Analytic Solution of a System of Linear Distributed Order Differential Equations in the Reimann-Liouville Sense

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Abstract

In this paper, solution of a system of linear differential equations of distributed order in the Riemann-Liouville sense is analytically obtained. The distributed order relaxation equation is a special case of the system investigated in this paper. The solution of the mentioned system is introduced on the basis of a function which can be considered as the distributed order generalization of the matrix Mittag-Leffler functions. It is shown that this generalized function in two special cases of the weight function can be expressed in terms of the multivariate Mittag-Leffler functions and the Wright functions.

Key words: Analytic solution; distributed order differential equation; Reimann-Liouville fractional derivative; Mittag-Leffler function; relaxation process.

1. Introduction

Fractional order calculus is used to model physical processes exhibiting anomalous dynamics, such as fractional order diffusion [1-2] and relaxation [3]. Relaxation processes modeled by integer order or fractional order derivatives, could be generalized by the concept of distributed order derivatives which allow to model processes with scaling law change [4]. This could also be done by utilizing either Riemann-Liouville or Caputo definition to construct a distributed order differentiation operator for modeling distributed order relaxation. Although the two approaches turn out to be equivalent in the fractional order case, this is not true in distributed order case. Since both lead to successful models describing the actual physical phenomena, a further study of both cases appeals. The typical approach to obtain the solution of such equations in time domain is based on direct inverse Laplace transform using

Fourier-Mellin formula or Titchmarsh theorem [5]. This approach which is considered in most papers [4, 6-9] in the literature, results in a solution expressed by a Laplace-type integral in both Riemann-Liouville and Caputo cases. As the link between this representation and the Fox-Wright functions used in fractional order differential equations is not clear, an alternative representation of the solution which incorporates Fox-Wright functions is proposed in [9] in which however, the Laplace-type integral still lingers. This problem has also been studied by using Laguerre series to give an approximation of the solution in [10]. In addition, the asymptotic of the solution is investigated in [6]. This problem is also treated in the case of triple impulses and double impulses as special cases of the weight function in [4] and [11] respectively. It is observed that a coherent extension of the Mittag-Leffler functions for distributed order calculus is in interest. In this regard, we present a new representation of the solution associated with general weight functions which excludes the Laplace-type integral. This representation consists of a series expansion which exhibits the impact of the weight function on the distribution of the orders in the time domain. The reduction of the solution to the Mittag-Leffler function is easily obtained in case of a single impulse weight function. Moreover, it is shown that the solution turns into the multivariate Mittag-Leffler function in case of several impulses as the weight function. Also, by choosing a unitary or exponential weight function, it is possible to express the solution in terms of the Wright functions.

This paper is organized as follows. In Section 2 some preliminaries are reviewed and some useful lemmas are presented with regard to distributed order calculus. Section 3 is devoted to main results where we present the solution of system of linear differential equations of distributed order in Riemann-Liouville sense in terms of what could be construed as the distributed order generalization of Mittag-Leffler functions. Some properties of these functions are stated and the problem is investigated further in two special cases of the weight functions. At the end, numerical examples are presented to simulate the solution of a distributed order relaxation equation and a system of differential equations with multiple fractional order operators as special cases in Section 4. Finally, the paper is concluded in Section 5.

2. Distributed order calculus

In this section some definitions and lemmas are presented that are essential for achieving the main results in the rest of the paper. Prior to commencement, some of the notations used in this paper are introduced in Table 1. The cornerstone of fractional calculus is based on the extension of integration order to real numbers in integral operators. This is realized by introducing fractional integration operator [12 p.65]

$$ {}_0I_t^{\alpha} f\left(t\right) = \int_0^t \left(\left(t-\tau\right)^{\alpha-1} / \Gamma(\alpha)\right) f(\tau) d\tau \tag{1} $$

Caputo fractional derivative is defined based on this definition as follows [12 p.78].

$$
{}_{0}^{C}D_{t}^{\alpha} f(t) = \begin{cases} f(t), & \alpha = 0 \\ {}_{0}I_{t}^{1-\alpha}\, \dot{f}\left(t\right), & 0 < \alpha < 1 \\ \dot{f}(t), & \alpha = 1 \end{cases} \tag{2}
$$

Table

In this paper, we consider the Riemann-Liouville fractional derivative which is defined as [12 p.62]

$$
{}_{0}^{RL}D_{t}^{\alpha} f(t) = \begin{cases} f(t), & \alpha = 0 \\ \dfrac{d}{dt}\, {}_{0}I_{t}^{1-\alpha}\, f\left(t\right), & 0 < \alpha < 1 \\ \dot{f}(t), & \alpha = 1 \end{cases} \tag{3}
$$

Riemann-Liouville fractional derivative (3) is equivalent to Caputo fractional derivative (2) if the function subject to the operator has zero initial conditions [12]. Integrating operator (1) over the order of integration with a weight function results in the so called distributed order integral [13]

$$
{}_{0}I_{t}^{w(\alpha)} f(t) = \int_{0}^{1} w(\alpha)\, {}_{0}I_{t}^{\alpha} f(t) d\alpha \tag{4}
$$

In the definition above, $w:\mathbb{R}\to\mathbb{R}$ denotes the weight function, for which $w(\alpha)=0$ holds for $\alpha \in (-\infty,0]\cup[1,+\infty)$. Distributed order derivative could also be defined in a similar way

$$
{}_{0}^{RL}D_{t}^{w(\alpha)} f(t) = \int_{0}^{1} w(\alpha)\, {}_{0}^{RL}D_{t}^{\alpha} f(t) d\alpha \tag{5}
$$

Power functions with real powers tend to appear in fractional calculus frequently. Laplace transform of these functions is given by [14]

$$
L_{t\to s}\left\{t^{a}\right\} = \Gamma\left(a+1\right)/ s^{a+1}, \quad a \in \left(-1,+\infty\right) \tag{6}
$$

Riemann-Liouville fractional derivative of a power function is given by [12]

$$
{}_{0}^{RL}D_{t}^{\alpha} t^{\mu} = \Gamma\left(\mu+1\right) t^{\mu-\alpha} / \Gamma\left(\mu-\alpha+1\right), \quad t>0, \mu>-1 \tag{7}
$$

According to (6), the representation of fractional integral of a function in the Laplace domain is given by [12]

$$
L_{t\to s}\left\{ {}_{0}I_{t}^{\alpha} f\left(t\right)\right\} = F\left(s\right)/ s^{\alpha}, \tag{8}
$$

where $F(s)=L_{t\to s}\{f(t)\}$. In addition, the representation of fractional derivative of a function with zero initial conditions in the Laplace domain is given by [12]

$$L_{t\to s}\left\{{}_{0}^{RL}D_t^{\alpha} f(t)\right\}=s^{\alpha}F(s) \tag{9}$$

We will also need the following Laplace transformation

$$L_{t\to s}\{-\ln t-\gamma\}=(\ln s)/s, \tag{10}$$

in which γ is the Euler-Mascheroni constant [14]. Before proceeding, we review some special functions used in fractional calculus. It has been shown that the solution of linear differential equations of fractional order with constant coefficients is expressed in terms of Mittag-Leffler functions [12 ch.4]. The Mittag-Leffler function is defined by the series [12 p.16]

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty} z^{\alpha k}/\Gamma(\alpha k+\beta),\qquad \alpha,\beta>0,\quad z\in\mathbb{C} \tag{11}$$

As a matter of fact, function (11) has been generalized in several ways to fit different problems in fractional calculus. A generalization of (11), which appears in the solution of fractional differential equations by the operational methods, is the multivariate Mittag-Leffler function [15]. This function is defined by

$$E_{(\beta_1,\beta_2,\dots,\beta_n),\sigma}(z_1,z_2,\dots,z_n)=\sum_{k=0}^{+\infty}\sum_{l_1+l_2+\cdots+l_n=k}\binom{k}{l_1,l_2,\cdots,l_n}\frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma\left(\sum_{i=1}^{n} l_i\beta_i+\sigma\right)}, \tag{12}$$

in which $\binom{k}{l_1,l_2,\cdots,l_n}=\dfrac{k!}{l_1!l_2!\cdots l_n!}$. The Wright function, which frequently appears in fractional order diffusion-wave equations, is defined by the series [12 p.37]

$$W_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)},\qquad \alpha,\beta>0,\quad z\in\mathbb{C} \tag{13}$$

Wright functions and Mittag-Leffler functions are both generalized by the following definition

$$\begin{aligned}{}_p\Psi_q\left[z;\begin{matrix}(a_1,a_1') & (a_2,a_2') & \cdots & (a_p,a_p')\\ (b_1,b_1') & (b_2,b_2') & \cdots & (b_q,b_q')\end{matrix}\right]\\ =\sum_{k=0}^{+\infty}\frac{\Gamma(a_1+a_1'k)\Gamma(a_2+a_2'k)\cdots\Gamma(a_p+a_p'k)z^k}{\Gamma(b_1+b_1'k)\Gamma(b_2+b_2'k)\cdots\Gamma(b_q+b_q'k)k!}\end{aligned} \tag{14}$$

which is called the Fox-Wright function [16].

Preposition 1

Representation of distributed order integral of a function in the Laplace domain is given by

$$L_{t\to s}\left\{ {}_0I_t^{w(\alpha)} f(t)\right\} = L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s} F(s), \tag{15}$$

where $F(s) = L_{t\to s}\left\{ f(t)\right\}$.

<u>Proof:</u>

Taking Laplace transform of (4) and using (8) result in

$$L_{t\to s}\left\{ {}_0I_t^{\alpha} f(t)\right\} = \int_0^1 w(\alpha)F(s)/s^{\alpha}d\alpha \tag{16}$$

Using the relation $1/s^{\alpha} = \exp(-\alpha\ln s)$ in (16), we obtain

$$\begin{aligned} L_{t\to s}\left\{ {}_0I_t^{w(\alpha)} f(t)\right\} &= \left(\int_0^{+\infty} w(\alpha)\exp(-\alpha\ln s)d\alpha\right)F(s) \\ &= L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s} F(s) \end{aligned} \tag{17}$$

□

Preposition 2

Representation of distributed order derivative of a function with zero initial conditions in the Laplace domain is given by

$$L_{t\to s}\left\{ {}_0^{RL}D_t^{w(\alpha)} f(t)\right\} = L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=-\ln s} F(s) \tag{18}$$

where $F(s) = L_{t\to s}\left\{ f(t)\right\}$.

Proof of Preposition 2 is easily obtained in a similar way to the proof of Preposition 1. We continue this section with an extension of the fractional power function $t^a, a\in(-1,+\infty)$ by introducing the distributed power function which turns out to be a remarkably general function, resulting in various well-known functions as its special cases. This function is defined by

$$p(t;f) \triangleq \int_0^{+\infty}\left(t^{\alpha-1}/\Gamma(\alpha)\right) f(\alpha)d\alpha \tag{19}$$

In which $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $f(\alpha)=0$ for $\alpha\in(-\infty,0]$. It is observed that $p\left(t;\Gamma(\alpha)\delta(\alpha-a-1)\right)=t^a$ which indicates how the fractional power function is a special case of this function. It is shown that this is also true for the more general Fox-Wright functions in the following preposition.

Preposition 3

It could be shown that

1. $p\left(t;\sum_{k=0}^{+\infty}\delta\left(\alpha-k\beta_1-\beta_2\right)\right)=t^{\beta_2-1}E_{\beta_1,\beta_2}\left(t^{\beta_1}\right),\quad \beta_1,\beta_2>0$

2. $p\left(t;\sum_{k=0}^{+\infty}\delta\left(\alpha-k\beta_1-\beta_2\right)/k!\right)=t^{\beta_2-1}W_{\beta_1,\beta_2}\left(t^{\beta_1}\right),\quad \beta_1,\beta_2>0$

3. $p\left(t;\sum_{k=0}^{+\infty}\delta\left(\alpha-k-1\right)\frac{\Gamma\left(a_1+a_1'k\right)\Gamma\left(a_2+a_2'k\right)\cdots\Gamma\left(a_p+a_p'k\right)}{\Gamma\left(b_1+b_1'k\right)\Gamma\left(b_2+b_2'k\right)\cdots\Gamma\left(b_q+b_q'k\right)}\right)=$

$$ {}_p\Psi_q\left[t;\begin{matrix}\left(a_1,a_1'\right) & \left(a_2,a_2'\right) & \cdots & \left(a_p,a_p'\right)\\ \left(b_1,b_1'\right) & \left(b_2,b_2'\right) & \cdots & \left(b_q,b_q'\right)\end{matrix}\right] $$

<u>Proof:</u>

Proof is directly obtained from definition (19). □

Laplace transform of distributed order power functions is given in the following lemma.

Lemma 1

Let $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $f\left(\alpha\right)=0$ for $\alpha\in\left(-\infty,0\right]$. Then, $L_{t\to s}\left\{p\left(t;f\left(\alpha\right)\right)\right\}=F\left(\ln s\right)$, where $F\left(s\right)=L_{t\to s}\left\{f\left(t\right)\right\}$.

<u>Proof:</u>

Taking Laplace transform of (19) using (6) gives

$$ L_{t\to s}\left\{p\left(t;f\left(\alpha\right)\right)\right\}=\int_0^{+\infty}\left(f\left(\alpha\right)/s^{\alpha}\right)d\alpha \tag{20} $$

Using the relation $1/s^{\alpha}=\exp\left(-\alpha\ln s\right)$ in (20) we obtain

$$ L_{t\to s}\left\{p\left(t;f\left(\alpha\right)\right)\right\}=\int_0^{+\infty}f\left(\alpha\right)\exp\left(-\alpha\ln s\right)d\alpha=F\left(\ln s\right) \tag{21} $$

in which $F\left(s\right)=L_{\alpha\to s}\left\{f\left(\alpha\right)\right\}$. □

Distributed order integrals of distributed order power functions could be calculated using the following lemma.

Lemma 2

Let $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $f(\alpha)=0$ for $\alpha\in(-\infty,0]$. Then,

$$ {}_0I_t^{w(\alpha)}p\big(t;f(\alpha)\big)=p\big(t;w(\alpha)*f(\alpha)\big), \tag{22}$$

in which $w(\alpha)*f(\alpha)=\int_0^{\alpha}w(\alpha-\tau)f(\tau)d\tau$.

Proof:

According to Preposition 1,

$$L_{t\to s}\left\{{}_0I_t^{w(\alpha)}p\big(t;f(\alpha)\big)\right\}=L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s}L_{t\to s}\left\{p\big(t;f(\alpha)\big)\right\} \tag{23}$$

Using Lemma 1, (23) yields

$$\begin{aligned}L_{t\to s}\left\{{}_0I_t^{w(\alpha)}p\big(t;f(\alpha)\big)\right\}&=L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s}F(\ln s)\\&=L_{t\to s}\left\{p\big(t;w*f(\alpha)\big)\right\},\end{aligned} \tag{24}$$

which results in (22). □

We can use the following lemma to determine the distributed order derivatives of distributed order power functions.

Lemma 3

Let $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $f(\alpha)=0$ for $\alpha\in(-\infty,1]$. Then,

$$ {}_0^{RL}D_t^{w(\alpha)}p\big(t;f(\alpha)\big)=p\big(t;w(-\alpha)*f(\alpha)\big), \tag{25}$$

in which $w(-\alpha)*f(\alpha)=\int_0^{1+\alpha}w(\tau-\alpha)f(\tau)d\tau$.

Proof:

Taking the Laplace transform of the left side of (25) and using Preposition 2 gives

$$L_{t\to s}\left\{{}_0^{RL}D_t^{w(\alpha)}p\big(t;f(\alpha)\big)\right\}=L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=-\ln s}L_{t\to s}\left\{p\big(t;f(\alpha)\big)\right\} \tag{26}$$

Using Lemma 1 yields

$$L_{t\to s}\left\{{}_0^{RL}D_t^{w(\alpha)}p\big(t;f(\alpha)\big)\right\}=L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=-\ln s}F(\ln s) \tag{27}$$

Regarding the relation $L_{\alpha\to s}\left\{w(-\alpha)*f(\alpha)\right\}=L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=-s}F(s)$ and using Lemma 1, it is concluded that

$$L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=-\ln s}F(\ln s)=L_{t\to s}\left\{p\left(t;w(-\alpha)*f(\alpha)\right)\right\}, \tag{28}$$

which is the Laplace transform of the right side of (25). □

3. Distributed order system of linear differential equations

Distributed order system of linear differential equations is supposed to be a generalization of fractional order system of linear differential equations. This generalization may be done in two ways, as extending fractional order system to distributed order one is possible by using Caputo or Riemann-Liouville operators. In fact, the actual physical processes in interest, such as diffusion phenomena and relaxation patterns, could be modeled by either of the operators successfully [6, 17]. We introduce distributed order system of linear differential equations in Riemann-Liouville sense as

$$\dot{x}(t)=A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha+Bg(t),\qquad x(0)=x_0, \tag{29}$$

in which $x(t)\in\mathbb{R}^n$ is the pseudo state and $g:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $g(t)=0$ for $t\in(-\infty,0)$. Also, $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times 1}$. Likewise, distributed order system described in Caputo sense is defined by

$$\int_0^1 w(\alpha)\,{}_0^{C}D_t^{\alpha}x(t)d\alpha=Ax(t)+Bu(t),\qquad x(0)=x_0, \tag{30}$$

which has been comprehensively studied in [18]. In this study we are concerned with the Riemann-Liouville based system (29) and the aim is to present the analytical solution for such a system. It is obvious that in the case of a scalar negative A and $g(t)\equiv 0$, this system is reduced to distributed order relaxation equation [6]. We recall that Riemann-Liouville derivative (3) is the left inverse of fractional integral operator (1). It is interesting to remark that this relationship is not true for distributed order operators (4) and (5) in general. This is the fundamental reason why modeling distributed order relaxation patterns by means of Caputo and Riemann-Liouville operators do not result in identical solutions.
We continue this section by the following definition which specifies the iterated self-convolution of a function

$$f^{*k}(t)=\int_0^t f(\tau)\,f^{*k-1}(t-\tau)\,d\tau,\qquad k\in\mathbb{N} \tag{31}$$

In (31), $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $f(t)=0$ for $t\in(-\infty,0]$ and for the initial function we define $f^{*0}(t)=\delta(t)$. Definition (31) is sometimes called the convolution power of function $f(t)$, which has found some applications in stochastic differential equations [19]. Using this definition, we introduce a function that could be considered as the distributed order version of the Mittag-Leffler function. This function, like the Fox Wright function, is also a special case of the distributed power function (19) and is defined by

$$E\left(t; Aw(\alpha)\right) \triangleq p\left(t; v(\alpha)\right), \tag{32}$$

in which

$$v(\alpha+1) = \sum_{k=0}^{+\infty} A^k w^{*k}(\alpha) \tag{33}$$

According to this definition and noting the fact that $w^{*k}(\alpha)$ is zero outside the range $(0,k)$, this function could be represented by

$$E\left(t; Aw(\alpha)\right) = \sum_{k=0}^{+\infty} \int_0^k A^k w^{*k}(\alpha)\left(t^\alpha / \Gamma(\alpha+1)\right) d\alpha \tag{34}$$

Lemma 4

The exact solution of the homogeneous system of differential equations

$$\dot{x}(t) = A\int_0^1 w(\alpha)\, {}_0^{RL}D_t^{1-\alpha} x(t) d\alpha, \quad x(0) = x_0 \tag{35}$$

is given by

$$x(t) = E\left(t; Aw(\alpha)\right) x_0, \quad t \geq 0, \tag{36}$$

in which $E\left(t; Aw(\alpha)\right)$ is defined by (32).

<u>Proof:</u>

Defining

$$v_1(\alpha+1) = \sum_{k=1}^{+\infty} A^k w^{*k}(\alpha) \tag{37}$$

allows us to write $v(\alpha+1) = \delta(\alpha) + v_1(\alpha+1)$. Thereby,

$$E\left(t; Aw(\alpha)\right) = p\left(t; \delta(\alpha-1) + v_1(\alpha)\right) \tag{38}$$

It is followed by (19) that the distributed power function is linear with respect to its second argument. Also, it is deduced from (19) that $p\left(t; \delta(\alpha-1)\right) = 1$. Therefore,

$$E\left(t; Aw(\alpha)\right) = 1 + p\left(t; v_1(\alpha)\right) \tag{39}$$

To show $x(t)=E(t;Aw(\alpha))x_0$ is the solution of (35), we need to prove

$$\frac{d}{dt}E(t;Aw(\alpha))x_0 = A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}E(t;Aw(\alpha))x_0 d\alpha \tag{40}$$

The right side of (40) equals

$$\begin{aligned}&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}E(t;Aw(\alpha))\,d\alpha x_0 = \\ &A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}(1)\,d\alpha x_0 + A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}p(t;v_1(\alpha))\,d\alpha x_0\end{aligned} \tag{41}$$

Using (7) for calculation of the first term in the second line of (41), we obtain

$$\begin{aligned}&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}(1)\,d\alpha x_0 = A\int_0^1 w(\alpha)\left(t^{\alpha-1}/\Gamma(\alpha)\right)d\alpha x_0 \\ &= p(t;Aw(\alpha))x_0\end{aligned} \tag{42}$$

For the second term in the second line of (41) we consider the variable change $\beta=1-\alpha$ which gives

$$A\int_0^1 w(1-\beta)\,{}_0^{RL}D_t^{\beta}p(t;v_1(\alpha))\,d\beta x_0 = A\,{}_0^{RL}D_t^{w(1-\alpha)}p(t;v_1(\alpha))x_0 \tag{43}$$

Note that $v_1(\alpha)=0$ for $0\le\alpha\le1$. So we can use Lemma 3 to calculate $A\,{}_0^{RL}D_t^{w(1-\alpha)}p(t;v_1(\alpha))x_0$ as

$$A\,{}_0^{RL}D_t^{w(1-\alpha)}p(t;v_1(\alpha))x_0 = Ap(t;w(1+\alpha)*v_1(\alpha))x_0 \tag{44}$$

Applying the time shift to the other function involved in the convolution, we obtain

$$A\,{}_0^{RL}D_t^{w(1-\alpha)}p(t;v_1(\alpha))x_0 = Ap(t;w(\alpha)*v_1(\alpha+1))x_0 \tag{45}$$

Thereby using relation (33) gives

$$\begin{aligned}&A\,{}_0^{RL}D_t^{w(1-\alpha)}p(t;v_1(\alpha))x_0 = Ap\left(t;w(\alpha)*\sum_{k=1}^{+\infty}A^k w^{*k}(\alpha)\right)x_0 \\ &= Ap\left(t;\sum_{k=1}^{+\infty}A^k w^{*k+1}(\alpha)\right)x_0 \\ &= p\left(t;\sum_{k=1}^{+\infty}A^{k+1} w^{*k+1}(\alpha)\right)x_0 \\ &= p\left(t;\sum_{k=2}^{+\infty}A^k w^{*k}(\alpha)\right)x_0\end{aligned} \tag{46}$$

By using (39) and the relations (42) and (46) we can write the right side of (40) as

$$A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}E\big(t;Aw(\alpha)\big)d\alpha x_0 = p\big(t;Aw(\alpha)\big)x_0 + p\Big(t;\sum_{k=2}^{+\infty}A^k w^{*k}(\alpha)\Big)x_0$$

$$= p\Big(t;\sum_{k=1}^{+\infty}A^k w^{*k}(\alpha)\Big)x_0 \tag{47}$$

On the other hand, for the left side of (40) one has

$$\frac{d}{dt}E\big(t;Aw(\alpha)\big)x_0 = \frac{d}{dt}p\big(t;v_1(\alpha)\big)x_0 \tag{48}$$

Due to the equality $\frac{d}{dt}p\big(t;v_1(\alpha)\big) = {}_0^{RL}D_t^{\delta(\alpha-1)}p\big(t;v_1(\alpha)\big)$, we can use Lemma 3 to write the left side of (40) as

$$\frac{d}{dt}E\big(t;Aw(\alpha)\big)x_0 = p\big(t;\delta(-\alpha-1)*v_1(\alpha)\big)x_0 \tag{49}$$

As $\delta(\alpha)$ is an even function we could write

$$\frac{d}{dt}E\big(t;Aw(\alpha)\big)x_0 = p\big(t;\delta(\alpha+1)*v_1(\alpha)\big)x_0$$
$$= p\big(t;v_1(\alpha+1)\big)x_0$$
$$= p\Big(t;\sum_{k=1}^{+\infty}A^k w^{*k}(\alpha)\Big)x_0 \tag{50}$$

which is equal to right side of equation given by (47). □

Lemma 5

The exact solution of the system of differential equations with the input function $g(t)$ and zero initial conditions

$$\dot{x}(t) = A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha + Bg(t), \qquad x(0)=0 \tag{51}$$

is given by

$$x(t) = E\big(t;Aw(\alpha)\big)*Bg(t), \quad t\geq 0, \tag{52}$$

in which and $E\big(t;Aw(\alpha)\big)$ is defined by (32).

<u>Proof:</u>

Proof of this lemma follows a quite similar procedure to the proof of Lemma 4. We simply show that (52) satisfies (51). From the relation (39) we could write the proposed solution as

$$x(t)=\left(1+p\left(t;v_1(\alpha)\right)\right)*Bg(t) \tag{53}$$

At first, we calculate the left side of (51) as

$$\begin{aligned}
\dot{x}(t)&=Bg(t)+\frac{d}{dt}p\left(t;v_1(\alpha)\right)*Bg(t)\\
&=Bg(t)+\frac{d}{dt}\int_0^{+\infty}p\left(t-\tau;v_1(\alpha)\right)H(t-\tau)Bg(\tau)d\tau\\
&=Bg(t)+\int_0^{+\infty}\dot{p}\left(t-\tau;v_1(\alpha)\right)H(t-\tau)Bg(\tau)d\tau+\int_0^{+\infty}p\left(t-\tau;v_1(\alpha)\right)\delta(t-\tau)Bg(\tau)d\tau\\
&=Bg(t)+\int_0^{+\infty}\dot{p}\left(t-\tau;v_1(\alpha)\right)H(t-\tau)Bg(\tau)d\tau+p\left(0;v_1(\alpha)\right)Bg(t)
\end{aligned} \tag{54}$$

Since $v_1(\alpha)=0$ for $0\le\alpha\le1$, it is observed from (19) that $p\left(0;v_1(\alpha)\right)=0$. Therefore,

$$\dot{x}(t)=Bg(t)+\dot{p}\left(t;v_1(\alpha)\right)*Bg(\tau) \tag{55}$$

The term $\dot{p}\left(t;v_1(\alpha)\right)$ has already been calculated in (50). Hence, for the left side of (51) we have

$$\begin{aligned}
\dot{x}(t)&=Bg(t)+p\left(t;v_1(\alpha+1)\right)*Bg(\tau)\\
&=Bg(t)+p\left(t;\sum_{k=1}^{+\infty}A^k w^{*k}(\alpha)\right)*Bg(t)
\end{aligned} \tag{56}$$

For calculation of the right side of (51) by using relation (39) we could write

$$\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha+Bg(t)=\\
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}\,{}_0I_t^1Bg(t)d\alpha+A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}\left(p\left(t;v_1(\alpha)\right)*Bg(t)\right)d\alpha+Bg(t)
\end{aligned} \tag{57}$$

Since for Riemann-Liouville derivative and fractional integral, there holds the relation ${}_0^{RL}D_t^{1-\alpha}\,{}_0I_t^1={}_0I_t^{\alpha}$ [12], by expanding convolution integrals we could write

$$\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha+Bg(t)=\\
&A\int_0^1 w(\alpha)\int_0^t\left((t-\tau)^{\alpha-1}/\Gamma(\alpha)\right)Bg(\tau)d\tau d\alpha+\\
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}\left(\int_0^{+\infty}p\left(t-\tau;v_1(\alpha)\right)H(t-\tau)Bg(\tau)d\tau\right)d\alpha+Bg(t)
\end{aligned} \tag{58}$$

By changing the order of integrations on the assumption that they exist, we obtain

$$
\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha + Bg(t) = \\
&A\int_0^t\int_0^1 w(\alpha)\Big((t-\tau)^{\alpha-1}/\Gamma(\alpha)\Big)d\alpha Bg(\tau)\,d\tau + \\
&A\int_0^{+\infty}\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}\Big(p\big(t-\tau;v_1(\alpha)\big)H(t-\tau)\Big)d\alpha Bg(\tau)\,d\tau + Bg(t)
\end{aligned}
\tag{59}
$$

(59) can be written it in a distributed order derivative form as

$$
\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha + Bg(t) = \\
&Ap\big(t;w(\alpha)\big)*Bg(t) + A\Big({}_0^{RL}D_t^{w(1-\alpha)}p\big(t;v_1(\alpha)\big)\Big)*Bg(t) + Bg(t)
\end{aligned}
\tag{60}
$$

Using Lemma 3 yields in

$$
\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha + Bg(t) = \\
&Ap\big(t;w(\alpha)\big)*Bg(t) + Ap\Big(t;w(\alpha+1)*v_1(\alpha)\Big)*Bg(t) + Bg(t) = \\
&Ap\big(t;w(\alpha)\big)*Bg(t) + Ap\Big(t;w(\alpha)*v_1(\alpha+1)\Big)*Bg(t) + Bg(t)
\end{aligned}
\tag{61}
$$

Finally, replacing $v_1(\alpha+1)$ with (37) we obtain the right side of (51) as

$$
\begin{aligned}
&A\int_0^1 w(\alpha)\,{}_0^{RL}D_t^{1-\alpha}x(t)d\alpha + Bg(t) = \\
&p\big(t;Aw(\alpha)\big)*Bg(t) + p\Big(t;\sum_{k=1}^{+\infty}A^{k+1}w^{*k+1}(\alpha)\Big)*Bg(t) + Bg(t) = \\
&p\Big(t;\sum_{k=1}^{+\infty}A^{k}w^{*k}(\alpha)\Big)*Bg(t) + Bg(t)
\end{aligned}
\tag{62}
$$

which is equal to (56). □

Theorem 1

The exact solution of the system of differential equations (29) is given by

$$
x(t) = E\big(t;Aw(\alpha)\big)x_0 + E\big(t;Aw(\alpha)\big)*Bg(t), \quad t \ge 0, \tag{63}
$$

in which $E\big(t;Aw(\alpha)\big)$ is defined by (32).

<u>Proof:</u>

As the equation (29) is linear, its solution is given by the summation of the homogeneous and particular solutions respectively presented by Lemmas 4 and 5. □

In order to highlight how the exact solution of a linear system of distributed order differential equations is affected by the definition used for its differential operators, let us also take a look at the system of differential equations in Caputo sense (30).

Considering (30) as an integral equation of convolution type, the resolvent formalism suggests the following representation for its solution

$$x(t)=\phi_1(t)x_0+\int_0^t \phi_2(t-\tau)Bu'(\tau)d\tau$$

where $\phi_1(t)=\int_0^t \sum_{k=0}^{+\infty} A^k i^{*k}(\tau)d\tau$, $\phi_2(t)=\int_0^t \sum_{k=0}^{+\infty} A^k i^{*k+1}(\tau)d\tau$, $u'(t)=\frac{d}{dt}u(t)$ and $i(t)=L^{-1}_{s\to t}\left\{1/\int_0^1 w(\alpha)s^{\alpha}d\alpha\right\}$. In fact, a variant of distributed order integration can be defined as a linear time-invariant operator with the impulse response $i(t)$. It can be shown that $i(t)$ cannot be expressed by elementary functions in general except a few special cases. For instance in case $w(\alpha)\geq 0$ and $w(\alpha)\not\equiv 0$, by using the Fourier-Mellin inverse formula together with the residue theorem, $i(t)$ can be computed through an improper integral as follows.

$$i(t)=\frac{1}{\pi}\int_0^{+\infty}\frac{\exp(-rt)\int_0^1 w(\alpha)r^{\alpha}\sin(\pi\alpha)d\alpha}{\left(\int_0^1 w(\alpha)r^{\alpha}\cos(\pi\alpha)d\alpha\right)^2+\left(\int_0^1 w(\alpha)r^{\alpha}\sin(\pi\alpha)d\alpha\right)^2}dr$$

This representation for $i(t)$ can be achieved by following a similar procedure to the one employed in deriving the analytic solution of the diffusion problem in [8]. On the other hand, using the initial value theorem reveals that $i(t)$ has a singularity at $t=0$. Based on this argument, comparing the exact solutions of (29) and (30) implies that using the RL definition for describing the distributed order system leads to an exact solution which can be computed relatively easier. This is due to the fact that calculation of convolution powers of the weight function which is usually just a polynomial in the range $[0,1]$ is obviously simpler than successive convolution powers of $i(t)$. In fact, $w^{*k}(\alpha)$ can be given explicitly for any $k\in\mathbb{N}$ in some usual cases of the weight function and it can be efficiently computed using quadrature formulas otherwise.

Lemma 6

The Laplace transform of function (32) is given by

$$L_{t\to s}\left\{E(t;Aw(\alpha))\right\}=\left(I-AL_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s}\right)^{-1}/s \tag{64}$$

Proof:

For convenience we use the result of Lemma 5 which states that the function (52) satisfies equation (51). Taking Laplace transform of (51) by using Preposition 2 yields in

$$sX(s) = AL_{\alpha\to\hat{s}}\{w(1-\alpha)\}\big|_{\hat{s}=-\ln s} X(s) + BG(s) \tag{65}$$

where $G(s) = L_{\alpha\to s}\{g(t)\}$. Since $L_{\alpha\to s}\{w(1-\alpha)\} = L_{\alpha\to\hat{s}}\{w(\alpha)\}\big|_{\hat{s}=-s}\exp(-s)$, relation (65) takes the form

$$sX(s) = AL_{\alpha\to\hat{s}}\{w(\alpha)\}\big|_{\hat{s}=\ln s} sX(s) + BG(s) \tag{66}$$

Solving (66) with respect to $X(s)$ yields

$$X(s) = \left(I - AL_{\alpha\to\hat{s}}\{w(\alpha)\}\big|_{\hat{s}=\ln s}\right)^{-1} BG(s)/s, \tag{67}$$

in which I is the identity matrix with the same dimensions as matrix A. Since (67) is the Laplace transform of (52) this lemma follows. □

In the following lemma some properties of function (32) is presented (In this lemma, it is shown that some properties of function (32) are analogous to the ones of the Mittag-Leffler functions).

Lemma 7

Function (32) satisfies the following relations.

1. $E(t; A\delta(\alpha-\beta)) = E_{\beta,1}(At^{\beta}), \quad 0<\beta<1$

2. ${}_0I_t^{w(\alpha)} E(t; Aw(\alpha)) = A^{-1}(E(t; Aw(\alpha)) - I)$

3. $E(at; Aw(\alpha)) = E(t; Aa^{\alpha}w(\alpha)), \quad a>0$

4. $E(t; Aw(\alpha)) + E(t; -Aw(\alpha)) = 2E(t; A^2 w^{*2}(\alpha))$

5. ${}_0I_t^{\frac{d}{d\alpha}w(\alpha)} E(t; Aw(\alpha)) = -A^{-1}(\ln t + \gamma) * \dot{E}(t; Aw(\alpha))$

Proof:

1. Noting that the iterated self-convolutions of the weight function becomes $w^{*k}(\alpha) = \delta(\alpha - \beta k)$ for this case, this result is followed by (34).

2. This result is followed by multiplying relation (64) by $\left(I - AL_{\alpha\to\hat{s}}\{w(\alpha)\}\big|_{\hat{s}=\ln s}\right)$ and rewriting the resultant expression in time domain.

3. By writing $a^{\alpha} = \exp(\alpha \ln \alpha)$ and using the relation $L_{\alpha\to s}\{a^{\alpha}w(\alpha)\} = L_{\alpha\to\hat{s}}\{w(\alpha)\}\big|_{\hat{s}=s-\ln a}$ we have

$$L_{\alpha\to s}\left\{E\left(t;Aa^{\alpha}w(\alpha)\right)\right\}=\frac{1}{s}\left(I-AL_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}_{|\hat{s}=\ln(s)-\ln(a)}\right)^{-1}$$
$$=\frac{1/a}{s/a}\left(I-AL_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}_{|\hat{s}=\ln(s/a)}\right)^{-1} \tag{68}$$

which is equal to $L_{\alpha\to s}\left\{E\left(at;Aw(\alpha)\right)\right\}$.

4. Proof of this property is on the basis of the proof of the similar property for the Mittag-Leffler functions in [12 p.23]. In addition it shall be noted that, although we have assumed the second argument of the function (32) to be a real matrix in our study, it is possible to define this function with complex arguments as well. The proof is presented for this generalized case. Since $\sum_{r=0}^{m-1}\exp\left(i2\pi kr/m\right)$ equals m if $k\equiv 0 \pmod m$ and zero otherwise, by using representation (34) we obtain

$$\begin{aligned}&\sum_{r=0}^{m-1}E\left(t;A\exp\left(i2\pi r/m\right)w(\alpha)\right)=\\&\int_0^{+\infty}\left(t^{\alpha}/\Gamma(\alpha+1)\right)\sum_{k=0}^{+\infty}A^k w^{*k}(\alpha)\sum_{r=0}^{m-1}\exp\left(i2\pi rk/m\right)d\alpha=\\&\int_0^{+\infty}\left(t^{\alpha}/\Gamma(\alpha+1)\right)\sum_{k=0}^{+\infty}mA^{mk}w^{*mk}(\alpha)d\alpha=\\&mE\left(t;A^m w^{*m}(\alpha)\right)\end{aligned} \tag{69}$$

As a special case, choosing $m=2$ in (69) gives the result of this part.

5. Multiplying and dividing (64) by $\ln s$ gives

$$L_{t\to s}\left\{E\left(t;Aw(\alpha)\right)\right\}=\ln s\left(\ln sI-A\ln s\,L_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}\Big|_{\hat{s}=\ln s}\right)^{-1}/s \tag{70}$$

Using the equality

$$\ln sL_{\alpha\to\hat{s}}\left\{w(\alpha)\right\}_{|\hat{s}=\ln s}=L_{\alpha\to\hat{s}}\left\{\frac{d}{d\alpha}w(\alpha)\right\}\Big|_{\hat{s}=\ln s} \tag{71}$$

we can rewrite (70) as

$$L_{t\to s}\left\{E\left(t;Aw(\alpha)\right)\right\}=\ln s\left(\ln sI-AL_{\alpha\to\hat{s}}\left\{\frac{d}{d\alpha}w(\alpha)\right\}\Big|_{\hat{s}=\ln s}\right)^{-1}/s \tag{72}$$

Multiplying (72) by $\left(\ln sI-A\,L_{\alpha\to\hat{s}}\left\{\frac{d}{d\alpha}w(\alpha)\right\}\Big|_{\hat{s}=\ln s}\right)$ gives

$$(\ln s)(s/s)L_{t\to s}\left\{E\left(t;Aw(\alpha)\right)\right\}-AL_{\alpha\to\hat{s}}\left\{\frac{d}{d\alpha}w(\alpha)\right\}\Big|_{\hat{s}=\ln s}L_{t\to s}\left\{E\left(t;Aw(\alpha)\right)\right\} \\ =I(\ln s)/s \tag{73}$$

Using (10) and the fact $E\left(0;Aw(\alpha)\right)=I$, the interpretation of (73) in time domain becomes

$$-(\ln t+\gamma)*\left(\dot{E}\left(t;Aw(\alpha)\right)+\delta(\alpha)I\right)-A\,{}_0I_t^{\frac{d}{d\alpha}w(\alpha)}E\left(t;Aw(\alpha)\right)=-(\ln t+\gamma)I \tag{74}$$

From (74), it is followed that

$$-(\ln t+\gamma)*\dot{E}\left(t;Aw(\alpha)\right)-(\ln t+\gamma)I-A\,{}_0I_t^{\frac{d}{d\alpha}w(\alpha)}E\left(t;Aw(\alpha)\right)=-(\ln t+\gamma)I$$

$${}_0I_t^{\frac{d}{d\alpha}w(\alpha)}E\left(t;Aw(\alpha)\right)=-A^{-1}(\ln t+\gamma)*\dot{E}\left(t;Aw(\alpha)\right) \tag{75}$$

□

In the following, the solution of (29) is obtained in three special cases of the weight function. In the first case we consider a weight function which consists of several weighted Dirac delta functions. This would turn equation (29) into a multi-term system of differential equations.

Theorem 2

Exact solution of (29) in the case $w(\alpha)=\sum_{i=1}^{n}r_i\delta(\alpha-\beta_i)$, where $0<\beta_i<1$ and $r_i>0$, is given by (63) in which

$$E\left(t;Aw(\alpha)\right)=E_{(\beta_1,\beta_2,\dots,\beta_n),1}\left(Ar_1t^{\beta_1},Ar_2t^{\beta_2},\dots,Ar_nt^{\beta_n}\right),\quad t\geq 0 \tag{76}$$

<u>Proof:</u>

The weight function in question has the following representation in the Laplace domain

$$L_{\alpha\to s}\left\{w(\alpha)\right\}=\sum_{i=1}^{n}r_i\exp(-s\beta_i) \tag{77}$$

Using multinomial theorem [20], it is found that

$$\left(L_{\alpha\to s}\left\{w(\alpha)\right\}\right)^k=\sum_{l_1+l_2+\cdots+l_n=k}\binom{k}{l_1,l_2,\cdots,l_n}\prod_{i=1}^{n}r_i^{l_i}\exp(-sl_i\beta_i)$$

$$= \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} \exp\left(-s\sum_{i=1}^{n} l_i\beta_i\right) \prod_{i=1}^{n} r_i^{l_i} \tag{78}$$

Thus, by taking the inverse Laplace transform, for iterated self-convolutions of the weight function it is found that

$$w^{*k}(\alpha) = \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} \prod_{i=1}^{n} r_i^{l_i} \delta\left(\alpha - \sum_{i=1}^{n} l_i\beta_i\right) \tag{79}$$

Thereby

$$\int_0^{+\infty} w^{*k}(\alpha)\left(t^{\alpha} / \Gamma(\alpha+1)\right) d\alpha = \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} \prod_{i=1}^{n} r_i^{l_i} \frac{t^{\sum_{i=1}^{n} l_i\beta_i}}{\Gamma\left(\sum_{i=1}^{n} l_i\beta_i + 1\right)} \tag{80}$$

Finally, from relation (34) it is concluded that

$$E\left(t; Aw(\alpha)\right) = \sum_{k=0}^{+\infty} \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} A^k \prod_{i=1}^{n} r_i^{l_i} \frac{t^{\sum_{i=1}^{n} l_i\beta_i}}{\Gamma\left(\sum_{i=1}^{n} l_i\beta_i + 1\right)} \tag{81}$$

Since $l_1 + l_1 + \cdots + l_n = k$ holds in the inner summation, it is possible to write

$$\begin{aligned} E\left(t; Aw(\alpha)\right) &= \sum_{k=0}^{+\infty} \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} \prod_{i=1}^{n} \left(Ar_i\right)^{l_i} \frac{t^{\sum_{i=1}^{n} l_i\beta_i}}{\Gamma\left(\sum_{i=1}^{n} l_i\beta_i + 1\right)} \\ &= \sum_{k=0}^{+\infty} \sum_{l_1+l_2+\cdots+l_n=k} \binom{k}{l_1, l_2, \cdots, l_n} \prod_{i=1}^{n} \left(Ar_i t^{\beta_i}\right)^{l_i} / \Gamma\left(\sum_{i=1}^{n} l_i\beta_i + 1\right) \\ &= E_{(\beta_1, \beta_2, \ldots, \beta_n),1}\left(Ar_1 t^{\beta_1}, Ar_2 t^{\beta_2}, \ldots, Ar_n t^{\beta_n}\right) \end{aligned} \tag{82}$$

□

Theorem 3

Solution of (29) in the case $w(\alpha) = \begin{cases} 1, & \alpha \in (0,1) \\ 0, & \alpha \in (-\infty, 0] \cup [1, +\infty) \end{cases}$ is given by (63) in which

$$\begin{aligned} &E\left(t; Aw(\alpha)\right) = \\ &I + \exp(A) t W_{1,2}(-At) - \ln t \int_0^{+\infty} W_{1,\beta+1}(-At\beta) \exp(A\beta) t^{\beta} d\beta, \quad t > 0 \end{aligned} \tag{83}$$

<u>Proof:</u>

First of all, we calculate the Laplace transform of self-convolutions of the weight function as

$$W(s) = \left(1 - \exp(-s)\right)/s \tag{84}$$

$$W^k(s) = \left(1 - \exp(-s)\right)^k / s^k \tag{85}$$

By using binomial theorem [20], we obtain

$$W^k(s) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \exp(-si)/s^k \tag{86}$$

In order to find the solution (63) expressed in terms of (34), we need to calculate $v(\alpha+1)$ by taking the inverse Laplace transform of (33) as follows.

$$v(\alpha+1) = \delta(\alpha) I + \sum\nolimits_{k=1}^{+\infty} A^k L_{s\to\alpha}^{-1}\left\{W^k(s)\right\} \tag{87}$$

From (86) and (87), it is deduced that

$$v(\alpha+1) = \delta(\alpha) I + \sum\nolimits_{k=1}^{+\infty} A^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i (\alpha - i)^{k-1} H(\alpha - i)/\Gamma(k) \tag{88}$$

By changing the order of series we could write

$$v(\alpha+1) = \delta(\alpha) I + \sum_{i=0}^{+\infty} \sum\nolimits_{k=i}^{+\infty} A^k \binom{k}{i} (-1)^i k (\alpha - i)^{k-1} H(\alpha - i)/k! \tag{89}$$

Note that in the range $\alpha \in (m, m+1]$, where $m \in \mathbb{Z}^{\geq 0}$, there holds

$$\begin{cases} H(\alpha - i) = 0, & i \geq m+1 \\ H(\alpha - i) = 1, & i \leq m \end{cases} \tag{90}$$

Thus we only need to consider $i \leq m$ in the series (89) which results in

$$\begin{gathered} v(\alpha+1) = \delta(\alpha) I + \sum_{i=0}^{m} \sum\nolimits_{k=i}^{+\infty} A^k \binom{k}{i} (-1)^i k (\alpha - i)^{k-1}/k!, \\ \alpha \in (m, m+1], \quad m \in \mathbb{Z}^{\geq 0} \end{gathered} \tag{91}$$

Changing the variables as $j = k - i$, we get

$$v(\alpha+1) = \delta(\alpha) I + \sum\nolimits_{i=0}^{m} \sum\nolimits_{j=0}^{+\infty} A^{i+j} \left((i+j)/i!\,j!\right)(-1)^i (\alpha - i)^{i+j-1}$$

$$=\delta(\alpha)I+\sum_{i=0}^{m}\left((-A)^{i}(\alpha-i)^{i-1}/i!\right)\sum_{j=0}^{+\infty}A^{j}(i+j)(\alpha-i)^{j}/j!,\qquad (92)$$
$$\alpha\in(m,m+1],\quad m\in\mathbb{Z}^{\geq 0}$$

The inner series in (92) equals

$$\sum_{j=0}^{+\infty}A^{j}(i+j)(\alpha-i)^{j}/j!=i\sum_{j=0}^{+\infty}A^{j}(\alpha-i)^{j}/j!+\sum_{j=1}^{+\infty}A^{j}(\alpha-i)^{j}/(j-1)!\qquad (93)$$

The first series in (93) is already expressed by the exponential function series. Changing the variable as $j'=j-1$ in the second series makes it possible to express both series by

$$\sum_{j=0}^{+\infty}A^{j}(i+j)(\alpha-i)^{j}/j!=\left(iI+A(\alpha-i)\right)\exp\left(A(\alpha-i)\right)\qquad (94)$$

Replacing (94) in relation (92) gives

$$v(\alpha+1)=\delta(\alpha)I+\sum_{i=0}^{m}g(\alpha,i),\qquad \alpha\in(m,m+1],\quad m\in\mathbb{Z}^{\geq 0},\qquad (95)$$

in which

$$g(\alpha,i)=(-A)^{i}(\alpha-i)^{i-1}\left(iI+A(\alpha-i)\right)\exp\left(A(\alpha-i)\right)/i!\qquad (96)$$

Therefore, according to relations (34) and (95), after a little manipulation it is possible to write the solution as

$$E\left(t;Aw(\alpha)\right)=I+\sum_{i=0}^{+\infty}\int_{i}^{+\infty}g(\alpha,i)\left(t^{\alpha}/\Gamma(\alpha+1)\right)d\alpha\qquad (97)$$

Changing the variable as $\beta=\alpha-i$ in the integral in the right side of (97) results in

$$E\left(t;Aw(\alpha)\right)=I+\int_{0}^{+\infty}\sum_{i=0}^{+\infty}g(\beta+i,i)\left(t^{\beta+i}/\Gamma(\beta+i+1)\right)d\beta\qquad (98)$$

Replacing $g(\beta+i,i)$ with its value obtained from (96) yields

$$E\left(t;Aw(\alpha)\right)=I+\int_{0}^{+\infty}\sum_{i=0}^{+\infty}(-A)^{i}\beta^{i-1}(iI+A\beta)\exp(A\beta)\left(t^{\beta+i}/\Gamma(\beta+i+1)\right)/i!\,d\beta\qquad (99)$$

Splitting the series in (99) gives

$$E\left(t;Aw(\alpha)\right)=$$
$$I+\int_{0}^{+\infty}t^{\beta}\exp(A\beta)\left(\sum_{i=1}^{+\infty}\frac{(-A)^{i}t^{i}\beta^{i-1}i}{i!\Gamma(\beta+i+1)}+\sum_{i=0}^{+\infty}\frac{(-A)^{i}t^{i}\beta^{i}A}{i!\Gamma(\beta+i+1)}\right)d\beta\qquad (100)$$

The second series in (100) is already a Wright function. Changing the variable as $i' = i - 1$ in the first series turns makes it possible to express both series in terms of the Wright function. By doing so we obtain

$$E\left(t; Aw(\alpha)\right) = I + \int_0^{+\infty} t^{\beta} \exp\left(A\beta\right)\left(-AtW_{1,\beta+2}\left(-A\beta t\right) + AW_{1,\beta+1}\left(-A\beta t\right)\right) d\beta \tag{101}$$

Since for the Wright function there holds $W_{\alpha,\beta+\alpha}\left(z\right) = \dfrac{d}{dz} W_{\alpha,\beta}\left(z\right)$, we could write (101) in the form

$$E\left(t; Aw(\alpha)\right) = I + \int_0^{+\infty} t^{\beta} \exp\left(A\beta\right) \frac{d}{d\beta}\left(W_{1,\beta+1}\left(-A\beta t\right)\right) d\beta + A\int_0^{+\infty} t^{\beta} \exp\left(A\beta\right) W_{1,\beta+1}\left(-A\beta t\right) d\beta \tag{102}$$

which allows us to integrate the first term by parts. Doing so results in (83). □

Figure

Corollary 1

Solution of (29) in the case $w(\alpha) = \begin{cases} ca^{\alpha}, & \alpha \in (0,1) \\ 0, & \alpha \in (-\infty, 0] \cup [1, +\infty) \end{cases}$, $a \in \mathbb{R}^{+}, c \in \mathbb{R}$ is given by (63) in which

$$E\left(t; Aw(\alpha)\right) = I + \exp\left(cA\right) atW_{1,2}\left(-caAt\right) - \ln at \int_0^{+\infty} W_{1,\beta+1}\left(-caAt\beta\right) \exp\left(cA\beta\right)\left(at\right)^{\beta} d\beta, \tag{103}$$

for $t > 0$.

<u>Proof:</u>

Let us denote the unitary weight function by $w_1\left(\alpha\right) = \begin{cases} 1, & \alpha \in (0,1) \\ 0, & \alpha \in (-\infty, 0] \cup [1, +\infty) \end{cases}$.

Then the relation $w\left(\alpha\right) = ca^{\alpha} w_1\left(\alpha\right)$ holds between the two weight functions. Thus, by using the third part of Lemma 7 it is deduced that

$$E\left(t; Aw(\alpha)\right) = E\left(at; cAw_1\left(\alpha\right)\right) \tag{104}$$

Using Theorem 3 for the term $E\left(at; cAw_1\left(\alpha\right)\right)$ in (104) concludes the proof. □

4. Numerical examples

In this section, at first, we will focus on distributed order relaxation equation in Riemann-Liouville sense as follows.

$$\dot{x}(t) = -\lambda \int_0^1 w(\alpha)\, {}_0^{RL}D_t^{1-\alpha} x(t)\, d\alpha, \quad x(0)=1, \quad \lambda > 0 \tag{105}$$

The solution of (105) in the cases $w(\alpha)=\begin{cases}1, & \alpha \in (0,1)\\ 0, & \alpha \in (-\infty,0]\cup[1,+\infty)\end{cases}$, $w(\alpha)=\begin{cases}\alpha, & \alpha \in (0,1)\\ 0, & \alpha \in (-\infty,0]\cup[1,+\infty)\end{cases}$, and $w(\alpha)=\begin{cases}\alpha^2, & \alpha \in (0,1)\\ 0, & \alpha \in (-\infty,0]\cup[1,+\infty)\end{cases}$ are respectively shown in Figures 1, 2, and 3 for $\lambda \in \{0.5,1,1.5,2\}$ (For numerically finding the solutions, at first their representations in the Laplace domain are derived by using Lemma 6. Talbot's method [21] is utilized afterwards, as a numerical technique for inversion of the solutions back to the time domain). As it can be seen in these figures, the solution decay is more intense for greater amounts of λ. Also, the solutions associated with different weight functions are evaluated together in figure 4 where $\lambda = 1$.

As another example, consider the system of linear differential equations

$$\begin{aligned}&\dot{x}(t) - A\left({}_0^{RL}D_t^{0.8}x(t) + {}_0^{RL}D_t^{0.6}x(t) + {}_0^{RL}D_t^{0.4}x(t) + {}_0^{RL}D_t^{0.2}x(t)\right) = Bg(t),\\ &x(0) = x_0,\end{aligned} \tag{106}$$

Figure

Figure

Figure

Figure

in which $A=\begin{bmatrix}1 & -0.8\\ 0.8 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0.25\\ 0.25\end{bmatrix}$. The system of linear differential equations (106) could be rewritten in the form of the distributed order system of linear differential equations (29) by choosing the weight function $w(\alpha)=\delta(\alpha-0.2)+\delta(\alpha-0.4)+\delta(\alpha-0.6)+\delta(\alpha-0.8)$. Therefore, its solution is given by Theorem 1. Assume that $x_0 = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^T$ and $g(t) = H(t)$. The solution in this case is indeed provided by the sum of homogeneous and particular solutions which are respectively shown in Figures 5 and 6.

Figure

5. Conclusion

The system of linear differential equations of distributed order defined in the Riemann-Liouville sense was studied in this paper. The analytic solution of such a system was presented in terms of what could be interpreted as the distributed order generalization of matrix Mittag-Leffler functions. Some interesting properties of this function was revealed and it was shown that this function turns into the multivariate Mittag-Leffler function when the weight function is made up by several impulses, and could be expressed in terms of the Wright function in case of an exponential weight function. Since a special case of the problem considered in this paper is the distributed order relaxation equation, a numerical simulation was performed to evaluate the solutions of such equations. As another example, the solution of a system of differential equations with multiple fractional order operators was numerically obtained.

References

[1] Rocca, M. C., Plastino, A. R., Plastino, A., Ferri, G. L. and de Paoli, A. "General solution of a fractional diffusion–advection equation for solar cosmic-ray transport", *Physica A: Statistical Mechanics and its Applications*, 447, pp. 402-410 (2016).

[2] Bellouquid, A., Nieto, J. and Urrutia, L. "About the kinetic description of fractional diffusion equations modeling chemotaxis", *Mathematical Models and Methods in Applied Sciences*, 26(02), pp. 249-268 (2016).

[3] Rosa, C. F. A. E. and Capelas de Oliveira, E. "Relaxation equations: fractional models", *Journal of Physical Mathematics*, 6(2), (2015).

[4] Saxena, R. K. and Pagnini, G. "Exact solutions of triple-order time-fractional differential equations for anomalous relaxation and diffusion I: The accelerating case", *Physica A: Statistical Mechanics and its Applications*, 390(4), pp. 602-613 (2011).

[5] Bobylev, A. V. and Cercignani, C. "The inverse Laplace transform of some analytic functions with an application to the eternal solutions of the Boltzmann equation", *Applied Mathematics Letters*, 15(7), pp. 807-813 (2002).

[6] Mainardi, F., Mura, A., Gorenflo, R. and Stojanović, M. "The two forms of fractional relaxation of distributed order", *Journal of Vibration and Control*, 13(9-10), pp. 1249-1268 (2007).

[7] Ansari, A. and Moradi, M. "Exact solutions to some models of distributed-order time fractional diffusion equations via the Fox H functions", *SCIENCEASIA*, 39, pp. 57-66 (2013).

[8] Naber, M. "Distributed order fractional sub-diffusion", *Fractals,* 12(01), pp. 23-32 (2004).

[9] Mainardi, F. and Pagnini, G. "The role of the Fox–Wright functions in fractional sub-diffusion of distributed order", *Journal of computational and applied mathematics*, 207(2), pp. 245-257 (2007).

[10] Stojanović, M. "Fractional relaxation equations of distributed order", *Nonlinear Analysis: Real World Applications*, 13(2), pp. 939-946 (2012).

[11] Langlands, T. A. M. "Solution of a modified fractional diffusion equation", *Physica A: Statistical Mechanics and its Applications,* 367, pp. 136-144 (2006).

[12] Podlubny, I. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Vol. 198), Academic press, San Diego, USA (1998).

[13] Li, Y., Sheng, H. and Chen, Y. Q. "On distributed order integrator/differentiator", *Signal Processing*, 91(5), pp. 1079-1084 (2011).

[14] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. *Tables of Integral Transforms*, (Vol. 1), (1954).

[15] Haubold, H. J., Mathai, A. M. and Saxena, R. K. "Mittag-Leffler functions and their applications", *Journal of Applied Mathematics*, (2011).

[16] Gorenflo, R., Kilbas, A. A., Mainardi, F. and Sergei V. Rogosin. *Mittag-Leffler functions, related topics and applications*, Springer, Berlin , Germany (2014).

[17] Sandev, T., Chechkin, A. V., Korabel, N., Kantz, H., Sokolov, I. M., and Metzler, R. "Distributed-order diffusion equations and multifractality: Models and solutions", *Physical Review E*, 92(4), 042117 (2015).

[18] Jiao, Z., Chen, Y. Q. and Podlubny, I. *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives*. SpringerBriefs in Electrical and Computer Engineering, (2012).

[19] Chrouda, M. B., El Oued, M. and Ouerdiane, H. "Convolution calculus and applications to stochastic differential equations", *Soochow Journal of Mathematics*, 28(4), pp. 375-388 (2002).

[20] Gossett, E. *Discrete mathematics with proof*. John Wiley & Sons, (2009).

[21] Abate, J. and Whitt, W. "A unified framework for numerically inverting Laplace transforms", *INFORMS Journal on Computing*, 18(4), pp. 408-421 (2006).

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Tables captions:

Table 1: Notations used in this paper.

Figures captions:

Figure 1: Solution of (105) with $w(\alpha)=\begin{cases}1, & \alpha\in(0,1)\\ 0, & \alpha\in(-\infty,0]\cup[1,+\infty)\end{cases}$ and $\lambda\in\{0.5,1,1.5,2\}$.

Figure 2: Solution of (105) with $w(\alpha)=\begin{cases}\alpha, & \alpha\in(0,1)\\ 0, & \alpha\in(-\infty,0]\cup[1,+\infty)\end{cases}$ and $\lambda\in\{0.5,1,1.5,2\}$.

Figure 3: Solution of (105) with $w(\alpha)=\begin{cases}\alpha^2, & \alpha\in(0,1)\\ 0, & \alpha\in(-\infty,0]\cup[1,+\infty)\end{cases}$ and $\lambda\in\{0.5,1,1.5,2\}$.

Figure 4: Solution of (105) for different weight functions where $\lambda=1$.

Figure 5: Solution of (106) where $x_0=(1/\sqrt{2},1/\sqrt{2})^T$ and $g(t)=0$.

Figure 6: Solution of (106) where $x_0=(0,0)^T$ and $g(t)=H(t)$.

Tables and figures:

Notation	Description	Notation	Description
w	Weight function	${}_0I_t^{\alpha}$	Fractional integral
$W_{\alpha,\beta}$	Wright function	${}_0^CD_t^{\alpha}$	Fractional derivative in Caputo sense
$E_{\alpha,\beta}$	Mittag-Leffler function	${}_0^{RL}D_t^{\alpha}$	Fractional derivative in Riemann-Liouville sense
$E_{(\beta_1,\beta_2,\ldots,\beta_n),\sigma}$	Multivariate Mittag-Leffler function	${}_0I_t^{w(\alpha)}$	Distributed order integral
${}_p\Psi_q$	Fox-Wright function	${}_0^{RL}D_t^{w(\alpha)}$	Distributed order derivative in Riemann-Liouville sense
δ	Dirac delta function	$*$	Convolution
H	Heaviside step function	f^{*k}	k -fold Iteration of convolution with itself
Γ	Complete Gamma function	$L_{t\to s}$	Laplace transform

Table 1

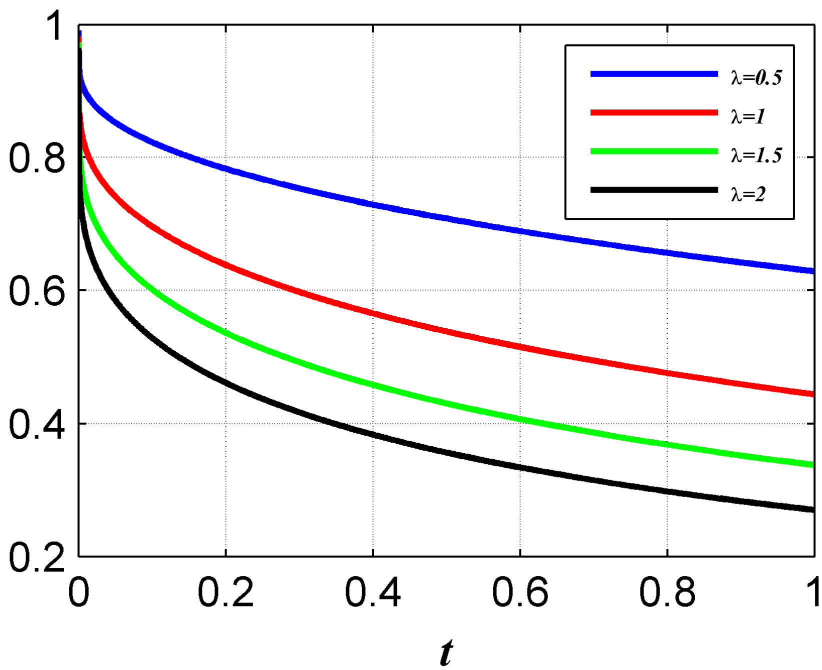

Figure 1

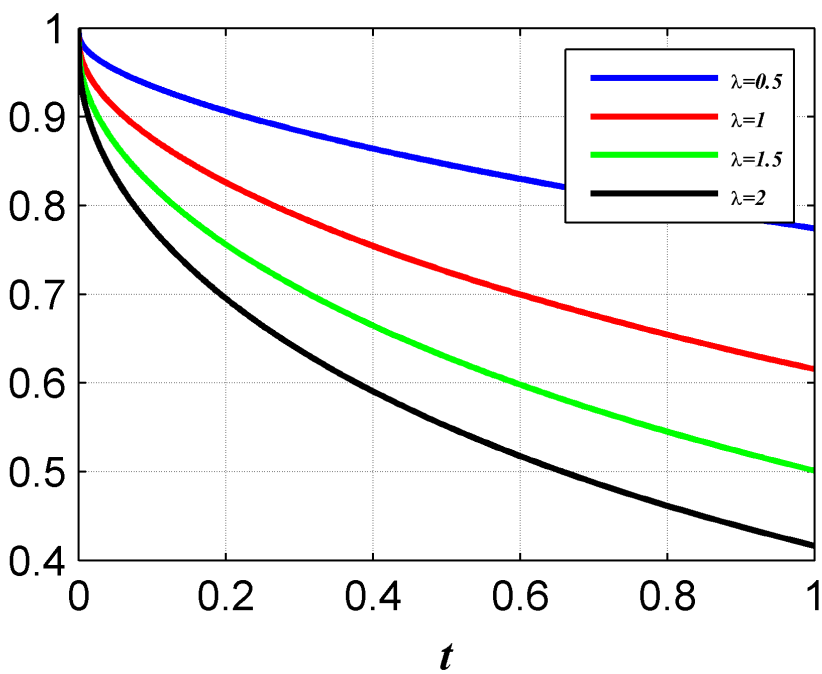

Figure 2

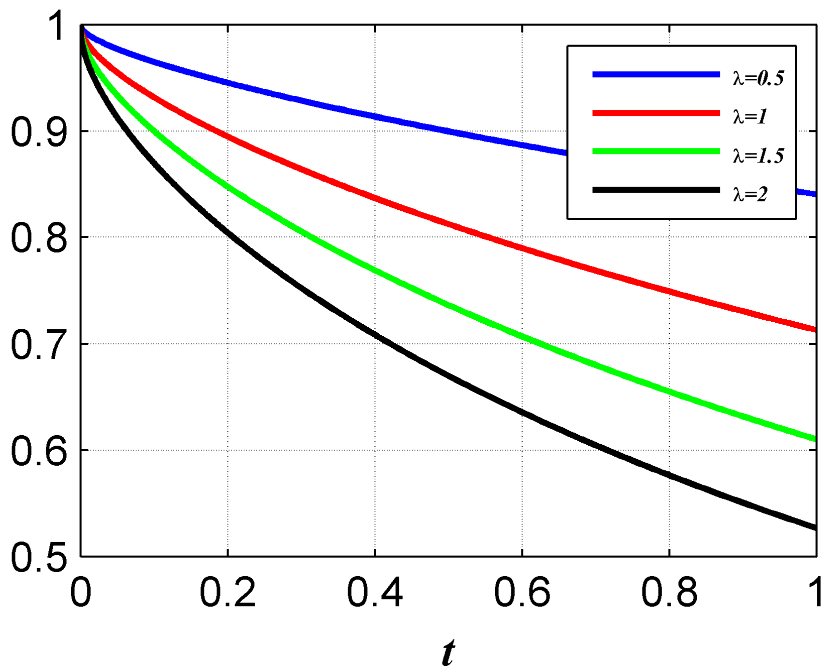

Figure 3

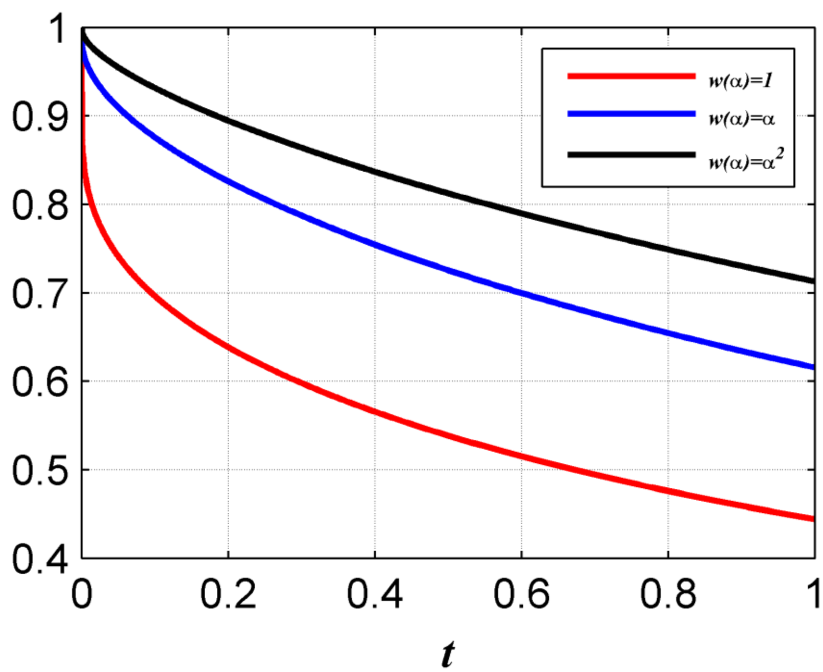

Figure 4

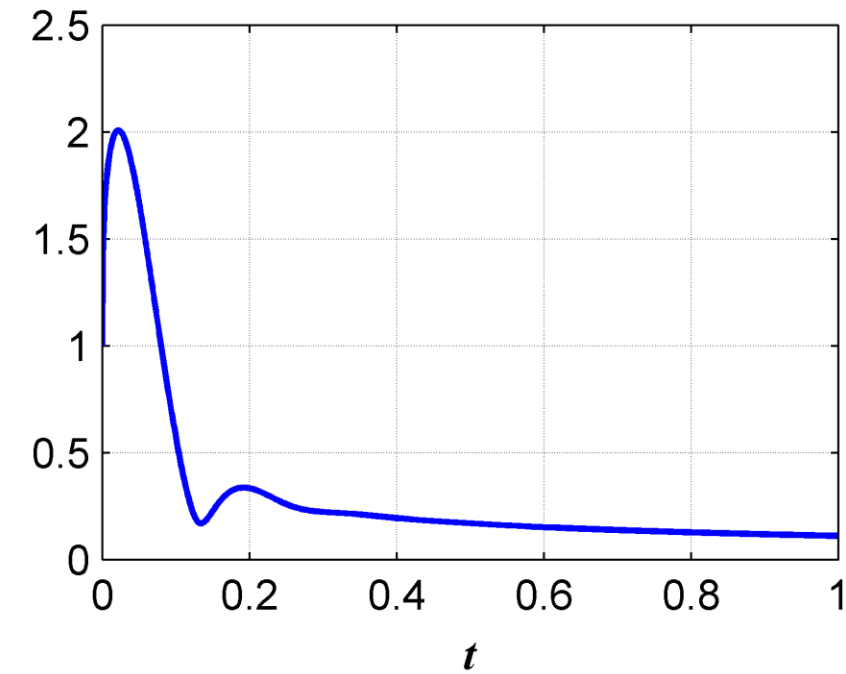

Figure 5

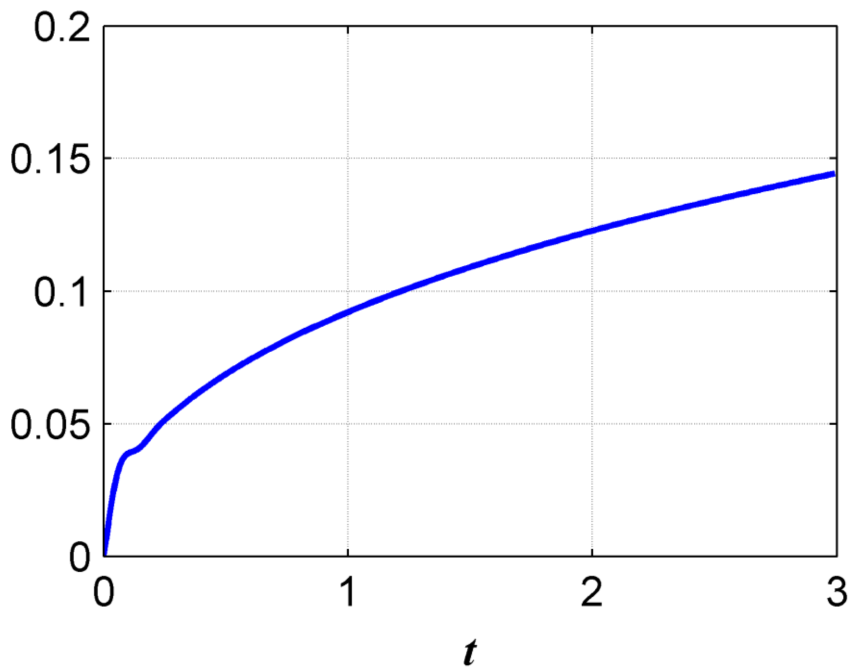

Figure 6