1) + 1)$$

where $Q = (1-q)Y$. Hence the expected total profit per unit time is given as follows:

$$E[TP_f]$$

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\[ E[T_{PU, i}] = \frac{E[T_{P, i}]}{E[T_i]} \]

By simplifying the expressions in Eq. 18, we reach to:

\[ E[T_{PU, i}] = \frac{D}{E[\beta](\alpha - 1) + 1} \]

\[ v \left\{ 1 - E[\beta] + \alpha E[\beta] \right\} + s \left\{ (1 - \alpha) E[\beta] \right\} + \frac{pq}{1 - q} - \frac{C_1}{1 - q} - C_2 \left( \frac{A_1 + A_2}{(1 - q) Y} \right) \]

\[- \frac{h_i}{1 - q} Y \left( \frac{(1 - q)^2}{2P_i} + \frac{q}{x} \right) - h_2 (1 - q) Y \]

\[ \left\{ \frac{E[G]^2}{2D} + \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha E[\beta]}{2P_2} \right\} \left\{ \left( 1 - \frac{D}{P_1} \right) - E[\beta] + E[G] \right\} - \frac{d_1}{1 - q} - d_2 - r \alpha E[\beta] \]

The above expected total profit per unit time \( E[T_{PU, i}] \) is concave, since:

\[ \frac{\partial^2 E[T_{PU, i}]}{\partial Y^2} = \frac{-2(A_1 + A_2) D}{(1 - q) Y (E[\beta](\alpha - 1) + 1)} \leq 0 \]

So we can set the first derivative of \( E[T_{PU, i}] \) to zero so as to reach the economic lot size, as follows.

\[ Y_i^* = \left[ \frac{2(A_1 + A_2) D}{h_i D \left( \frac{(1 - q)^2}{P_1} + \frac{2q}{x} \right) + h_2 (1 - q)^2 \left( E[G]^2 + \frac{D}{P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{\alpha D E[\beta]}{P_2} \left( \left( 1 - \frac{D}{P_1} \right) - E[\beta] + E[G] \right) \right)} \right]^{\frac{1}{2}} \]

2.3. Case II: When shortage occurs and is fully backordered

In addition, Figure 2 depicts the inventory level in case II. Similar to previous case, we consider two scenarios of this case in the sequel. In first scenario, we just consider the finished product cycle, and do not consider
the raw material cycle, while in second scenario, we consider both raw material and finished product cycles, simultaneously.

Here we address the first scenario of case II. Note that $t_{R1}$ and $t_{R2}$ represent the rework period when inventory level is less than and greater than zero, respectively ($t_{R} = t_{R1} + t_{R2}$). The total revenue per cycle in this case $TR_{II}$ is similar to that of case I, i.e., sum of sales of perfect and reworked items and sales of scrapped items, which is given by $TR_{II} = v \{Q - \beta Q + \alpha \beta Q\} + s \{(1 - \alpha) \beta Q\}$. The total cost in this case involves production, setup, holding, shortage, screening and reworking costs. The production and setup costs per cycle are calculated as $PC_{II} = C_{Q}$ and $SC_{II} = A_{2}$. The area under inventory level is classified into three periods, i.e., production period $S_{1}$, rework period when inventory level is positive $S_{2}$ and depletion period $S_{3}$. The first area is calculated as $S_{1}$ is similar to that of case I which is given as $S_{1} = \frac{Q}{2P_{1}} (1 - D / P_{1})$. To calculate $S_{2}$, we should first obtain the inventory level at the end of rework period (start of depletion period) via $I_{2} = Q (1 - D / P_{1}) - \beta Q + (P_{2} - D) t_{R}$. By substituting $t_{R} = \alpha \beta Q / P_{2}$ into $I_{2}$, it simplifies to $I_{2} = Q (1 - D / P_{1}) - \beta Q + \alpha \beta Q (1 - D / P_{2})$. Therefore, the area under inventory level in rework period is calculated as $S_{2} = I_{2} t_{R2} / 2$. Since $t_{R1}$ is the time in which the shortage quantity $b = \beta Q - Q (1 - D / P_{1})$ is fully covered by reworked items, it can be calculated as $t_{R1} = b / (P_{2} - D)$ which can be re-written as $t_{R1} = \{\beta Q - Q (1 - D / P_{1})\} / (P_{2} - D)$. So $t_{R2}$ can be attained by $t_{R} - t_{R1}$ which is summarized as $t_{R2} = \alpha \beta Q / P_{2} - \{\beta Q - Q (1 - D / P_{1})\} / (P_{2} - D)$. Hence $S_{2}$ can be expressed as:
Moreover, the area under inventory level of depletion period is formulated as $S_3 = \frac{I_2}{2} \cdot t_D$, where depletion period $t_D$ is $I_2 / D$, hence, we have $S_3 = \frac{I_2^2}{2D}$. By substituting $I_2$ into $S_3$, it can be re-written as $S_3 = \frac{1}{2D} \left[ Q \left( 1 - \frac{D}{P_1} \right) - \beta Q + \alpha \beta Q \left( 1 - \frac{D}{P_2} \right) \right]^2$.

By using $S_1$, $S_2$ and $S_3$, the holding cost is formulated as follows.

$$HC_{II} = h_1 Q^2 \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{G}{2(P_2 - D)} \left[ \frac{\alpha \beta}{P_2} - \beta + \left( 1 - \frac{D}{P_1} \right) \right] + \frac{G^2}{2D} \right\}$$

(23)

In addition, the shortage cost is calculated as $KC_{II} = \pi \cdot \beta \cdot t_{r1} / 2$ where $\pi$ denotes the shortage cost per unit time per item. It can be re-expressed as follows:

$$KC_{II} = \pi \frac{Q^2}{P_2 - D} \left\{ \beta - \left( 1 - \frac{D}{P_1} \right) \right\}^2$$

(24)

The screening and reworking costs per cycle are computed as $WC_{II} = d_2 Q$ and $RC_I = r \cdot \alpha \beta Q$, similar to those of case I.

So the total cost per cycle in case II is attained by $PC_{II} + SC_{II} + HC_{II} + KC_{II} + WC_{II} + RC_{II}$ as follows.

$$TC_{II} = C_2 Q + A_2 + h_2 Q^2 \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{G}{2(P_2 - D)} \left[ \frac{\alpha \beta}{P_2} - \beta + \left( 1 - \frac{D}{P_1} \right) \right] + \frac{G^2}{2D} \right\}$$

$$+ \pi \frac{Q^2}{P_2 - D} \left\{ \beta - \left( 1 - \frac{D}{P_1} \right) \right\}^2 + d_2 Q + r \alpha \beta Q$$

(25)

Here, the total profit per cycle in case II is calculated as follows.
\[ TP_h = v \{ Q - \beta Q + \alpha \beta Q \} + s \{ (1 - \alpha) \beta Q \} - C_2 Q - A_2 \]
\[ -h_Q \left\{ \frac{1}{2P_1} \left[ 1 - \frac{D}{P_1} \right] + \frac{G}{2(P_2 - D)} \left[ \frac{\alpha \beta}{P_2} - \beta + \left( 1 - \frac{D}{P_1} \right) \right] + \frac{G^2}{2D} \right\} \]
\[ -\pi \frac{Q^2}{P_2 - D} \left\{ \beta - \left( 1 - \frac{D}{P_1} \right) \right\}^2 - d_2 Q - r \alpha \beta Q \]

Since \( \beta \) is random variable, it should be replaced with expected value \( E[\beta] \) in \( TP_h \) to calculate expected total profit \( E[TP_h] \) as follows.

\[ E[TP_h] = v \{ Q - E[\beta] Q + \alpha E[\beta] Q \} + s \{ (1 - \alpha) E[\beta] Q \} - C_2 Q - A_2 \]
\[ -h_Q \left\{ \frac{1}{2P_1} \left[ 1 - \frac{D}{P_1} \right] + \frac{E[G]}{2(P_2 - D)} \left[ \frac{\alpha E[\beta]}{P_2} - E[\beta] + \left( 1 - \frac{D}{P_1} \right) \right] + \frac{E[G]^2}{2D} \right\} \]
\[ -\pi \frac{Q^2}{P_2 - D} \left\{ E[\beta] - \left( 1 - \frac{D}{P_1} \right) \right\}^2 - d_2 Q - r \alpha E[\beta] Q \]

The cycle time \( T_h \) in this cases is obtained similar to that of case I as
\[ T_h = \frac{Q}{D} \left( \beta (\alpha - 1) + 1 \right) \]
whose expected value is calculated as follows
\[ E[T_h] = \frac{Q}{D} (E[\beta] (\alpha - 1) + 1). \]
Hence the expected total profit per unit time is calculated as follows:

\[ E[TPU_h] = \frac{E[TP_h]}{E[T_h]} \]

By simplifying the expressions in Eq. 28, we reach to:

\[ E[TPU_h] = \frac{D}{E[\beta](\alpha - 1) + 1} \left\{ v \left[ 1 - E[\beta] + \alpha E[\beta] \right] + s \left[ (1 - \alpha) E[\beta] \right] - C_2 \right\} \]
\[ -A_2 \left( \frac{1}{Q} - h_Q \left\{ \frac{1}{2P_1} \left[ 1 - \frac{D}{P_1} \right] + \frac{E[G]}{2(P_2 - D)} \left[ \frac{\alpha E[\beta]}{P_2} - E[\beta] + \left( 1 - \frac{D}{P_1} \right) \right] + \frac{E[G]^2}{2D} \right\} \right) \]
\[ -\pi \frac{Q^2}{P_2 - D} \left\{ E[\beta] - \left( 1 - \frac{D}{P_1} \right) \right\}^2 - d_2 - r \alpha E[\beta] \}

This is also a concave function, since:
\[
\frac{\partial^3 \text{E}[TPU_{II}]}{\partial Q^2} = \frac{-2A_2D}{Q^3\{E[\beta](\alpha-1)+1\}} \leq 0 \tag{30}
\]

So we can set the first derivative of \( \text{E}[TPU_{II}] \) to zero so as to obtain the economic lot size, as follows.

\[
Q_{II}^* = \left[ h_2 \left\{ \frac{D}{P_1} \left(1 - \frac{D}{P_1} \right) + \frac{DE[G]}{(P_2 - D)} \left\{ \frac{\alpha E[\beta]}{P_2} - E[\beta] + \left\{1 - \frac{D}{P_1}\right\} \right\} + \frac{E[G]^2}{2D} \right\} + \frac{2\pi D}{P_2 - D} \left\{ E[\beta] - \left\{1 - \frac{D}{P_1}\right\} \right\} \right]^{\frac{1}{2}} \tag{31}
\]

Now, we consider second scenario of case II. Similar to second scenario of case I, the order quantity of raw material \( Y \) is decision variable, and production quantity of finished product \( Q \) is dependent variable that calculated by \( Q = (1-q)Y \). The total revenue per cycle \( TR_{II} \) is given as

\[
TR_{II} = v \left\{(1-q)Y - \beta Y + \alpha \beta (1-q)Y\right\} + s \left\{(1-\alpha)\beta (1-q)Y\right\} + pqY \tag{32}
\]

The total cost of raw material \( TC_{m} \) is similar to that of second scenario of first case. The total cost of finished product \( TC_{fp} \), similar to that of first scenario of this case, where order quantity is replaced with \( Q = (1-q)Y \) as follows.

\[
TC_{fp} = C_2(1-q)Y^2 + A_2 + h_2(1-q)^2Y^2 \left\{ \frac{1}{2P_1} \left\{1 - \frac{D}{P_1}\right\} + \frac{G}{2(P_2 - D)} \left\{ \frac{\alpha \beta - \beta}{P_2} + \left\{1 - \frac{D}{P_1}\right\} \right\} + \frac{G^2}{2D} \right\} + \pi \left\{1 - \frac{D}{P_1}\right\} \right\} + d_2(1-q)Y^2 + r \alpha \beta (1-q)Y \tag{32}
\]

Therefore the total cost per cycle of second scenario is obtained by \( TC_{fp} + TC_{m} \) as follows.
\[
TC_{II} = \left\{ C_1 + C_2 (1-q) \right\} Y + A_1 + A_2 + hY^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} \\
+h_2 (1-q)^2 Y^2 \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{G}{2(P_2-D)} \left( \frac{\alpha\beta}{P_2} - \beta + \left( 1 - \frac{D}{P_1} \right) \right) + \frac{G^2}{2D} \right\} \\
+\pi \frac{(1-q)^2 Y^2}{P_2-D} \left\{ \beta - \left( 1 - \frac{D}{P_1} \right) \right\}^2 + dY + d_2 (1-q)Y + r_2 \alpha\beta (1-q)Y
\]

(33)

Then, the total profit per cycle in second scenario of case II can be calculated by \( TR_{II} - TC_{II} \) as follows.

\[
TP_{II} = v \left\{ (1-q)Y - \beta (1-q)Y + \alpha\beta (1-q)Y \right\} + s \left\{ (1-\alpha) \beta (1-q)Y \right\} + pqY \\
-\left\{ C_1 + C_2 (1-q) \right\} Y - A_1 - A_2 - hY^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} \\
-h_2 (1-q)^2 Y^2 \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{G}{2(P_2-D)} \left( \frac{\alpha\beta}{P_2} - \beta + \left( 1 - \frac{D}{P_1} \right) \right) + \frac{G^2}{2D} \right\} \\
-\pi \frac{(1-q)^2 Y^2}{P_2-D} \left\{ \beta - \left( 1 - \frac{D}{P_1} \right) \right\}^2 -dY - d_2 (1-q)Y - r_2 \alpha\beta (1-q)Y
\]

(34)

Since \( \beta \) is random variable, it should be replaced with expected value \( E[\beta] \) in \( TP_{II} \) to calculate expected total profit \( E[TP_{II}] \) as follows.
\[ E[TP_u] = v \left\{ (1-q)Y - E[\beta](1-q)Y + \alpha E[\beta](1-q)Y \right\} \\
+ s \left\{ (1-\alpha)E[\beta](1-q)Y \right\} + pqY - \left\{ C_1 + C_2(1-q) \right\}Y \]

\[-A_1 - A_2 - hY^2 \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} \]

\[-h_2(1-q)^2Y^2 \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{E[G]}{2(P_2 - D)} \left\{ \frac{\alpha E[\beta]}{P_2} - E[\beta] + \left( 1 - \frac{D}{P_1} \right) \right\} + \frac{E[G]^2}{2D} \right\} \]

\[-\pi \frac{(1-q)^2Y^2}{P_2 - D} \left\{ E[\beta] - \left( 1 - \frac{D}{P_1} \right) \right\}^2 - dY - d_2(1-q)Y - r\alpha E[\beta](1-q)Y \]

Moreover, the expected cycle time \( E[T_u] \), similar to that of first scenario, is

\[ E[T_u] = \frac{Q}{D} (E[\beta](\alpha - 1) + 1) \] where \( Q = (1-q)Y \). Hence the expected total profit per unit time is given as follows:

\[ E[TPU_u] = \frac{E[TP_u]}{E[T]} \] (36)

By simplifying the expressions in Eq. 36, we reach to:

\[ E[TPU_u] = \frac{D}{E[\beta](\alpha - 1) + 1} \left\{ v \left\{ 1 - E[\beta] + \alpha E[\beta] \right\} + s \left\{ 1 - \alpha E[\beta] \right\} \right\} \]

\[ + pq - C_1 - C_2 - \frac{A_1 + A_2}{(1-q)Y} - hY \left\{ \frac{(1-q)^2}{2P_1} + \frac{q}{x} \right\} \]

\[-h_2(1-q)Y \left\{ \frac{1}{2P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{E[G]}{2(P_2 - D)} \left\{ \frac{\alpha E[\beta]}{P_2} - E[\beta] + \left( 1 - \frac{D}{P_1} \right) \right\} + \frac{E[G]^2}{2D} \right\} \]

\[-\pi \frac{(1-q)^2Y^2}{P_2 - D} \left\{ E[\beta] - \left( 1 - \frac{D}{P_1} \right) \right\}^2 - \frac{d_1}{1-q} - d_2 + \alpha E[\beta] \]

The above expected total profit per unit time \( E[TPU_u] \) is concave, since:
\[
\frac{\partial^2 E[TPU_\mu]}{\partial Y^2} = \frac{-2(A_1 + A_2)D}{(1-q)Y^3 \{E[\beta](\alpha-1)+1\}} \leq 0
\]  

(38)

So we can set the first derivative of \( E[TPU_\mu] \) to zero so as to attain the economic lot size, as follows.

\[
Y_{\mu}^* = \left[ h_1D \left( \frac{(1-q)^2}{P_1} + \frac{2q}{x} \right) + h_2(1-q)^2 \left( \frac{D}{P_1} \left( 1 - \frac{D}{P_1} \right) + \frac{DE[G]}{P_2 - D} \left( \frac{\beta}{P_1} - E[\beta] + \left( 1 - \frac{D}{P_1} \right) \right) \right) + E[G]^2 \right] + 2\pi D \left( \frac{1-q)^2}{P_2 - D} \left( E[\beta] - \left( 1 - \frac{D}{P_1} \right) \right) \right]^2
\]

(39)

2.4. Case III: When shortage occurs and is partially backordered

Here, we analyze the case III as shown by Figure 3. Similar to previous cases, we consider two scenarios of this case in this section. In first scenario, we just consider the finished product cycle, and do not consider the raw material cycle, while in second scenario, we consider both raw material and finished product cycles, simultaneously.

>> Please insert Figure 3 here <<

First, first scenario of this case will be discussed. As can be seen, all of the shortages cannot be backordered by reworked items in this case and, hence, some amount of shortage is backordered \( (b) \) and others are not satisfied at this moment \( (l) \). To avoid lost sale, we use a special order of product at the end of rework period, whenever this case occurs. Indeed, the manufacturer uses a service from external producer to fulfill the unsatisfied demand. This special order is received gradually within interval \( t_{sp} \). In this
case, the total revenue per cycle in this case \( TR_{III} \) equals to sum of sales of perfect and reworked items and sales of scrapped items, which is given by

\[
TR_{III} = v \{Q - \beta Q + \alpha \beta Q\} + s \{(1 - \alpha) \beta Q\}.
\]

The total cost involves production, setup, holding, shortage, screening, and reworking costs. The production and setup costs per cycle are calculated as

\[
PC_{III} = C_\beta Q \quad \text{and} \quad SC_{III} = A_2.
\]

The area under inventory level at production period \( S_1 \) is similar to those of cases I and II which is given as

\[
S_1 = \frac{Q^2}{2P_1} \left(1 - \frac{D}{P_1}\right).
\]

Then the holding cost is calculated as follows.

\[
HC_{III} = h_2 \frac{Q^2}{2P_1} \left(1 - \frac{D}{P_1}\right).
\]

(40)

In addition, the shortage cost is calculated as

\[
KC_{III} = \pi \{S_2 + S_3\} \]

where \( S_2 \) and \( S_3 \) represent the rework and special order period respectively. To calculate \( S_2 \), we should first obtain the backordered and unsatisfied demand quantities. As can be seen in Figure 3, sum of backordered and unsatisfied demand is

\[
b + l = \beta Q - l_{\max}\]

which simplifies to

\[
\beta Q - Q \left(1 - \frac{D}{P_1}\right).
\]

On the other hand, \( b \) equals to \((P_2 - D) t_R\) which simplifies by substituting \( t_R = \alpha \beta Q / P_2 \) to

\[
b = \alpha \beta Q \left(1 - \frac{D}{P_2}\right).
\]

Therefore the unsatisfied demand at the end of rework period is attained as

\[
l = Q \left(1 - \frac{D}{P_1}\right) - \beta Q + \alpha \beta Q \left(1 - \frac{D}{P_2}\right).
\]

Here, we can calculate the shortage area under inventory level in rework period as

\[
S_2 = (b + 2l) t_R / 2.
\]

By substituting \( b \), \( l \) and \( t_R \) into \( S_2 \), it can be expressed as:

\[
S_2 = \frac{\alpha \beta Q}{2P_2} \left\{\alpha \beta Q \left(1 - \frac{D}{P_2}\right) - 2GQ\right\}.
\]

(41)

In addition, the third area under curve is calculated as

\[
S_3 = l t_{sp} / 2
\]

where special order period \( t_{sp} \) is \( l / D \), hence, we reach \( S_3 = l^2 / (2D) \). By substituting \( l \) into \( S_3 \), it can be re-written as:
\[ S_3 = \frac{G^2 Q^2}{2D} \]  

(42)

By using \( S_2 \) and \( S_3 \), the shortage cost is calculated as:

\[ KC_m = \pi Q^2 \left\{ \frac{\alpha \beta}{2P_2} \left\{ \alpha \beta \left(1 - \frac{D}{P_2}\right) - 2G \right\} + \frac{G^2}{2D} \right\} \]  

(43)

The screening and reworking costs per cycle are computed as \( WC_m = d_s Q \) and \( RC_m = r \alpha \beta Q \), similar to those of cases I and II.

So the total cost per cycle in case III is calculated by \( PC_m + SC_m + HC_m + KC_m + WC_m + RC_m \) as follows.

\[ TC_m = C_m Q + A_2 + h_2 \frac{Q^2}{2P_1} \left(1 - \frac{D}{P_1}\right) + \pi Q^2 \left\{ \frac{\alpha \beta}{2P_2} \left\{ \alpha \beta \left(1 - \frac{D}{P_2}\right) - 2G \right\} + \frac{G^2}{2D} \right\} + d_s Q + r \alpha \beta Q \]  

(44)

Furthermore, the total profit per cycle in case III is attained as follows.

\[ TP_m = v \left\{ Q - \beta Q + \alpha \beta Q \right\} + s \left\{ (1 - \alpha) \beta Q \right\} - C_m Q - A_2 - h_2 \frac{Q^2}{2P_1} \left(1 - \frac{D}{P_1}\right) - \pi Q^2 \left\{ \frac{\alpha \beta}{2P_2} \left\{ \alpha \beta \left(1 - \frac{D}{P_2}\right) - 2G \right\} + \frac{G^2}{2D} \right\} \]  

(45)

\[ -d_s Q - r \alpha \beta Q \]

The defective rate is replaced with expected value \( E[\beta] \) to calculate expected total profit \( E[TP_m] \) as follows.

\[ E[TP_m] = v \left\{ Q - E[\beta]Q + \alpha E[\beta]Q \right\} + s \left\{ (1 - \alpha) E[\beta]Q \right\} - C_m Q - A_2 - h_2 \frac{Q^2}{2P_1} \left(1 - \frac{D}{P_1}\right) - \pi Q^2 \left\{ \frac{\alpha E[\beta]}{2P_2} \left\{ \alpha E[\beta] \left(1 - \frac{D}{P_2}\right) - 2E[G] \right\} + \frac{E[G]^2}{2D} \right\} \]  

(46)

\[ -d_s Q - r \alpha E[\beta]Q \]

The cycle time \( T_m \) is obtained by \( T_m = t_p + t_r + t_{sp} \) as:

\[ T_m = t_p + t_r + t_{sp} \]
\[ T_{m} = \frac{Q}{D} (\beta (\alpha - 1) + 1) \]  

(47)

Note that \( T_{m} \geq 0 \). The expected total profit per unit time is given as follows:

\[ E[TPU_{m}] = E\left[ \frac{TP_{m}}{E[T_{m}]} \right] \]  

(48)

By simplifying the expressions in Eq. 48, we reach to:

\[
E[TPU_{m}] = \frac{D}{E[\beta](\alpha - 1) + 1} \left\{ \nu \left[ 1 - E[\beta] + \alpha E[\beta] \right] + s \left[ (1 - \alpha) E[\beta] \right] \right\} 
- C_{2} - \frac{A_{2}}{Q} - h_{2} \frac{Q}{2P_{1}} \left( 1 - \frac{D}{P_{1}} \right) - \pi Q \left\{ \frac{\alpha E[\beta]}{2P_{2}} \left\{ \alpha E[\beta] \left( 1 - \frac{D}{P_{2}} \right) - 2E[G] \right\} + \frac{E[G]^{2}}{2D} \right\} 
- d_{2} - r\alpha E[\beta] \]

This is a concave function, since:

\[
\frac{\partial^{2}E[TPU_{m}]}{\partial Q^{2}} = \frac{-2A_{2}D}{Q^{3}\left( E[\beta](\alpha - 1) + 1 \right)} \leq 0
\]

(50)

So we can set the first derivative of \( E[TPU_{m}] \) to zero so as to calculate the economic lot size, as follows.

\[
Q_{m}^{*} = \left[ h_{2} \frac{D}{P_{1}} \left( 1 - \frac{D}{P_{1}} \right) + \pi \left\{ \frac{\alpha DE[\beta]}{P_{2}} \left\{ \alpha E[\beta] \left( 1 - \frac{D}{P_{2}} \right) - 2E[G] \right\} + \frac{E[G]^{2}}{2D} \right\} \right]^{\frac{1}{2}} 
\]

(51)

Now, we consider second scenario of case III. Similar to second scenario of previous cases, the order quantity of raw material \( Y \) is decision variable, and production quantity of finished product \( Q \) is dependent variable that calculated by \( Q = (1 - q)Y \). The total revenue per cycle \( TR_{m} \) and the total cost of raw material \( TC_{m} \) is similar to those of second scenario of previous cases.

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The total cost of finished product $TC_{fp}$ is similar to that of first scenario of current case, where order quantity is replaced with $Q = (1-q)Y$ as follows.

$$TC_{fp} = C_1 (1-q)Y + A_2 + h_2 \frac{(1-q)^2 Y^2}{2P_1} \left(1 - \frac{D}{P_1}\right) + \pi (1-q)^2 Y^2 \left\{ \frac{\alpha \beta}{2P_2} \left( \frac{1}{x} \right) + \left(1 - \frac{D}{P_2}\right) - 2G \right\} + d_2 (1-q)Y + r \alpha \beta (1-q)Y$$

(52)

Therefore the total cost per cycle of second scenario is obtained by $TC_{fp} + TC_{rm}$ as follows.

$$TC_{III} = \{C_1 + C_2 (1-q)\}Y + A_1 + A_2 + hY^2 \left\{ \frac{(1-q)^2 Y^2}{2P_1} + \frac{q}{x} \right\} + h_2 \frac{(1-q)^2 Y^2}{2P_1} \left(1 - \frac{D}{P_1}\right) + \pi (1-q)^2 Y^2 \left\{ \frac{\alpha \beta}{2P_2} \left( \frac{1}{x} \right) + \left(1 - \frac{D}{P_2}\right) - 2G \right\} + dY + d_2 (1-q)Y + r \alpha \beta (1-q)Y$$

(53)

Then, the total profit per cycle in second scenario of case III can be calculated by $TR_{III} - TC_{III}$ as follows.

$$TP_{III} = v \left\{ (1-q)Y - \beta (1-q)Y + \alpha \beta (1-q)Y \right\} + s \left\{ (1-\alpha)(1-q)Y \right\} + pqY - \{C_1 + C_2 (1-q)\}Y - A_1 - A_2 - hY^2 \left\{ \frac{(1-q)^2 Y^2}{2P_1} + \frac{q}{x} \right\} - h_2 \frac{(1-q)^2 Y^2}{2P_1} \left(1 - \frac{D}{P_1}\right) - \pi (1-q)^2 Y^2 \left\{ \frac{\alpha \beta}{2P_2} \left( \frac{1}{x} \right) + \left(1 - \frac{D}{P_2}\right) - 2G \right\} - dY - d_2 (1-q)Y - r \alpha \beta (1-q)Y$$

(54)

The expected total profit $E[TP_{III}]$ is calculated as follows.
Moreover, the cycle time is similar to that of first scenario where $Q = (1-q)Y$. Hence the expected total profit per unit time is given as follows:

$$E[T_{PU_{III}}] = E[T_{P_{III}}]$$

By simplifying the expressions in Eq. 56, we reach to:

$$E[T_{PU_{III}}] = \frac{D}{E[\beta](\alpha-1)+1} \left\{ v \left[ 1 - E[\beta] + \alpha E[\beta] \right] + s \left\{ (1-\alpha) E[\beta] \right\} + \frac{pq}{1-q} - \frac{C_1}{1-q} \right\}$$

The above expected total profit per unit time $E[T_{PU_{III}}]$ is concave, since:

$$\frac{d^2E[T_{PU_{III}}]}{dY^2} = \frac{-2(A_1 + A_2)D}{(1-q)Y^3 \{ E[\beta](\alpha-1)+1 \}} \leq 0$$

So we can set the first derivative of $E[T_{PU_{III}}]$ to zero so as to gain the economic lot size, as follows.
$$Y_m^* = \left[ \frac{2(A_1 + A_2)D}{h_1D \left( \frac{(1-q)^2}{P_1} + \frac{2q}{x} \right) + h_2D \left( \frac{1-D}{P_1} \right) + \pi (1-q)^2 \left\{ \frac{\alpha DE[\beta]}{P_2} \left( \frac{1}{P_2} \right) - 2E[G] \right\} + E[G]^2} \right]^{\frac{1}{2}}$$

(59)

3. Illustrative Experiments

We are going to present and discuss a numerical example in this section.
Let us consider a manufacturer which orders raw materials from external supplier with purchase cost $C_1 = 10$ units of money per item and ordering cost $A_1 = 250$ units of money per order. Then, manufacturer produces a finished product via a finite production rate $P_1 = 200$ units per month with production cost $C_2 = 20$ units of money per item. The demand rate for this product is $D = 100$ units per month. In addition, assume that the machine setup cost is $A_2 = 150$ units of money per production cycle. Moreover the holding costs for raw materials and finished products are $h_1 = 2$ and $h_2 = 5$ units of money per item per unit time. The raw materials contain imperfect quality items with the rate of $q = 0.12$ and a 100% screening process is carried out upon receive of lot size with the rate of $x = 100$ units per month. On the other hand, the production system is also defective and a fraction of finished products are imperfect. It is assumed that the defective rate follows a uniform distribution on interval $0.08 - 0.12$ ($\beta \sim \text{Unifom}(0.08, 0.12)$). Therefore, the expected value of the defective rate $\beta$ is calculated as:

$$E[\beta] = \int_{a}^{b} \beta f(\beta) \, d\beta = \int_{a}^{b} \beta \frac{1}{b-a} \, d\beta = \frac{a+b}{b-a} \frac{1}{2} = \frac{0.08 + 0.12}{2} = 0.10$$

The screening cost for raw materials and finished products are $d_1 = 5$ and $d_2 = 10$ units of money per item. The perfect finished products are sold at
price \( v = 50 \) units of money per item. The imperfect raw materials are sold after screening process with price \( p = 2 \) units of money per item, while the defective finished products go under a further rework process with rework rate \( p_2 = 250 \) units per month and rework cost \( r = 5 \) units of money. A percent of defective products are reworkable \( (\alpha = 0.8) \) and have potential to become perfect after reworking process, while others are scrapped items and sold at a lower price \( s = 8 \). Moreover, the backorder shortage cost is \( \pi = 4 \) units of money per item per unit time. Since defective rate of machine is a random variable, three cases are possible regarding occurrence of backordering shortage. Table 1 presents the optimal results for decision variables and total profits.

>> Please insert Table 1 here <<

To select the optimal solution for this example, the optimal case should be selected. As we discussed before, if defective rate falls within \( 0 \) and \( 1 - D / P_i \) (0.50), the first case is satisfied, else if defective rate falls within \( 1 - D / P_i \) (0.50) and \( \{1 - \alpha(1 - D / P_i)\} \) (0.9615), the second case is true, and otherwise, the third case is selected. Since the expected value of defective rate is 0.1 which falls within first interval, the optimal solution is that of the first one. To further analyze the obtained results, three cases are compared together, here. As can be seen, the optimal cycle time \( T \) decreases from first case to third one. The third case have higher optimal order and production quantities \( (Y \text{ and } Q) \), while the first and second ones have next higher quantities respectively. As an important observation of optimal solution of this example is that the third case shows best achievement in terms of both total profit \( (TP) \) and total profit per unit time \( (TPU) \) among all cases. Of course, this result is just true for this special parameters, and further results can be obtained via an analysis of sensitivity. Therefore, we
perform an analysis by changing the value of input parameters, in order to assess the outputs under various inputs.

>> Please insert Figure 4 here <<
>> Please insert Figure 5 here <<
>> Please insert Figure 6 here <<
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>> Please insert Figure 11 here <<
>> Please insert Figure 12 here <<
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>> Please insert Figure 14 here <<
>> Please insert Figure 15 here <<
>> Please insert Figure 16 here <<
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>> Please insert Figure 18 here <<
>> Please insert Figure 19 here <<

The consideration of raw materials, backordering shortage and reworking are essential issues in our proposed manufacturing model. Thus, some analysis are carried out here by fluctuating the imperfect rate $q$, holding cost $h$, of raw materials, shortage cost of finished products $\pi$, and percentage of reworkable items $\alpha$, while all other parameters are kept unchanged. The result are depicted by Figures 4-19. Moreover, Tables 2-4 summarize the results for each case, depicted by these figures, at a glance. From Figures 4-11, it is clear that when the imperfect rate for raw materials...
$(q)$ and holding cost $(h)$ increase, the total profit of three cases become close to each other. Moreover, when the imperfect rate for raw materials $(q)$ tends to 1, the problem tends to become unbeneficial. From Figures 8 and 9, it can be seen that when holding cost increases, all of outputs (order size, production size and total profits) decreases significantly. Since in Case I there is no shortage of finished products, the order size, production size, and total profits do not change in this case. For this case, the raw material order size is fixed at $Y_i = 160.5249$ (Figure 12), the finished product production size $Q_i$ is fixed at 141.2619 (Figure 13), the total profit $TP_i$ is fixed at 13324.3934 (Figure 14), and the total profit per unit time $TPU_i$ is fixed at 9624.9014. Moreover, the total profits in this case ($TP_i$ and $TPU_i$) is less than those of Case III, and greater than those of Case II. The total profit of second case $TP_{II}$ starts from 12574.5545 and then decreases ultimately, while that of third case $TP_{III}$ starts from 15894.0332, increases to 16029.0845 and then decreases. In addition, the total profit per unit of second case $TPU_{II}$ starts from 9592.5027 and then decreases, while that of third case $TPU_{III}$ starts from 9713.8521 and then increases. From Figures 16-19, it is clear that when the imperfect rate for raw materials percentage of reworkable items $\alpha$ increases, the total profit of three cases become far away from each other. Generally, the third case is more beneficial among all cases in terms of total profit and total profit per unit time.

>> Please insert Table 2 here <<

>> Please insert Table 3 here <<

>> Please insert Table 4 here <<
4. Conclusions

In real-world manufacturing systems, there exist imperfect raw materials and defective products. In order to address these issues, this paper proposed a manufacturer system in which raw materials are supplied from external source, and a finished product is produced. The structure of this manufacturing planning problem is based on EPQ framework. The demand rate for an item is pre-known and constant, all quantity of an order are received instantaneously and all products are produced gradually, via a finite production rate, products are entirely consumed when the next order is commenced, no safety stock is allowed, there is no quantity discount, and ordering/setup cost is fixed per order/production. In addition to basic assumptions, we consider there is a fraction of imperfect raw materials, as well as a defective rate of production system. A 100% screening process is carried out upon receive of raw materials, and imperfect items are sold at a discounted price. A percent of defective products are reworkable and have potential to become perfect after reworking process, while others are scrapped items and sold at a lower price. The reworkable finished products are go under a reworking process on same machine. The defective rate is assumed to be a random variable, resulting three possible cases regarding occurrence of backordering shortage. We design two scenarios for each case, resulting six total states. The concavity of total profit per unit times are derived for each case separately. Then the optimal closed form solution is derived for each case separately. The proposed manufacturing model is illustrated via a numerical example. An extensive sensitivity analysis is done to assess the impact of input changes on the outputs variations.

As an interesting opportunity for future research is to adopt the proposed model for a more general situation where partial backordering shortage is permitted. Moreover, one may consider a sampling inspection instead of 100% inspection, the learning and forgetting effects in inspection, or quantity discount as future researches of this study.
REFERENCES


Hadi Mokhtari is currently an Assistant Professor of Industrial Engineering in University of Kashan, Iran. His current research interests include the applications of operations research and artificial intelligence techniques to the areas of project scheduling, production scheduling, and manufacturing supply chains. He also published several papers in international journals such as Computers and Operations Research, International Journal of Production Research, Applied Soft Computing, Neurocomputing, International Journal of Advanced Manufacturing Technology, IEEE Transactions on Engineering Management, and Expert Systems with Applications.

Captions:

Figure 1: The inventory level for manufacturing system in case I
Figure 2: The inventory level for manufacturing system in case II
Figure 3: The inventory level for manufacturing system in case III
Figure 4: Behavior of raw material order size $Y$ for each case for different values of imperfect rate $q$
Figure 5: Behavior of finished product production size $Q$ for each case for different values of imperfect rate $q$
Figure 6: Behavior of total profit $TP$ for each case for different values of raw material imperfect rate $q$
Figure 7: Behavior of total profit per unit time $TPU$ for each case for different values of raw material imperfect rate $q$
Figure 8: Behavior of raw material order size $Y$ for each case for different values of holding cost of raw materials $h_1$
Figure 9: Behavior of finished product production size $Q$ for each case for different values of holding cost of raw materials $h_1$
Figure 10: Behavior of total profit $TP$ for each case for different values of holding cost of raw materials $h_1$
Figure 11: Behavior of total profit per unit time $TPU$ for each case for different values of holding cost of raw materials $h_1$
Figure 12: Behavior of raw material order size $Y$ for each case for different values of shortage cost $\pi$
Figure 13: Behavior of finished product production size $Q$ for each case for different values of shortage cost $\pi$

Figure 14: Behavior of total profit $TP$ for each case for different values of shortage cost $\pi$

Figure 15: Behavior of total profit per unit time $TPU$ for each case for different values of shortage cost $\pi$

Figure 16: Behavior of raw material order size $Y$ for each case for different values of percentage of reworkable items $\alpha$

Figure 17: Behavior of finished product production size $Q$ for each case for different values of percentage of reworkable items $\alpha$

Figure 18: Behavior of total profit $TP$ for each case for different values of percentage of reworkable items $\alpha$

Figure 19: Behavior of total profit per unit time $TPU$ for each case for different values of percentage of reworkable items $\alpha$

Table 1: Optimal decision variables and total profits for numerical example

Table 2: Results of sensitivity analysis for case I

Table 3: Results of sensitivity analysis for case II

Table 4: Results of sensitivity analysis for case III
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Figure 7: Behavior of total profit per unit time $TPU$ for each case for different values of raw material imperfect rate $q$
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Figure 9: Behavior of finished product production size $Q$ for each case for different values of holding cost of raw materials $h_i$. 


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Figure 12: Behavior of raw material order size $Y$ for each case for different values of shortage cost $\pi$

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Figure 15: Behavior of total profit per unit time $TPU$ for each case for different values of shortage cost $\pi$
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Figure 17: Behavior of finished product production size $Q$ for each case for different values of percentage of reworkable items $\alpha$
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Figure 19: Behavior of total profit per unit time $TPU$ for each case for different values of percentage of reworkable items $\alpha$
Table 1: Optimal decision variables and total profits for numerical example

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Table 2: Results of sensitivity analysis for case I

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Table 3: Results of sensitivity analysis for case II

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Table 4: Results of sensitivity analysis for case III

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