A fuzzy network assignment model based on user equilibrium condition

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Abstract. In this paper, a new traffic assignment model is proposed based on fuzzy equilibrium condition where perceived travel times of users are assumed to follow fuzzy values. First, a new method is proposed to determine the membership function based on link congestion levels using probabilistic models. Then, a new index is presented based on percentage of users' risk-acceptance for comparison of fuzzy numbers. Using this index, two approaches, fuzzy Dijkestra shortest path algorithm and defuzzification method, are established for solving the shortest path problem. Fuzzy equilibrium condition is defined based on the two proposed fuzzy shortest path methods, and a traffic assignment model is developed with consideration of fuzzy equivalency equilibrium condition. Frank-Wolfe algorithm and fuzzy shortest path method are combined to solve the proposed traffic assignment problem. The assignment model is applied to a small and medium-sized network. Sensitivity analysis for link flows is performed under different levels of users' risk-acceptance to understand the route choice of different types of users. To apply the model to a large-scale network, the network of Mashhad, Iran is considered as a case study. The fuzzy traffic assignment model provides more accurate estimation of volume compared to conventional traffic assignment.

1. Introduction

Traffic assignment problem determines flow patterns in a transportation network, and is essential to many transportation network planning applications. Numerous efforts with different approaches have been made to estimate the flows in a network. All of them deal with two important assumptions. The first one is about the nature of link travel time, and the second is about travelers' route choice behaviors. The Wardrop's principles are widely-used assumptions on travelers' route choice behaviors.

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Wardrop [1] proposed two principles for link flow prediction, known as the User Equilibrium (UE) and the System Optimum (SO) for traffic assignment problems. UE assumes that each user tries to minimize his or her own travel time from origin to destination. SO flow pattern minimizes total network travel time. The UE model is widely used in practice because it considers user behavior, is computationally convenient, and leads to a relatively reasonable approximation of link flows in the network over the long run. However, a number of studies have shown that the user equilibrium models are not able to represent real situations [2]. The UE model is based on some ideal assumptions which further models have tried to relax. For example, UE assumes that travelers have perfect information about travel times. However, travelers' perceived travel times are subject to multiple sources of variability. Stochastic User Equilibrium (SUE) is commonly used

to account for imperfect information. SUE considers the perceived travel times that follow a certain distribution. However, it still assumes that travelers' perceived travel times have exact crisp values and thus travelers choose paths with perfect rationality. In an effort to relax this assumption and model the behavior of users more realistically, many researchers investigated the concept of bounded rational user equilibrium where travelers choose acceptable routes, whose travel times can be higher than the travel time of the shortest path up to a certain value. Based on this assumption, they developed a bounded rational user equilibrium model [3,4].

This study adopts fuzzy and probability theories to relax user equilibrium assumptions. The fuzzy theory can be considered as an alternative way to model the route choice decision-making process, since it is proposed to represent real-world human perception. The concept of fuzzy theory was proposed in 1965. Fuzzy logic is a very powerful tool for drawing definite conclusions from complex systems that generate vague, imprecise, or ambiguous information. This theory has been applied in different fields of transportation to capture more realistic behavior of travelers and improve the predicted results. For example, Foulds et al. [5] proposed the fuzzy approach to capture the imprecision in the link count to estimate the Origin-Demand (O-D) traffic demands. EL-Rashidy and Grant-Muller [6] employed a fuzzy logic to measure the mobility of transportation network in terms of physical connectivity and level of service. Fuzzy logic is capable of accommodation of relative importance of each element under different traffic conditions, i.e. free flow condition. The tasks of determining the path travel time and finding the shortest path are other examples of complex systems. In fuzzy traffic assignment, perceived travel time of a specific route is defined as a set of values with specific possibilities.

Many efforts have been made to combine the notion of fuzzy logic with traffic assignment. These studies can be categorized into two groups. The first group maintains a fuzzy approach throughout the process. The work of Wang and Liao [7] is an example of the first class of fuzzy models. In their work, they proposed a model with fuzzy travel cost and demand function and, consequently, their optimal solution was a fuzzy set. Zhang et al. [8] developed a hierarchical fuzzy rule base system to increase the accuracy in prediction of short-term traffic congestion using a large number of input variables, i.e. vehicle speed measured by loop detector. They employed the genetic algorithm to optimize the system and maintain a high degree of accuracy. The second group, which is more common, has a defuzzification approach toward the assignment. Given a fuzzy set and its corresponding membership function, models with defuzzification approach attempt

to derive crisp numbers for the final assignment result. Decreasing the computational burden is the main objective of defuzzification of fuzzy models. Wang and Liao [9] formulated a model with defuzzification approach. They considered the node-arc incidence matrix with fuzzy nodes due to arc uncertainties in membership of paths. In contrary to previous studies, they assumed travel cost and demand as crisp variables and functions. Chang and Chen [10] assumed that link travel times are fuzzy numbers. They proposed a fuzzy assignment model by applying the concept of α-cut. In other words, they utilized the crisp assignment model based on travel times with certain possibility. Chen and Tzeng [11] emphasized that since travelers are not able to precisely determine the increase in link travel time due to link congestion, perceived link travel times should be fuzzy. They utilized the concepts of fuzzy measure and fuzzy integral to construct the perceived link costs and, subsequently, flows are determined by the diagonalization approach.

Liu et al. [12] proposed a fuzzy dynamic network assignment model based on linguistic descriptions. They modeled travelers' perception of travel time using linguistic descriptions, or in other words, discrete fuzzy sets with certain membership values. These models are called fuzzy rule-based models. In their study, fuzzy shortest path algorithm is implemented using defuzzification. Ridwan [13] formulated a traffic assignment model within fuzzy preference. Fuzzy preference is a method to compare the alternatives different from the traditional binary preference (YES/NO). He employed fuzzy concept to consider both rational travelers who always follow the shortest path and bounded rational travelers who would accept paths that are in a certain range (bound) of the shortest path. Ghatee and Hashemi [14] introduced a network assignment model with a fuzzy level of travel demand. They suggested a fuzzy model with binary variables to maximize the degree of certainty. By fuzzy comparison of possible paths and fuzzy subsets for travel time, Henn [15] derived a degree of attractiveness for each path. The risk attitudes of travelers are considered toward path travel time, and an assignment model that is similar to the Logit model is proposed based on fuzzy indices.

In a follow-up study, Henn [16] formulated a method for constructing the fuzzy set, based on a more detailed discussion on the reasons for adopting fuzzy concepts. He argued that travel time is a fuzzy set due to two reasons: travelers' imprecise perception and uncertain path travel times. Subsequently, a fuzzy traffic assignment model that combines decision theory and probability measure is developed. Ramezani and Shafahi [17] suggested a fuzzy shortest path algorithm by modifying the Dijkestra algorithm and utilizing fuzzy indices. They considered both path travel time and degree of saturation for congested networks in their

algorithm. In the line of their research, Ramezani et al. [18] proposed fuzzy user equilibrium to determine the flows in a network by implementation of incremental traffic assignment.

In this study, fuzzy perceived travel times would be considered in traffic assignment with defuzzification approach. A new method is proposed to construct fuzzy travel times based on congestion levels in a network. It allows us to calibrate the fuzzy travel time of different links based on a proposed distribution of the random actual travel time. Triangular membership function is utilized herein to define fuzzy numbers which model the perceived travel time. Fuzzy shortest path algorithm of Ramezani and Shafahi [17] is extended to include the risk-taking behavior in route decision making.

The rest of the paper is organized as follows. After defining the membership function for perceived travel time in Section 2, the ranking method of fuzzy numbers is explained in Section 3. Fuzzy assignment model and the concept of fuzzy user equilibrium condition are presented in Section 4. Section 5 defines the extension of fuzzy shortest path problem, new defuzzification method for simplifying the algorithm, and reducing the arithmetic calculations. In order to examine the efficiency of model in Section 6, the fuzzy assignment model is applied to three networks with different sizes. Section 7 concludes the paper.

2. Fuzzy perceived link travel time

In the first step of fuzzy network assignment, the membership function should be defined as a tool for modeling uncertainty in travel time with fuzzy concept. Flat L-R type fuzzy number is one of the simplest forms for representation of Fuzzy Perceived Link Travel Time (FPLTT) which was first proposed by Dubois and Prade [19]. Since our major interest is traffic assignment rather than individual's route choice, the membership functions of individual traveler have to be simple enough so that the aggregation of individual choices in a macroscopic assignment is possible [12]. Triangular fuzzy numbers are used as a special case of the L-R fuzzy numbers in this study. Triangular membership function of a fuzzy number, $\tilde{m}$, can be written as:

$$\mu_{\tilde{m}}(x) = \begin{cases} m_L\left(\frac{m-x}{L}\right) & \text{if } x < m \\ 1 & \text{if } x = m \\ m_R\left(\frac{m-x}{R}\right) & \text{if } x > m \end{cases} \quad (1)$$

where $L > 0$ and $R > 0$ are the left and right spreads of $\tilde{m}$. Figure 1 shows the shape of a triangular membership function and its parameters. It should be noted that $L - R$ fuzzy number can be denoted
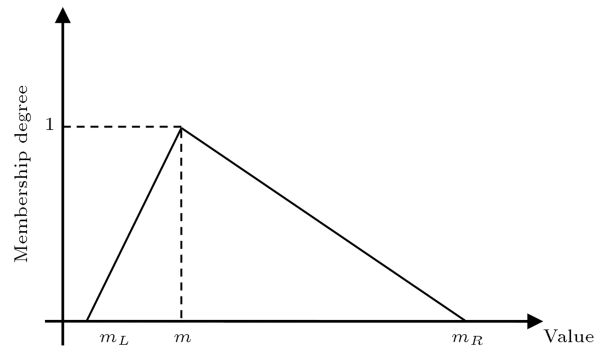

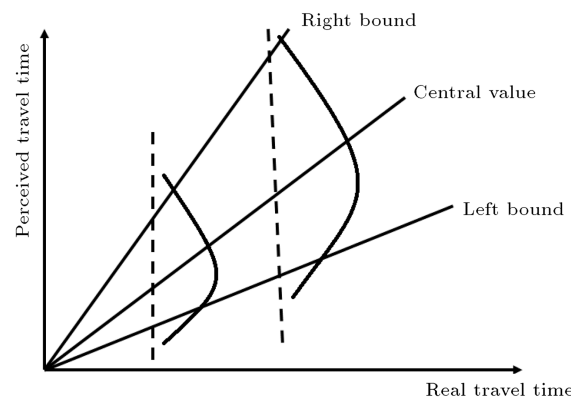**Figure 1.** Shape of triangular membership function and its parameters.

Figure 2. Range of perceived travel time with respect to real travel time.

by (m_L, m, m_R). Left bound, center and right bound of fuzzy number $\tilde{m}$ are denoted by m_L, m, and m_R, respectively. The fuzzy sets of perceived link travel times should be defined to reflect travelers' perception of link travel time. Left and right bounds of fuzzy perceived travel time are the minimum and maximum perceptions of the users in respect to their travel time.

With the consideration of a variety of users, left and right bounds of fuzzy travel time are derived by using stochastic distributions. The schematic figure of parameters of FPLTT is presented in Figure 2. The aggregate value of central value has the highest probability to perceive by travelers where this probability reduces by decreasing/increasing the value of perceived travel time. In this paper, travel times are considered as fuzzy numbers and their boundaries depend on real travel time and vary for different users.

2.1. Weibull distribution for establishing boundaries of membership function

In the absence of necessary information like perceived travel time in different types of links for different travelers, Weibull distribution is used as a rough model for following the perception of users [20]. This distribution has some distinct characteristics. First, it has simple closed form of distribution function. Second, it can be calibrated such that it includes only positive numbers

which results in flexibility due to different values of shape parameter. Let's refer to x as the perceived value of travel time. The cumulative distribution function of random variable x, which has a Weibull distribution, is given by:

$$F(x) = \begin{cases} 1 - \exp\left(\frac{x}{\lambda}\right)^k & x > v \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

where v is the location parameter, λ is the scale parameter, and k is the shape parameter. To model FPLTT, it is assumed that v is equal to zero.

If we want to model the FPLTT of P percent of users, the lower bound can be given by:

$$1 - \exp\left(-\left(\frac{x}{\lambda}\right)^k\right) = \frac{1-p}{2} \rightarrow$$

$$x_l = \lambda * \left(\ln \frac{2}{1+p}\right)^{\frac{1}{k}}. \qquad (3)$$

Considering the equation of the mode of Weibull's distribution, we have:

$$\mu = \lambda \left(\frac{k-1}{k}\right)^{\frac{1}{k}}. \qquad (4)$$

The lower bound of the fuzzy number can be written as:

$$x_l = \mu \left(\frac{\ln\left(\frac{2}{1+p}\right)}{k-1} * k\right)^{\frac{1}{k}} \equiv x_l = \mu.\alpha_l. \qquad (5)$$

Similarly, the upper bound is:

$$x_R = \mu \left(\frac{\ln\left(\frac{2}{1-p}\right)}{k-1} * k\right)^{\frac{1}{k}} \equiv x_R = \mu.\alpha_R, \qquad (6)$$

where μ is mode value of distribution and α_l, α_R are the left and right coefficients of the fuzzy number. In this study, mode would be used as a center value of fuzzy set and computed from the performance function.

2.2. Analysis of membership function parameters

In order to make the proposed model more realistic, parameters of the distribution function should be analyzed. At first, skewness of FPLTT as one of criteria of its shape is examined. Figure 3 presents the shape of the difference between right and left spreads versus k when P is set to 95%. The curve's equation is as follows:

$$R - L = \left(\left(\frac{3.69}{k-1} * k\right)^{\frac{1}{k}} + \left(\frac{0.025}{k-1} * k\right)^{\frac{1}{k}} - 2\right).\mu$$

$$\equiv \alpha_R + \alpha_l - 2. \qquad (7)$$

As can be seen in the Figure 3, FPLTT is a right
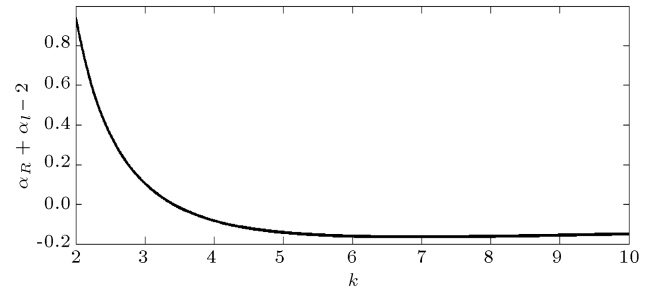
Figure 3. Relation between difference of right and left spreads and shape parameter k.

skewed fuzzy set as it has a wider right spread than left spread when the value of k is between 1 and 3.41. This assumption is valid in the roads with prevailing recurrent traffic congestion in which travelers expect to face a longer time (compared to actual commuter experience) [7]. When k equals 3.41, it has the same right and left spreads and it is applicable to links with few opportunities of congestion occurrence. When the value of k is more than 3.41, it is a left skewed set and it is appropriate for road segments where traffic congestion hardly appears [7].

After determination and analysis of skewness, coefficients α_R and α_l are examined. Figure 4 presents the behavior of α_R and α_l under different values of k, respectively. It can be seen that as k approaches infinity, α_l and α_R approach one. This presents the decrease in the left and right spread of FPLTT. In other words, less uncertainty is presented in the users' perceived travel time. Clearly, α_l and α_R should take values less and more than one, respectively. In this study, it is supposed that α_R is less than two. This indicates that the right bound of FPLTT is less than two times of the mode of travel time distribution. Results can be summarized as follows:

$$\alpha = \begin{cases} 0 < \alpha_k < 1 & \rightarrow & k > 1.03 \\ 1 < \alpha_R < 2 & \rightarrow & k > 2.6 \end{cases} \qquad (8)$$

Combining conditions in Eq. (8), value greater than 2.6 should be selected for k. Finally, Criteria (1) is suggested by taking advantage of $\frac{V}{c}$ as a criterion for traffic congestion in links. For modeling the FPLTT in the absence of data:

$$\text{Criteria (1)}: \begin{cases} \text{For links with } \frac{v}{c} > 1 \\ \quad \rightarrow 2.6 < k < 3.41 \\ \text{For links with } 0.5 < \frac{v}{c} < 1 \\ \quad \rightarrow 3.41 < k < 6.92 \\ \text{For links with } \frac{v}{c} < 0.5 \\ \quad \rightarrow 6.9 < k < 10 \end{cases} \qquad (9)$$

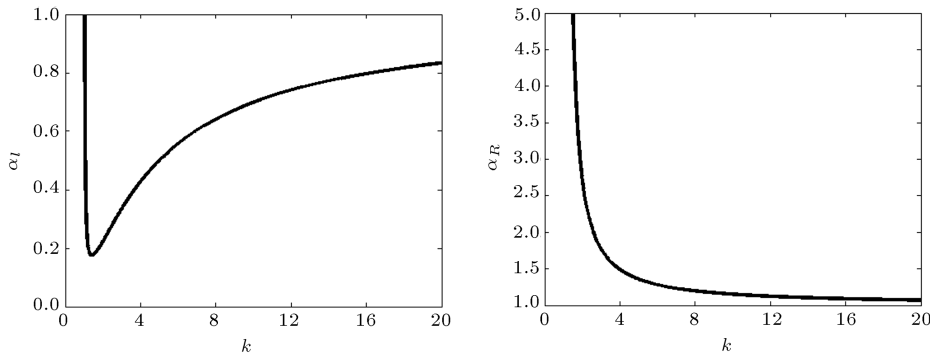In the formulated fuzzy travel perception time in

Figure 4. Behavior of left and right coefficients under different values of k.

Eq. (9), the second rule is proposed because the curve has a steeper gradient between 3.41 and 6.92 in the Figure 3. This gives the idea to break the interval of $[3.41, \infty)$ to two intervals. Using the observed volume from different networks, we can calibrate parameter k for different links. In the next section, a method is proposed to compare the fuzzy numbers.

3. Ranking fuzzy numbers

Ranking of fuzzy numbers is one of the most important steps in the fuzzy traffic assignment. Dubois and Prade [19] introduced four indexes to describe the location of fuzzy numbers as below:

$$I_1(M) = \mathrm{Poss}(M \leq N),$$

$$I_2(M) = \mathrm{Poss}(M < N),$$

$$I_3(M) = \mathrm{Nec}(M \leq N),$$

$$I_4(M) = \mathrm{Nec}(M < N). \tag{10}$$

In Eq. (10), Poss and Nec denote possibility and necessity indexes. Each of these indexes has its own advantages and drawbacks. Figure 5 illustrates the positions of these indexes. Remarking the drawback of these indexes, Ramezani et al. [18] proposed the using of I_2 for risk-seeking users and I_3 for risk-averse ones. The results are established under the assumption that users are pessimistic since they choose the route with less uncertainty in travel time. However, some of

the users are optimistic in their route choice decision making. In other words, they seek more risk than others in the route choice process. Obviously, the combination of possibility and necessity indexes leads to a more comprehensive index.

Let us assume that $\tilde{m} = (m^l, m, m^R)$ and $\tilde{n} = (n^l, n, n^R)$ are two fuzzy numbers. Figure 5 illustrates that $I_3 (\tilde{m}, \tilde{n})$ is the cross-point of $]\tilde{n}, +\infty)$ and the right side of the $\tilde{m}$. The membership degree of $]\tilde{n}, +\infty)$ and the right side of the $\tilde{m}$ are determined as follows [18]:

$$I_3(\tilde{m}, \tilde{n}) = \frac{n^R - m}{(m^R + n^R) - (m + n)}. \tag{11}$$

Similarly, $I_2 (\tilde{m}, \tilde{n})$ is derived by:

$$I_2(\tilde{m}, \tilde{n}) = \frac{n - m^l}{(m + n) - (m^l + n^l)}. \tag{12}$$

Combination of I_2 and I_3 in Eqs. (11) and (12) can be expressed as:

$$I(\tilde{m}, \tilde{n}) = \alpha.I_2(\tilde{m}, \tilde{n}) + \beta.I_3(\tilde{m}, \tilde{n}), \tag{13}$$

where, α and β are the percent of proportional weights of optimistic and pessimistic users. The summation of α and β equals 1. The difficulty caused by this combination is the equation's complexity. However, this complexity can be simplified under some assumptions, as it is explained in the next section. For comparison of the two fuzzy numbers$(\tilde{m}, \tilde{n})$, the combination index can be extended as:
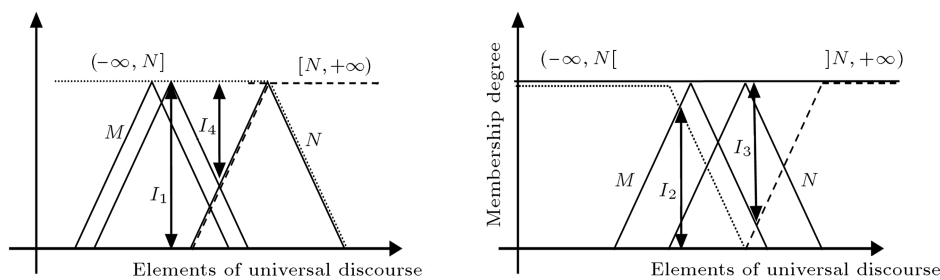

Figure 5. Schematic definition of Dubois and Prade indexes.

$$I(\tilde{m}, \tilde{n}) = \alpha . \frac{n - m^l}{(m + n) - (m^l + n^l)}$$

$$+ \beta . \frac{n^R - m}{(m^R + n^R) - (m + n)}. \qquad (14)$$

Simply, it can be proven that according to this index, the two fuzzy numbers $\tilde{m}$ and $\tilde{n}$ are equal if and only if:

$$I(\tilde{m}, \tilde{n}) = I(\tilde{n}, \tilde{m}) = 0.5. \qquad (15)$$

And $\tilde{m}$ is greater than $\tilde{n}$, if and only if:

$$I(\tilde{m}, \tilde{n}) < 0.5 < I(\tilde{n}, \tilde{m}). \qquad (16)$$

If we assume that the membership functions of path travel times are symmetric, by substituting Eq. (14) in Eq. (16), it can be concluded that $\tilde{m}$ is greater than $\tilde{n}$ if and only if:

$$I(\tilde{n}, \tilde{m}) - 0.5 \geq 0 \rightarrow DN(\tilde{m})$$

$$= \left(\alpha . m^l + m + \beta . m^R \right) \geq DN(\tilde{n})$$

$$= \left(\alpha . n^l + n + \beta . n^R \right). \qquad (17)$$

$DN(\tilde{m})$ stands for the defuzzification number of fuzzy number $\tilde{m}$. This assumption is redundant if α or β are equal to one; in other words, it is not required if all users are optimistic or pessimistic.

4. Fuzzy user equilibrium model

In this section, the rule of route choice by each motorist is specified for determination of its formulation. In the crisp UE, equilibrium condition is reached only when no traveler can improve his travel time by unilaterally changing routes. This is the characterization of crisp user equilibrium condition [21]. However, the drawback in this definition is the assumption that each motorist has the perfect information about the routes and, hence they can exactly predict their travel times. Therefore, travel time should be replaced by fuzzy set. Because of the fuzziness of user's perception of routes travel times, possibility and necessity theory can be used for estimation of FPRTT's utility.

By assuming that FPLTTs are symmetric, the fuzzy equilibrium assignment program can be written as:

$$\min \sum_a \int_0^{x_\alpha} \left(\alpha . t_a^l(w) + t_a(w) + \beta . t_a^R(w) \right) dw$$

$$\equiv \sum_a \int_a^{x_\alpha} \left(DN(\tilde{t}_a(w)) \right) dw, \qquad (18)$$

$$\sum f_m^{r,s} = q_{rs}, \qquad (19)$$

$$f_m^{r,s} \geq 0, \qquad (20)$$

$$x_a = \sum_r \sum_s \sum_m f_m^{r,s} . \delta_{a,m}^{rs}, \qquad (21)$$

where $t_a^l(w)$, $t_a(w)$, and $t_a^R(w)$ are the left, center, and right bounds of fuzzy travel time-value function $\tilde{t}_a(w)$; α is the percent of optimistic users; β is the percent of pessimistic users; and $DN((\tilde{t}_a)(w))$ is the defuzzified number of travel time of link 'a'.

Proposition 1. Fuzzy and conventional traffic assignments are identical if and only if fuzzy travel time is singleton.

Proof. To compare the crisp with fuzzy equilibrium conditions (18), let us assume that fuzzy travel time is singleton. In other words, the values of left, center and right bounds of fuzzy travel time are assumed to be equal. Then, the objective function (18) is reduced to:

$$\sum_a \int_0^{x_a} \left(\alpha . t_a^l(w) + t_a(w) + \beta . t_a^R(w) \right) dw$$

$$= 2 \sum_a \int_0^{x_a} \left(t_a(w) \right) dw. \qquad (22)$$

Therefore, the proposed fuzzy equilibrium assignment program and crisp user equilibrium condition are identical, if and only if fuzzy travel time is singleton.

Using the Lagrangian of the minimization problem, it is simple to show that the first order conditions for the model are identical to the below equilibrium condition, such that:

$$f_m^{r,s} \left(\alpha . t_{m^{rs}}^l + t_{m^{rs}} + \beta . t_{m^{rs}}^R \right) - (u'_{rs})) = 0, \qquad (23)$$

$$\left(\alpha . t_{m^{rs}}^l + t_{m^{rs}} + \beta . t_{m^{rs}}^R \right) - (u'_{rs}) \geq 0, \qquad (24)$$

$$\sum f_m^{r,s} = q_{rs}, \qquad (25)$$

$$f_m^{r,s} \geq 0. \qquad (26)$$

If u_{rs}^l, u_{rs}, and u_{rs}^R are chosen such that "u'_{rs}" Lagrangian multiplier satisfies the below condition:

$$u'_{rs} = \alpha . u_{rs}^l + u_{rs} + \beta . u_{rs}^R, \qquad (27)$$

u'_{rs} can be considered as defuzzificated number of fuzzy set $\tilde{u}_{rs} = (u_{rs}^l, u_{rs}, u_{rs}^R)$. Based on the assumption in Section 3, these formulations can be expressed as:

$$f_m^{r,s} \left(0.5 - I^{r,s}(\tilde{t}_{m^{rs}}, \tilde{u}_{rs}) \right) = 0 \qquad \forall m, r, s \qquad (28)$$

$$0.5 - I^{r,s} \left(\tilde{t}_{m^{rs}}, \tilde{u}_{rs} \right) \geq 0 \qquad \forall r, s \qquad (29)$$

$$\sum f_m^{r,s} = q_{rs} \qquad \forall m, r, s \qquad (30)$$

$$f_m^{r,s} \geq 0 \qquad \forall m, r, s \qquad (31)$$

where:

$\tilde{u}_{rs}$:   Fuzzy set with the most utility;

$\tilde{t}_{m^{rs}}$:   Fuzzy perceived travel time of each route between 'r' and 's';

q_{rs} :   Travel demand between 'r' and 's';

$I^{r,s}(\tilde{t}_{m^{rs}}, \tilde{u}_{rs})$: Utility (or comparison's index) of $\tilde{t}_{m^{rs}}$ with respect to $\tilde{u}_{rs}$.

Eq. (28) implies that travelers choose the route m if and only if it has combination index value of 0.5. Based on Eq. (29), $\tilde{u}_{rs}$ is less than or equal to any fuzzy perceived travel time of any path between origin "r" and destination "s". In other words, if $\tilde{t}_{m^{rs}} = \tilde{u}_{rs}$ according to index $I^{r,s}$, flow on path 'm' can be positive. Thus, $\tilde{u}_{rs}$ equals the minimum fuzzy path travel time between origin "r" and destination "s" according to index $I^{r,s}$. In the special cases, α and β should be replaced by one for totally pessimistic and optimistic users, respectively.

The Frank-Wolfe algorithm is modified to solve the model. To do so, it is required to replace the function of $t_a(w)$ with $DN((\tilde{t}_a)(w))$ in the traditional Frank-Wolfe algorithm. The Bureau of Public Roads (BPR) equation for road travel time is used in this study. Since it is monotonically increasing convex function and depends only on the flows of links, there is unique global solution for the proposed model. Fuzzy equilibrium assignment program can be solved by using any convex combination methods such as the Frank-Wolfe algorithm. Two different approaches for solving fuzzy shortest path by using the Frank-Wolfe algorithm are developed in this study. The most important step in Frank-Wolfe algorithm is finding the shortest path between origins and destinations for finding improving direction in each iteration. This step will extensively be explained in the next section.

5. Fuzzy shortest path algorithm

Finding a path with minimum fuzzy travel time is an important issue in solution algorithm of any type of traffic assignment model. Up to now, most of the methods fall into two categories of approaches toward this problem. In the first approach, fuzzy shortest path is defuzzified into crisp ones like Blue et al. [22] and in second approach, it is solved by using fuzzy index in spite of crisp comparison like Ramezani et al. [18]. In this study, both approaches are employed. Furthermore, level of risk-acceptance is used to gain the set of fuzzy paths. Here, these two methods are briefly explained.

5.1. Fuzzy Dijkestra shortest path algorithm
Ramezani et al. [18] proposed a fuzzy Dijkestra shortest path algorithm to find the best path based on fuzzy comparison of travel times. They used the index I_3

to find a fuzzy shortest path, assuming all users are pessimistic. With assumption of equal left and right coefficients for fuzzy travel time of different paths, Dijkestra fuzzy algorithm can be used with employing the index I to compare different paths if and only if it holds the transitive property. In other words, if fuzzy travel time $\tilde{m}$ is smaller than $\tilde{n}$ and fuzzy travel time $\tilde{n}$ is smaller than $\tilde{w}$, then fuzzy travel time $\tilde{m}$ is smaller than $\tilde{w}$.

Proposition 2. Index I is transitive over triangular fuzzy numbers with equal left and right coefficients.

Proof. To prove that I is transitive, let's assume that fuzzy travel time $\tilde{m}$ is smaller than fuzzy travel time $\tilde{n}$. From the definition:

$$I(\tilde{m}, \tilde{n}) < I(\tilde{n}, \tilde{m}). \tag{32}$$

Using Eq. (14), it follows:

$$\alpha \frac{m + m^l}{(m+n) - (m^l + n^l)} + \beta . \frac{m^R + m}{(m^R + n^R) - (m+n)}$$
$$< \alpha . \frac{n + n^l}{(m+n) - (m^l - n^l)}$$
$$+ \beta . \frac{n^R + n}{(m^R + n^R) - (m+n)}. \tag{33}$$

Left and right bounds of fuzzy numbers are derived using Eqs. (5) and (6) with assuming the same left coefficient, α_l, and right coefficient, α_R, for fuzzy numbers $\tilde{m}$ and $\tilde{n}$. Combining Eqs. (5), (6) and (33), we have:

$$\alpha . \frac{m(1+\alpha_l)}{(1-\alpha_l)} + \beta . \frac{m(1+\alpha_R)}{(1-\alpha_R)} < \alpha . \frac{n(1+\alpha_l)}{(1-\alpha_l)}$$
$$+ \beta . \frac{n(1+\alpha_R)}{(1-\alpha_R)}. \tag{34}$$

Similarly, fuzzy travel time $\tilde{n}$ is smaller than $\tilde{w}$, if and only if:

$$\alpha . \frac{n(1+\alpha_l)}{(1-\alpha_l)} + \beta . \frac{n(1+\alpha_R)}{(1-\alpha_R)} < \alpha . \frac{w(1+\alpha_l)}{(1-\alpha_l)}$$
$$+ \beta . \frac{w(1+\alpha_R)}{(1-\alpha_R)}. \tag{35}$$

Consequently:

$$\alpha . \frac{m(1+\alpha_l)}{(1-\alpha_l)} + \beta . \frac{m(1+\alpha_R)}{(1-\alpha_R)} < \alpha . \frac{w(1+\alpha_l)}{(1-\alpha_l)}$$
$$+ \beta . \frac{w(1+\alpha_R)}{(1-\alpha_R)}. \tag{36}$$

So, fuzzy travel time $\tilde{m}$ is smaller than $\tilde{w}$ and index

I holds the transitive property. Index I provides the chance to find a set of shortest paths based on different levels of risk-acceptance.

5.2. Defuzzification method

Applying a convex combination algorithm to solve the model requires a solution of a linear program and, it can be shown that this is achieved by assigning all motorists to the path connecting origin to their destinations with the shortest travel-defuzzified time. For this purpose, fuzzy travel times of network link, $\tilde{m} = (m_l, m, m_R)$, is defuzzified into a crisp number such as:

$$DN(\tilde{m}) = \alpha.m_l + m + \beta.m_R. \tag{37}$$

These crisp numbers, in Eq. (37), are assigned to each link based on their FPLTTs. A conventional shortest path algorithm, such as regular Dijkestra, can be employed to find a shortest path based on these crisp values. Defuzzification method is simpler than fuzzy Dijkestra shortest path to be implemented because it does not need the path enumeration

6. Numerical examples

Three different sized networks are represented to show how the fuzzy network assignment works. In Section 6.1, fuzzy network assignment is applied to a small network to show the implementation of the model and to prove that results of fuzzy model obtained in Section 4 can be the estimations of fuzzy user equilibrium. Fuzzy shortest path algorithm is applied to Sioux-Falls city test network in Section 6.2, to show the performance of this important part of assignment procedure. Finally, as a case study, link flows of a large network are calculated by using this model in order to show the performance of the model by comparing the result of the model with real data.

6.1. Example 1: Synthetic network

To investigate whether the solution of fuzzy traffic assignment model satisfies the fuzzy user equilibrium condition, the proposed model is tested on the small network in Figure 6. This network consists of four nodes and six links.

BPR function is used for travel time function as follows:

$$t(x) = t_0 \left(1 + 0.15(\frac{v}{c})^4\right), \tag{38}$$

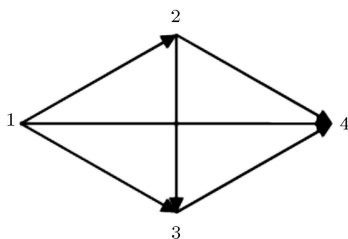**Figure 6.** Synthetic network of Example 1.

Table 1. Characteristics of links.

i	j	$t_0(i,j)$	$c(i,j)$	$V(i,j)$	V/c
1	2	4	200	312.66	1.56
1	3	5	150	233.89	1.56
1	4	17	300	153.51	0.51
2	3	7	250	0	0
2	4	7	250	312.6	1.25
3	4	7	250	233.89	0.94

where:

t_0 :   Free flow travel time, and

c :   Practical capacity of each link.

These two are the link parameters and "V" is the traffic volume on it. Table 1 shows the values of link parameters, "t_0" and "c", of the network as well as link flows under the crisp user equilibrium circumstance and $\frac{V}{c}$ ratios. The travel times of links which are obtained based on conventional user equilibrium condition are used as center of fuzzy travel time of each link. The user equilibrium flows, presented in Table 1, are used for determination of the condition of links, $\frac{V}{c}$. Based on this index, it can be shown that:

1. In the path (1-2-4), recurrent traffic congestion prevails and according to Criteria (1), $k = 3$ is used for this path;

2. In the path (1-3-4), traffic congestion hardly appears but, still, it occurs more in comparison with path (1-4). So, $k = 6$ and $k = 15$ are used for paths (1-3-4) and (1-4), respectively.

We check the sensitivity of the link flows according to the different levels of risk-acceptance by using model (18)-(21) for five combinations of risk-acceptance scenario of users. The results are reported in Table 2. Figure 7 shows the flows on the three specified paths according to the different levels of risk-acceptance. As can be seen, more users choose the path (1-2-4) by increasing the level of risk-acceptance. This means that this path is more attractive for optimistic
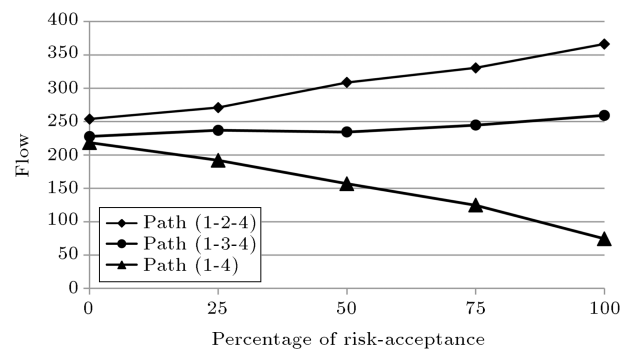

Figure 7. Link flow values according to different levels of risk-acceptance.

Table 2. Variation of link flows according to different levels of risk-acceptance.

Origin node	Destination node	Percentage of risk-acceptance									
		0%		25%		50%		75%		100%	
		$t(i,j)$	$f(i,j)$	$t(i,j)$	$f(i,j)$	$t(i,j)$	$f(i,j)$	$t(i,j)$	$f(i,j)$	$t(i,j)$	$f(i,j)$
1	2	5.56	253.86	6.02	271.07	7.41	308.72	8.47	330.49	10.75	366.29
1	3	8.98	227.73	9.68	237.03	9.47	234.37	10.32	244.77	11.71	259.4
1	4	18.49	218.41	17.89	191.89	17.4	156.91	17.16	124.74	17.02	74.31
2	3	7	0	7	0	7	0	7	0	7	0
2	4	8.12	253.86	8.45	271.07	9.44	308.72	10.21	330.49	11.84	366.29
3	4	7.72	227.73	7.85	237.03	7.81	234.37	7.96	244.77	8.22	259.4

Table 3. Fuzzy index values of different paths in 75% of risk-acceptance.

|	Path (1-2-4)	Path (1-3-4)	Path (1-4)	Path (1-2-3-4)
Lower bound of fuzzy travel time	6.27	10.21	13.49	12.80
Center value of fuzzy travel time	18.68	18.28	17.15	23.43
Upper bound of fuzzy travel time	33.04	23.42	18.63	32.86
Value of fuzzy indexes in comparison to path (1-2-4)	-	0.500	0.506	0.292

users. This can be related to the lower free flow travel time of link (1-2) in comparison to links (1-3) and (1-4) where optimistic traveler can experience a lower travel time in absence of congestion.

Table 3 represents the fuzzy travel time of four possible paths under the fuzzy equilibrium condition for 75% risk-acceptance scenario. It is clear that fuzzy travel time of used paths is equal to each other and less than used paths according to index I'. Values of fuzzy indexes in comparison to path (1-2-4) denote that we can derive an approximate solution using model (18)-(21) instead of model (28)-(31) without relaxing index I'.

6.2. Example 2: Sioux-Falls network
The second test network used in this study is the Sioux-Falls city network, which is the most extensively used test network for the traffic assignment in the literature. The Sioux-Falls network, as shown in Figure 8, has 24 nodes, 76 links, and 528 nonzero O/D pairs. This network is used to show the different paths chosen according to the risk-acceptance of users. First, conventional user equilibrium traffic assignment model is solved in order to determine $\frac{V}{c}$ due to lack of observed link volumes and, consequently, "k" is calculated similar to the previous example. Results are presented in Table 4. Then fuzzy shortest path with indexes I_2 and I_3 is conducted.

Results prove that chosen paths between 40 nonzero O/D pairs are sensitive to risk-acceptance of users. For example, pessimistic users choose the path $6 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 4 \rightarrow 5$, while optimistic users prefer the link $6 \rightarrow 5$ to reach their destination. In detail, pessimistic users prefer fuzzy perceived travel
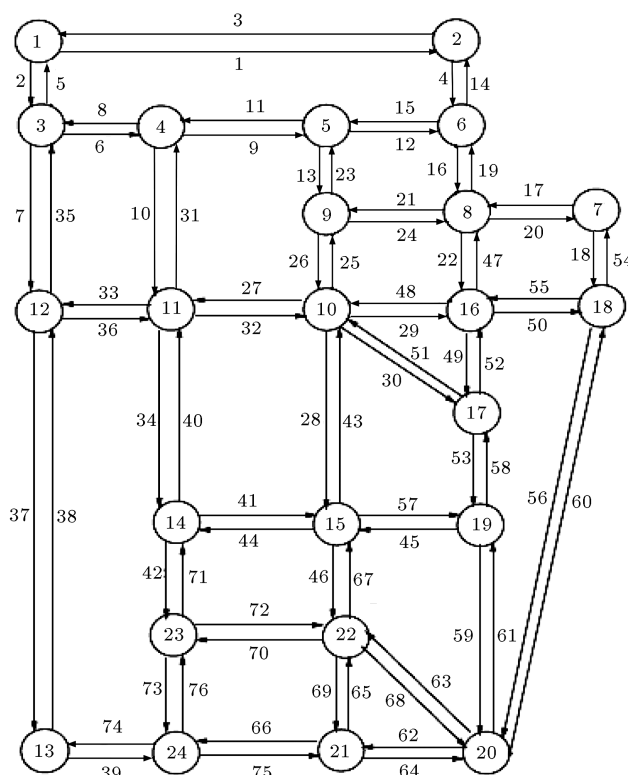
**Figure 8.** Network of Example 2: Sioux-Falls network.

time of first path $'(0.24, 0.36, 0.43)'$ to second one $'(0.12, 0.35, 0.62)'$.

6.3. Case study: Mashhad city network
The objective of the case study is to demonstrate the efficiency as well as accuracy of the fuzzy traffic assignment model in estimating the better flows in real large networks. The network of Mashhad city in

Table 4. Example of different chosen paths according to different levels of risk-acceptance.
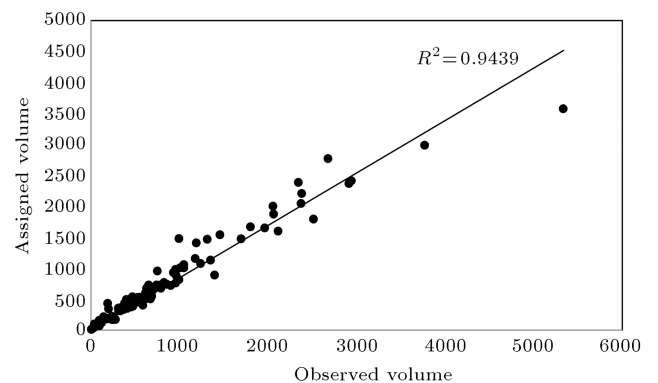
	Type of users O-D	Link sets of path between O-D	Type of users O-D	Link sets of path between O-D
1	Pessimistic 22-1	$22 \to 15 \to 10 \to 11 \to 4 \to 3 \to 1$	Optimistic 22-1	$22 \to 23 \to 24 \to 13 \to 12 \to 3 \to 1$
2	Pessimistic 22-2	$22 \to 15 \to 10 \to 11 \to 4 \to 3 \to 1 \to 2$	Optimistic 22-2	$22 \to 23 \to 24 \to 13 \to 12 \to 3 \to 1 \to 2$
3	Pessimistic 22-3	$22 \to 15 \to 10 \to 11 \to 4 \to 3$	Optimistic 22-3	$22 \to 23 \to 24 \to 13 \to 12 \to 3$
4	Pessimistic 21-4	$21 \to 20 \to 18 \to 7 \to 8 \to 9 \to 5 \, to4$	Optimistic 21-4	$21 \to 24 \to 13 \to 12 \to 3 \to 4$
5	Pessimistic 22-4	$22 \to 15 \to 10 \to 11 \to 4$	Optimistic 22-4	$22 \to 23 \to 24 \to 13 \to 12 \to 3 \to 4$
6	Pessimistic 6-5	$6 \to 2 \to 1 \to 3 \to 4 \to 5$	Optimistic 6-5	$6 \to 5$
7	Pessimistic 13-20	$13 \to 12 \to 3 \to 4 \to 5 \to 9 \to 8 \to 7 \to 18 \to 20$	Optimistic 13-20	$13 \to 24 \to 21 \to 20$
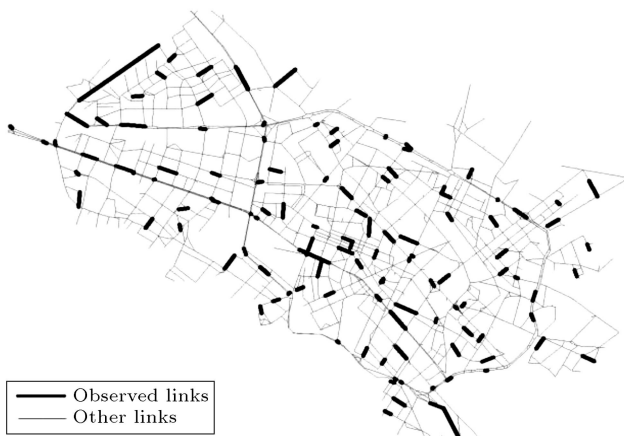
Figure 9. Mashhad city network and observed links.

Figure 10. Relationship between observed and assigned volumes.

Khorasan state, Iran, is considered as the case study. This network contains 935 nodes, 2538 links, and 7157 non-zero O/D pairs. The city is divided into 141 traffic zones. A one-day origin-destination survey with house interviewing was conducted. Data were collected from 4% of the households and validated by observation from several screen lines in the study area. Traffic volumes of 112 links were simultaneously counted with data collected in the same day. Figure 9 shows Mashhad city network and observed links, which are highlighted in the black.

Using the fuzzy traffic assignment, the graph, depicted in Figure 10, compares the estimated volumes with observed volumes on the 112 links with observed volumes. The x-axis of the graph corresponds to the observed volumes, whereas the y-axis corresponds to the estimated volumes. A linear trend-line, $y = x$, passes through the points and the R^2 value is included in Figure 10. The graph shows the high correlation between assigned and observed volume. The analysis shows that the users of the Mashhad network are mostly optimistic rather than pessimistic in their route choice behavior. If Mashhad users are assumed to be optimistic, 41% improvement can be achieved in relative error of assigned volume to observed ones in comparison to the conventional traffic assignment.

All numerical examples are performed on a computer Intel core i5 CPU processor with 4 GB of RAM.

In Mashhad Network, computation of traffic volumes in fuzzy traffic assignment takes about 1 minute, while it takes approximately 0.9 minutes for conventional user equilibrium model. With consideration of calibration ability of fuzzy user equilibrium model, it is clear that there is no significant difference between these two models in terms of computational efficiency. In terms of accuracy, the result of fuzzy assignment shows 41% improvement comparing to traditional user equilibrium.

7. Conclusion remarks

The paper has developed and demonstrated a fuzzy traffic assignment model and its solution for modeling of fuzzy equilibrium condition. In the first step, a new method based on stochastic distribution for construction of boundaries of fuzzy sets is presented. Then, a new index is formulated based on the possibility and necessity theory of Dubois and Prade [19]. Different levels of risk-acceptance of users are involved in choosing paths according to index "I, since it is an important factor in the fuzzy comparison. Two approaches are proposed in this study. The first one is based on the fuzzy comparison by index I which is adopted into the fuzzy Dijkestra shortest path. The second approach is defuzzification of the first approach with assuming the equality of left and right spreads of fuzzy perceived travel time.

The validity and robustness of the proposed model is tested with a hypothetical small network. Using this example, the effect of user's risk-acceptance level on link's flow pattern in the network is demonstrated. Next, a medium sized network, Sioux Falls, is tested under different scenarios using fuzzy equilibrium. Results show that level of risk-acceptance is an important factor in determination O/D shortest path. Finally, a case study is considered to achieve the objective of developing a fuzzy traffic assignment model. Accordingly, the fuzzy network assignment of Mashhad city, Iran, is solved. This network contains 935 nodes and 2538 links. The results of fuzzy network assignment are compared with observed ones. The significant correlation of observed and assigned volumes and reduction of relative error of assigned volume in comparison to conventional traffic assignment encourages one to use this fuzzy model.

This model has some potential future works. First, link-based assignment algorithm is applied with assumption of equality of left and right spreads, in this study, to solve the fuzzy user equilibrium condition. For relaxation of this assumption, a path based assignment algorithm should be adopted to obtain a solution for fuzzy traffic assignment model. Different complex membership functions can be customized for modeling sets of fuzzy travel time and, finally, this problem can be modeled as multiclass user equilibrium in future studies since there are both optimistic and pessimistic users in traffic network.

Acknowledgment

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