MHD PERISTALTIC SLIP FLOW OF CASSON FLUID AND HEAT TRANSFER IN CHANNEL FILLED WITH A POROUS MEDIUM

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Abstract

We examine the effect of velocity slip on hydromagnetic peristaltic flow of a Casson fluid and heat transfer through an asymmetric channel fluid filled a porous medium. The model governing equations are obtained, simplified using long wavelength and low Reynolds number assumptions and then tackled analytically. Numerical result for effects of embedded parameters on the stream function, axial velocity, pressure drop, temperature, skin friction and Nusselt number are presented graphically and discussed. It is found that the permeability parameter enhances the size of the trapped bolus while velocity slip diminishes it. A rise in magnetic field intensity and Casson fluid parameters decrease the both velocity and temperature profiles. The present problem is of substantial importance in crude oil refinement and biomedical engineering.

Keywords: Peristaltic flow; MHD; Casson fluid; Heat transfer; Porous medium; Partial slip
Nomenclature

\[ a_1, b_1 \] the amplitudes of the waves
\[ c \] velocity of propagation
\[ C_f \] skin-friction coefficient
\[ d_1 + d_2 \] width of the channel
\[ e_{ij} \] \((i,j)\)th component of deformation rate
\[ Ec \] Eckert number
\[ h_1 \] upper wall
\[ h_2 \] lower wall
\[ K \] permeability parameter
\[ L \] slip parameter
\[ M \] Hartman number
\[ Nu \] Nusselt number
\[ p \] pressure in wave frame of reference
\[ P \] pressure in fixed frame of reference
\[ Pr \] Prandtl number
\[ q \] flux in the wave frame
\[ s_y \] yield stress of the fluid
\[ t \] time
\[ u \text{ and } v \] velocity components in the wave frame \((x, y)\)
\[ X \] the direction of the wave propagation
\[ (x, y) \] wave frame
\[ (X, Y) \] laboratory frame
\[ Y \] perpendicular to \(X\)

Greek symbols

\[ \beta \] heat source parameter
\[ \lambda \] wave length
\[ \phi \] phase difference
\[ \pi_c \] a critical value of \(\pi\)
\[ \mu_B \] plastic viscosity of the fluid
\[ \zeta \] Casson fluid parameter
1. Introduction

Peristaltic motion describes the fluid transport by contraction and relaxation of an extensible channel or tube walls. It is the pumping process of fluid motion along the wave propagation route. Peristaltic motion plays a vital role in physiological fluids transport and can be found in several biological and biomedical systems such as gastrointestinal tract, hemolysis, bypass surgery, esophagus, lymph, small intestine, peristaltic pump and heart-lung machines [1-7]. Moreover, studies related to complex interaction of peristaltic motion of a conducting fluid with externally imposed magnetic field will enhance a better understanding of the performance of conductive physiological fluids like blood flow in small vessels, blood pump machines and MHD compressor cilia transport. Stud et al. [8] reported that present of magnetic field may enhance the arterial blood flow. Magnetic field effects on movement of erythrocytes in plasma were investigated by Srivastava and Agrawal [9]. Agrawal and Anwaruddin [10] theoretically investigated the influence of an externally imposed magnetic field on peristaltic motion of electrically conducting liquid in an equally branched stenosed channel. Their results reveal the suitability of magnetic field for cardiac operations blood flow control mechanism. Hydromagnetic peristaltic flow have been discussed by many researchers [11-14] with a view to understand some practical phenomena such as blood pump machine and magnetic resonance imaging (MRI) which is used for diagnosis of brain, vascular diseases and all the human body. Sarkar et al.[15] presented a numerical result for the influence of thermal radiation and buoyancy force on hydromagnetic peristaltic motion of nanofluids in a heated asymmetric channel. Meanwhile, the presence of surface lubrication in peristaltic fluid motion through microchannels or small vessels do lead to velocity slip, hence, the application of no-slip condition may not be realistic for this type of flow [16-23]. Srinivas et al. [24] investigated the effects of velocity slip and wall properties on hydromagnetic peristaltic transport with heat transfer. Similar flow problem in a vertical annulus was numerically investigated by Mekheimer and Elmaboud [25]. Other relevant work on the combined effects of velocity slip and magnetic field on peristaltic motion of a conducting fluid with heat transfer can be found [26-32]. Bhatti and co-authors [33-40] have studied on the peristalsis with different aspects and conditions. Ijaz et al. analyzed the effects of suspended solid particles on hydromagnetic peristaltic motion and heat transfer of Ree-Eyring fluid in a channel. Their results revealed diminish in velocity profiles due to magnetic field.
The study of non-Newtonian fluids has attracted more attention due to their extensive applications in applied engineering and industry namely extraction of crude oil from petroleum oil products, and syrup drugs and production of plastic materials. Casson fluid is one of the non-Newtonian fluids with a distinct feature and was its rheological model was pioneered by Casson [41]. The main objective of this present study is to extend the peristaltic flow analysis of Akbar [42] to include the effects of heat transfer with magnetic field, velocity slip, porous medium and uniform heat source. The effect of magnetic field on the flow and heat transfer in a peristaltic motion was examined by Nadeem and Akram [43]. Ijaz [44] studied the effects of nanoparticles on hydromagnetic flow of non-Newtonian. Srinivas and Muthuraj [45] reported the analytical solution for peristaltic flow of a Jeffrey fluid in an inclined asymmetric channel with slip. The model equations are obtained and solved analytically. Pertinent results are displayed graphically and discussed.

2. Mathematical Formulation

Consider the motion of an incompressible, electrically conducting Casson fluid in an irregular channel filled with a saturated porous medium with heat transfer under the influence of a transversely imposed magnetic field of strength $B_0$. The magnetic Reynolds number is assumed to be small so that induced electric field is negligible. The flow is driven by sinusoidal wave trains propagating with constant speed $c$ over the channel walls. Fig. 1 below describes the physical geometry of the problem.

![Fig. 1. Geometry of the problem.](image-url)
The wall surfaces expressions for the retained flow in the irregular channel flow are:

\[ Y = H_1(X,t) = d_1 + a_1 \cos \left( \frac{2\pi}{\lambda}(X - ct) \right), \text{ upper wall} \]  

\[ Y = H_2(X,t) = -d_2 - b_1 \cos \left( \frac{2\pi}{\lambda}(X - ct) + \phi \right), \text{ lower wall} \]  

where \(a_1, b_1\) are the amplitudes of the waves, \(\lambda\) is the wave length, \(d_1 + d_2\) is the width of the channel, \(c\) is the velocity of propagation, \(t\) is the time, \(X\) is the direction of the wave propagation, \(Y\) is perpendicular to \(X\) and \(\phi\) is the phase difference which varies in the range \(0 \leq \phi \leq \pi\). In addition, \(a_1, a_2, b_1, b_2\) and \(\phi\) should satisfy the following condition

\[ a_1^2 + b_1^2 + 2a_1b_1 \cos \phi \leq (a_1 + a_2)^2. \]

The governing equations for mass, momentum and heat conservation equations for 2D (two dimensional) unsteady flow of incompressible fluid considering of magnetic field, porous medium, joule heating impacts are given as:

\[ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \]

\[ \rho \left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} \right] U = -\frac{\partial P}{\partial X} + \frac{\partial}{\partial X} (S_{xx}) + \frac{\partial}{\partial Y} (S_{xy}) - \sigma B_0^2 U - S_{xy} \frac{\mu}{K} U, \]

\[ \rho \left[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} \right] V = -\frac{\partial P}{\partial Y} + \frac{\partial}{\partial X} (S_{xy}) + \frac{\partial}{\partial Y} (S_{yy}), \]

\[ \frac{\rho c_p}{c_p} \left( \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} \right) = k \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right) + \nu S_{xy} \left[ 2 \left( \frac{\partial U}{\partial X} \right)^2 + 2 \left( \frac{\partial V}{\partial X} \right)^2 + \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right)^2 \right] \]

\[ + \left[ \sigma B_0^2 + S_{xy} \frac{\mu}{K} \right] U^2 + Q_0 \]  

The stress tensor for an isotropic flow of a Casson fluid [41] can be expressed as

\[ S_{ij} = \begin{cases} 2\epsilon_{ij} \left( \mu_h + s_y / \sqrt{2\pi} \right), & \pi > \pi_c, \\ 2\epsilon_{ij} \left( \mu_h + s_y / \sqrt{2\pi} \right), & \pi < \pi_c, \end{cases} \]

where \(s_y\) is the yield stress of the fluid, \(\pi = e_{ij} e_{ij}\) (product of the component of deformation rate with itself) and \(e_{ij}\) is the \((i, j)\)-th component of deformation rate expressed as
\( e_y = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \), \( \pi_c \) is a critical value of \( \pi \), \( \pi \) based on the non-Newtonian model, \( \mu_B \) is the plastic viscosity of the fluid. This model reduces to Newtonian model for vanishing yield stress i.e. when \( s_y = 0 \). We introduce the transformation between the wave frame \((x,y)\) and laboratory frame \((X,Y)\) in order to facilitate the analytical solutions:

\[
x = X - ct, \quad y = Y, \quad u = U - c, \quad p(x) = P(X,t),
\]

where \( u \) and \( v \) are the velocity components in the wave frame \((x,y)\), \( p \) and \( P \) are pressure in wave and fixed frame of references respectively. The following dimensionless variables and quantities are introduce:

\[
\bar{x} = \frac{2\pi x}{\lambda}, \quad \bar{y} = \frac{y}{d_1}, \quad \bar{u} = \frac{u}{c}, \quad \bar{v} = \frac{v}{\bar{c}}, \quad \bar{d} = \frac{d_2}{d_1}, \quad \bar{\bar{d}} = \frac{2\pi d_1^2 p}{\mu c \lambda}, \quad \bar{T} = \frac{2\pi ct}{\lambda}, \quad \bar{h}_1 = \frac{H_1}{d_1},
\]

\[
h_2 = \frac{H_2}{d_2}, \quad a = \frac{a_1}{d_1}, \quad b = \frac{b_1}{d_1}, \quad \text{Re} = \frac{cd_1}{\gamma} \frac{d_2}{\gamma}, \quad M = \sqrt{\frac{\sigma}{\mu} B_0 d_1}, \quad K = \frac{k_0}{d_2}, \quad \bar{S} = \frac{Sd_1}{\mu c}, \quad \zeta = \mu_y \sqrt{2\pi_c} / \sigma,
\]

\[
Ec = \frac{c^2}{C'(T_1 - T_0)}, \quad \text{Pr} = \frac{\mu_c}{k}, \quad \theta = \frac{T - T_0}{T_1 - T_0}, \quad \beta = \frac{Q \bar{d}^2}{k (T_1 - T_0)}, \quad \bar{u} = \frac{\partial \psi}{\partial \bar{x}}, \quad \bar{v} = -\frac{\partial \psi}{\partial \bar{x}}.
\]

We employing the low Reynolds number and long wavelength suppositions to the Eqs. (4) - (7) with the aid of Eqs. (8) – (10), the dimensionless model equations for momentum and energy balance can be written as

\[
\left( 1 + \frac{1}{\zeta} \right) \frac{\partial^2 \psi}{\partial \bar{y}^2} - \left[ M^2 + \left( 1 + \frac{1}{\zeta} \right) \frac{1}{K} \right] \frac{\partial^2 \psi}{\partial \bar{y}^2} = 0,
\]

\[
\frac{1}{\text{Pr} \frac{\partial^2 \theta}{\partial \bar{y}^2}} + Ec \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial^2 \psi}{\partial \bar{y}^2} \right)^2 + \left[ M^2 + \left( 1 + \frac{1}{\zeta} \right) \frac{1}{K} \right] \left( \frac{\partial \psi}{\partial \bar{y}} \right)^2 + \beta = 0,
\]

The reduced boundary conditions for the flow in the wave frame are

\[
\psi = \frac{q}{2}, \quad \frac{\partial \psi}{\partial \bar{y}} + L \left( 1 + \frac{1}{\zeta} \right) \frac{\partial^2 \psi}{\partial \bar{y}^2} = -1, \quad \theta = 0, \quad \text{at} \quad h_1 = 1 + \cos(x),
\]

\[
\psi = -\frac{q}{2}, \quad \frac{\partial \psi}{\partial \bar{y}} - L \left( 1 + \frac{1}{\zeta} \right) \frac{\partial^2 \psi}{\partial \bar{y}^2} = -1, \quad \theta = 1, \quad \text{at} \quad h_2 = -d - \cos(2\pi x + \phi).
\]

Here \( \zeta, \ M, \ K, \ \text{Pr}, \ Ec, \ L, \ \beta \) are the Casson fluid parameter, Hartman number, permeability parameter, Prandtl number, Eckert number, slip parameter and heat source parameter respectively, and \( q \) is the flux in the wave frame and \( a, b, \phi \) and \( d \) satisfies the relation,
\[a^2 + b^2 + 2ab \cos \phi \leq (1 + d)^2. \quad (15)\]

### 3. Solution of the problem

Now solving the equations (11) and (12) along the boundary conditions (13) and (14), the exact solution of stream function, and non-dimensional temperature can be obtained as

\[\psi = F_0 + F_1 y + F_2 \cosh Ny + F_3 \sinh Ny \quad (16)\]

\[\theta = -\frac{1}{8N} \left[ 16BF_1F_3 \cosh \left( Ny \right) + N(F_2^2 + F_3^2)(B + AN^2) \cosh \left( 2Ny \right) \right. \]
\[+ \left. 2(8BF_1F_2 \sinh \left( Ny \right) + \left( N(2Cy^2 + 2BF_1^2y^2 - BF_2^2N^2y^2 + BF_2^2N^2y^2 + AF_3^2y^2 + AF_3^2N^4y^2 - AF_3^2N^4y^2 - 4c_1 - 4yc_2 + F_2F_3(B + AN^2) \sinh(2Ny) \right) \right] \quad (17)\]

The unknowns involved in the equations (16) and (17) are defined in Appendix.

The volume flux at any channel cross-section in the fixed frame is

\[Q = \int_{y_2}^{y_1} \left( \frac{\partial \psi}{\partial y} + 1 \right) dy = h_1 - h_2 + q. \quad (18)\]

The time-mean flow over a period \(T\) is defined as

\[\Theta = \frac{1}{T} \int_0^T Q dt = \frac{1}{T} \int_0^T (q + h_1 - h_2) dt = q + 1 + d. \quad (19)\]

The pressure gradient is obtained from the dimensionless momentum equation for the axial velocity as

\[\frac{dp}{dx} = \frac{\partial}{\partial y} \left[ \left( 1 + \frac{1}{\zeta} \right) \frac{\partial^3 \psi}{\partial y^3} - \left( M^2 + \left( 1 + \frac{1}{\zeta} \right) \frac{1}{K} \right) (\psi + 1) \right]. \quad (20)\]

The dimensionless pressure rise \(\Delta p\) is

\[\Delta p = \int_0^1 \left( \frac{dp}{dx} \right) dx, \quad (21)\]

where \(dp/dx\) is defined in (20). The shear stress and heat transfer rate at the walls of the channel are defined in terms of skin friction coefficient \((C_f)\) and Nusselt number \((Nu)\) as follows:

\[C_f = \left( 1 + \frac{1}{\zeta} \right) \frac{\partial^2 \psi}{\partial y^2} \bigg|_{y=h_1,h_2}, \quad Nu = \frac{\partial \theta}{\partial y} \bigg|_{y=h_1,h_2}. \quad (22)\]
4. Results and discussion

In this section, we explore the influence of sundry emerging physical parameters on the quantities of interest to the present problem such as axial velocity, temperature, pressure gradient, pressure rise and streamlines are examined through Figs 2-20. In the present study following default parameter values are adopted for numerical computations: $\zeta = 0.5, M = 1.0, K= 0.5, L= 0.1, Pr= 0.7, Ec= 1.0, \beta=0.5, x=0.2, a=0.3, b= 0.5, d= 1.0, \phi = \pi/6$. All graphs therefore correspond to these values unless specifically indicated on the appropriate graph. To study the behavior of the distributions of the axial velocity ($u$) numerical calculations for the influence of slip parameter ($L$), Hartmann number ($M$), permeability parameter ($K$) and Casson fluid parameter ($\zeta$) are presented through Figs. 2-5 respectively. Fig. 2 displays the axial velocity for various value of slip parameter $L$. It can be revealed that the velocity mountains due to increase in $L$ near the channel walls but the velocity field decreases at the centre of the channel. The influence of Hartmann number on the axial velocity is analyzes in Fig.3. It can be evident that the axial velocity field increases near the channel walls while the axial velocity decreases in the middle of the channel. This is due to fact for the reason that the magnetic field acts in the transverse direction to the flow and magnetic force resists the flow. Smaller increase in the maximum value of the fluid velocity when the permeability parameter is assigned higher values and which is evident from Fig.4. It also can be concluded that the axial velocity increases with an increase in the permeability of the porous medium in the middle of the channel. This observation is physically sustain by the fact that a more permeable porous medium will provide less resistance to the fluid flow and consequently there is an increase in the value of maximum velocity of fluid. Fig. 5 depicts that the axial velocity profile increases with an increase in Casson fluid parameter $\zeta$ at the centre of the channel while the velocity field decreases near the channel walls. Our results are in good agreement with that given by Akbar [42].

The dimensionless temperature field to perceive the behaviors of pertinent parameters like slip parameter, Hartmann number, permeability parameter; Casson fluid parameter, Prandtl number and heat source parameter $\beta$ are plotted in Figures 6 - 9. The variation of temperature profile for various values of Hartmann number $M$ is depicted in Fig.6. It can be
conclude from Fig.6 that the temperature decreases with an increase in Hartmann number $M$. Fig. 7 displays the influence of permeability parameter on the temperature distribution. From this figure found that a small decrease in the fluid temperature with an increase in the permeability parameter. When fluid moves through a porous medium, internal energy is dissipated in form of heat due to friction between fluid and the porous medium. Such energy loss is small when the porous medium possesses larger permeability. The temperature distribution to perceive the behavior of Casson fluid parameter $\zeta$ is prepared Fig. 8. From this figure shows that the temperature field decreases due to increase for increasing values of Casson fluid parameter. The influence of heat source parameter $\beta$ on $\theta(y)$ is presented in Fig. 9. It can be found that the dimensionless fluid temperature rises, as the value of the heat source parameter $\beta$ escalates. Furthermore, an important note that the energy boundary layer thickness enhances for higher values in $\beta$.

The axial pressure gradient for various values of $M$, $K$, $\zeta$ and $\phi$ are presented in Figs. 10-13. From Fig 10, it can be observed that the pressure gradient increases with increasing $M$ and pressure gradient in case of MHD fluid is greater as compared to hydrodynamic fluid. In Fig.11, the variation in pressure gradient along axial distance due to the influence of permeability parameter $K$ can be observed. It is found that the value of pressure gradient decreases by a small amount when permeability parameter rises in the centre part of channel. The pressure gradient for different values of the Casson fluid parameter $\zeta$ is shown in Fig. 12. The magnitude of pressure gradient reduces with an increase in $\zeta$. Fig. 13 depicts that the amplitude of the pressure gradient decreases with increasing $\phi$ and the point of the maximum decreases with increasing $\phi$. The pressure rise (i.e. $\Delta p$ versus $Q$) for different values of governing physical parameters are plotted in Figs.14-16. It is found that pressure rise increases with the increase in Hartmann number $M$ and amplitudes $\phi$ (see Figs. 14). From Figs. 15& 16, it is found that in the pumping region the pressure rise decreases with the increase of Casson fluid parameter $\zeta$ and permeability parameter $K$ because rise in viscosity and amplitudes flow becomes slow and pressure rise decreases.

Figs. 17–20 are capture to examine the effect of for different values of the flow parameters on trapping. Fig. 17 portrays that the trapped bolus is less in size for the hydromagnetic fluid. In fact, the hydromagnetic characteristic arises because of an applied magnetic field. Consequently, the Lorentz force acts as a retarding force. This helps in
reducing the size of trapped bolus. Fig. 18 illustrates the effect of the permeability parameter $K$ on the streamlines for fixed values of other parameters. It is observed that the size of trapped bolus increases with increasing $K$. Fig. 19 reveals that with an increase in Casson fluid parameter $\zeta$ trapping bolus reduces in the lower and upper parts of the channel. Fig. 20 illustrates the effects of $\phi$ on trapping. It can be noticed that the bolus appearing in the central region for $\phi = 0$ moves towards left and decreases in size as $\phi$ increases.

Comparison of the present solution in limiting case of obtained results with the ones from the published literature for some particular values of the governing parameters is shown in Table 1. It is found to be good agreement with the existing literature. The influence of various flow parameters on the skin friction coefficient and Nusselt number at the lower and upper walls are displayed in Table 2 and Table 3. Table 2 displays the variation in skin friction coefficient at the walls for various values of governing flow pertinent parameters. It is evident that the magnitude of the skin friction coefficient is rises with strengthen in the Hartmann number while it depress with an increase in the slip parameter, permeability parameter and Casson fluid parameter. The variation in Nusselt number at the walls for different values of governing flow pertinent parameters is depicted in Table 3. From this table, it is found that the magnitude of Nusselt number at the lower and upper walls is enhances with an increase in the Hartmann number, Prandtl number and heat source parameter while it diminishes with an increase in the slip parameter, permeability parameter and Casson fluid parameter.

5. Conclusions

The heat transfer analysis on magnetohydrodynamic peristaltic Casson fluid flow filled a porous medium in asymmetric channel in presence of partial slip was presented. The interaction of various pertinent governing flow parameters with peristaltic Casson fluid flow is analyzed. The important key findings of this work are summarized is as follows:

- The axial velocity increases with increase in slip parameter $L$ and Hartmann number $M$ near the channel walls while velocity decreases at the centre of the channel and the opposite behavior for the permeability parameter $K$ and Casson fluid parameter $\zeta$.
- The temperature distribution reduces with an increase in $M$ and $\zeta$ while the temperature profile enhances with a rise in $K$ and $\beta$. 
• It is observed that an increase in the slip parameter decreases the magnitude of the pressure gradient and the magnitude of pressure gradient decreases with an increase in $\zeta$.
• The pressure rise increases with the increase in Hartmann number $M$ and amplitudes $\phi$.
• The size of trapped bolus increases with increasing $K$.
• The magnitude of Nusselt number at the lower and upper walls is enhances with an increase in the Hartmann number and heat source parameter while it diminishes with an increase in the permeability parameter and Casson fluid parameter.

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References


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Fig. 1. Geometry of the problem.

Fig. 2. Impact of $L$ on $u(y)$.

Fig. 3. Impact of $M$ on $u(y)$.
Fig. 4. Impact of $K$ on $u(y)$.

Fig. 5. Impact of $\zeta$ on $u(y)$.

Fig. 6. Impact of $M$ on $\theta(y)$.

Fig. 7. Impact of $K$ on $\theta(y)$.
Fig. 8. Impact of $\zeta$ on $\theta(y)$.

Fig. 9. Impact of $\beta$ on $\theta(y)$.

Fig. 10. Impact of $M$ on pressure gradient.

Fig. 11. Impact of $K$ on pressure gradient.
Fig. 12. Impact of $\zeta$ on pressure gradient.

Fig. 13. Impact of $\phi$ on pressure gradient.

Fig. 14. Impact of $M$ on pressure rise.

Fig. 15. Impact of $K$ on pressure rise.
Fig. 16. Impact of $\zeta$ on pressure rise.

Fig. 17. Stream lines for different values of $M$. 
Fig. 18. Stream lines for different values of $K$.

Fig. 19. Stream lines for different values of $\zeta$. 
Fig. 20. Stream lines for different values of $\phi$.

**Table 1:** Comparison of pressure rise for volume flow rate when $\zeta \to \infty$, $L = 0.1$.

<table>
<thead>
<tr>
<th>$Q$</th>
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Table 2: Variation in Skin-friction coefficient $C_f$ at the wall for different non-dimensional governing flow parameters.

| $L$ | $M$ | $K$ | $\zeta$ | $C_f \big|_{y=h_1}$ | $C_f \big|_{y=h_2}$ |
|-----|-----|-----|--------|----------------|----------------|
| 0.1 | 1.0 | 0.5 | 0.5    | -2.8593        | 2.8593         |
| 0.2 | 2.0 |     |        | -1.9621        | 1.9621         |
| 0.3 |     | 3.0 |        | -1.4934        | 1.4934         |
|     | 1.0 |     |        | -2.9712        | 2.9712         |
|     | 3.0 |     |        | -3.1284        | 3.1284         |
|     |     | 1.0 |        | -2.7295        | 2.7295         |
|     |     | 3.0 |        | -2.6304        | 2.6304         |
|     |     |     | 1.0    | -2.2673        | 2.2673         |
|     |     |     | 2.0    | -1.8863        | 1.8863         |
Table 3: Variation in Nusselt Number $Nu$ at the lower and upper walls for different non-dimensional governing flow parameters.

<table>
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<tr>
<th>$L$</th>
<th>$M$</th>
<th>$K$</th>
<th>$\zeta$</th>
<th>$Pr$</th>
<th>$\beta$</th>
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Appendix

\[ N^2 = \left(\frac{\zeta}{1+\zeta}\right)\left[M^2 + \frac{1}{K}\right], \quad N_1 = L\left(1 + \frac{1}{\zeta}\right), \quad A = Pr\ E c\left(1 + \frac{1}{\zeta}\right), \quad B = Pr\left[M^2 + \left(1 + \frac{1}{\zeta}\right)\frac{1}{K}\right], \]

\[ C = Pr\beta, \]

\[ F_0 = \frac{(h_1 + h_2)\left[Nq\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(2 + N_1N^2q\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)\right]}{2\left(-(h_1-h_2)N\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(2 + A(h_1 + h_2)N^2\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)\right)}, \]

\[ F_1 = \frac{Nq\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(2 + N_1N^2q\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)}{(h_1-h_2)N\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(-2 + N_1(h_1-h_2)N^2\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)}, \]

\[ F_2 = \frac{(h_1-h_2+q)\left(\cosh(h_1N) - \cosh(h_2N) + NN_1\left(\sinh(h_1N) + \sinh(h_2N)\right)\right)}{2\left(NN_1\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \sinh\left(\frac{1}{2}(h_1-h_2)N\right)\right)\left((h_1-h_2)N\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(-2 + N_1(h_1-h_2)N^2\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)\right)}, \]

\[ F_3 = -\frac{(h_1-h_2+q)\cosh\left(\frac{1}{2}(h_1+h_2)N\right)}{(h_1-h_2)N\cosh\left(\frac{1}{2}(h_1-h_2)N\right) + \left(-2 + N_1(h_1-h_2)N^2\right)\sinh\left(\frac{1}{2}(h_1-h_2)N\right)}. \]

\[ c_1 = -\frac{1}{8(h_1-h_2)N}\]

\[ c_1 = \begin{cases} 
16BF_1F_3\cosh(h_1N) + \left(F_2^2 + F_3^2\right)N\left(B + N_1N^2\right)\cosh(2h_1N) \\
+2 \left(8BF_1F_2\sinh(h_1N) + N\left(\frac{2C + N_1\left(F_2^2 - F_3^2\right)N^4}{h_1^2} + B\left(2F_1^2 + \left(F_3 - F_2^2\right)N^2\right)\right)\right) \\
16BF_1F_3\cosh(h_2N) + \left(F_2^2 + F_3^2\right)N\left(B + N_1N^2\right)\cosh(2h_2N) \\
-\frac{h_1}{2} \left(4 + 2Ch_1^2 + 2BF_1^2h_1^2 - BF_2^2h_1^2N^2 \right) \\
+ 2 \left(8BF_1F_2\sinh(h_2N) + N\left(\frac{BF_1^2h_2^2N^2 + N_1F_2^2h_2^4N^4 - N_1F_3^2h_2^4N^4}{h_1^2} + F_2F_3\left(B + N_1N^2\right)\sinh(2h_2N)\right)\right) \\
\end{cases} \]
\[
c_2 = -\frac{1}{8(h_1 - h_2) N} \begin{bmatrix}
8N - 4Ch_1^2 N - 4BF_1^2 \cosh^2 h_1 N + 4Ch_2^2 N + 4BF_1^2 \cosh^2 h_2 N + 2BF_1^2 \cosh^2 h_1^2 N^3 - 2BF_3^2 \cosh^2 h_1^3 N^3 - 2BF_2^2 \cosh^2 h_1^2 N^3 + 2BF_3^2 \cosh^2 h_2^2 N^3 - 2AF_1^2 \cosh^2 h_1^2 N^5 + 2AF_3^2 \cosh^2 h_2^2 N^5 - 2AF_2^2 \cosh^2 h_1^2 N^5 - 2AF_1^2 \cosh^2 h_2^2 N^5 - 16BF_1F_3 \cosh (h_1 N) - \left(F_1^2 + F_3^2 \right) N \left(B + N, N^2 \right) \cosh (2h_1 N) + 16BF_1F_3 \cosh (h_1 N) + BF_1^2 N \cosh (2h_1 N) + BF_3^2 N \cosh (2h_2 N) + N, F_2^2 N^3 \cosh (2h_2 N) - 2N, F_2 F_3 N^3 \cosh (2h_2 N) - 16BF_2 F_2 \sinh (h_1 N) + 2BF_2 F_2 \sinh (2h_1 N) - 2N, F_2 F_3 N^3 \sinh (2h_1 N) + 16BF_2 F_2 \sinh (h_2 N) + 2N, F_2 F_3 N^3 \sinh (2h_2 N)
\end{bmatrix}
\]