Numerical analysis of the flow over weirs and labyrinth side-weirs

M. Nabatian\textsuperscript{a}\textsuperscript{*}, M. Amiri\textsuperscript{a}, M.R. Hashemi\textsuperscript{b} and N. Talebeydokhti\textsuperscript{a}

\textsuperscript{a}. Department of Civil and Environmental Engineering, Shiraz University, Shiraz, Iran.
\textsuperscript{b}. Department of Water Engineering, Shiraz University, Shiraz, Iran.

Received 20 January 2014; received in revised form 27 September 2014; accepted 30 June 2015

KEYWORDS
Labyrinth side-weir; Lateml flow; Spatially varied flow; Differential quadrature; Unsteady flow.

Abstract. The study is undertaken on Differential Quadrature Method (DQM) with the aim of developing a 1-D numerical analysis of the flow over side-weirs. The numerical results were compared with relevant experimental data for both the simple and labyrinth side-weirs. The results showed that the numerical methodology can effectively predict the discharge and flow profile associated with labyrinth side-weirs. Deviation (from experimental data) was found not exceeding 4.5%. Furthermore, the contributions of different terms of the governing equation were assessed through a comprehensive sensitivity analysis. The results show that in order to simplify the governing equation, the channel slope and the friction slope can be eliminated.

\textcopyright 2015 Sharif University of Technology. All rights reserved.

1. Introduction

Side weirs are traditionally adopted to protect hydraulic structures and sometimes function as an outlet in irrigation canals. Computation of the water surface profile along a side weir and its outlet discharge are the main data required for design of this structure.

Flow over a side weir is a Spatially Varied Flow (SVF) with decreasing discharge. Due to nonlinearity of the governing equation, a closed form solution is not available for the general case \cite{1} and numerical methods alongside experimental data are used to quantify the flow characteristics \cite{2-7}. In particular, many researchers have tried to provide empirical relations for the weir coefficient \cite{4,5,6,7}.

When there is a limitation on the length of a side weir, the Labyrinth Side Weir (LSW) is recommended, because it allows for a greater discharge by increasing the effective length of weir. Some researchers have claimed that the outlet discharge of labyrinth side-weir can be four times as much as that of the side-weirs \cite{15}. In terms of the previous research, most of the studies have focused on quantification of the LSW coefficient \cite{15-21} and less attention has been paid to numerical solution of the water surface profile over this type of weir.

The results of this method are verified by experimental data. Many numerical techniques are used for analyzing steady SVF, such as Finite Difference (FD) and Boundary Element (BE). DQM can get more accurate results, because the grids are directly calculated and also it can match with a different boundary zone easier than other methods; also, it requires less formulations and calculations in comparison with other methods \cite{22-27}. DQM was used for modeling gradually varied flow \cite{26} and in this study, it is used for modeling SVF on prismatic and non-prismatic canals.

In this study, the numerical solution of flow over LSW has been investigated using DQM. In terms of the weir coefficient, several formulae were compared against the experimental data \cite{8} and the formula which had the most efficient parameters was proposed.
2. Material and methods

In this part, numerical methodology of the model is introduced. After introducing the governing equations, the solution procedure, in numerical aspect, has been explained in detail.

2.1. Governing equations

Conservations of mass, momentum, and energy in the fluid are the most utilized principles for mathematical modeling of the hydraulic phenomena. That which of the equations are suitable for a certain problem mainly depends on its physical features as well as the ability of each equation set. In view of the fact that the longitudinal velocity of lateral outflow in the SVF with decreasing discharge over side-weirs cannot be easily estimated employing momentum equation [28], the set of energy and continuity equations is more appropriate and thus it is proposed as the mathematical tool. However, it should be noted that in derivation of the energy equation, it is assumed that longitudinal velocity is equal to the cross-sectional mean flow velocity which is a reasonable hypothesis [29]. The set can be presented as follows [29]:

\[ \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} + q_i = 0, \]  

\[ \frac{1}{g} \frac{\partial V}{\partial t} + \frac{\partial H_a}{\partial x} + \frac{V^2}{2g} - \frac{q_i}{A} = \frac{\varepsilon}{\gamma} \frac{\partial A}{\partial t} - \frac{V}{\alpha \beta} \frac{\partial Q}{\partial x}, \]  

where \( A \) is cross section; \( Q \) is discharge; \( V \) is velocity; \( V_i \) is velocity of lateral flow; \( q_i \) is discharge per unit channel width; \( \alpha \) is energy coefficient; \( H_a \) is total head; \( \varepsilon \) is rate of work done by shear force per unit channel length; \( g \) is acceleration due to gravity; \( y \) is water depth; \( x \) is distance measured along the channel; and \( t \) is time.

After some manipulations, the governing equation for non-prismatic cross section channels can be written as:

\[ \frac{dy}{dx} = \frac{S_0 - S_f - \frac{\alpha Q}{\beta} \frac{dQ}{dx} + \frac{n Q^{2/3}}{A^{1/3}} \beta^2}{1 - \frac{P_0}{A}}, \]  

where \( S_0 \) is channel slope; \( S_f \) is friction slope; and \( P_0^2 = \frac{2g Q^2 \frac{dA}{dy} \frac{dy}{dx}}{A^3} \). The friction slope \( (S_f) \) can be estimated by Manning equation which reads [29]:

\[ S_f = \frac{n^2 Q^2 P^{1/3}}{A^{10/3}}, \]  

where \( n \) is manning coefficient, and \( P \) is wetted perimeter. Eq. (3) can be reduced to the following form for prismatic channels:

\[ \frac{dy}{dx} = \frac{S_0 - S_f - \frac{\alpha Q}{\beta} \frac{dQ}{dx}}{1 - \frac{P_0}{A}}. \]  

For side-weir channels, the rate of flow varies with distance along the main channel. The discharge per unit length (\( \frac{dQ}{dx} \)) can be computed as [29]:

\[ \frac{dQ}{dx} = \frac{2}{3} C_w \sqrt{2g(y - p)^{3/2}}, \]  

in which flow discharges over a side-weir, \( p \)=weir height; \( C_w \)=side-weir coefficient. This equation was obtained by [30].

2.2. Numerical solution

With reference to Eqs. (3) and (5), a non-linear ordinary differential equation should be solved to obtain the flow field flow over a side-weir. The analytical solutions for these equations exist only for special cases, but have no generality. To overcome this shortage, a numerical solution can be utilized. Typical solution strategies include single-step, predictor-corrector or modified Hind's methods. In this investigation, DQM is employed as an influential efficient numerical scheme. The concept of DQM, as a numerical tool for solving differential equations, was emerged from the integral quadrature technique. The definite integral of a typical function \( f(x) \) over a specific interval can be approximated as [26]:

\[ \int_a^b f(x)dx = \sum_{i=1}^{N} w_i f(x_i), \]  

where \( w_i \) is weighting coefficients, and \( f(x_i) \) is the function value at \( x_i \). In the same manner, the \( n \)th order differentiation of the function with respect to the independent variable can be estimated applying a linear weighted sum of the function values [26]:

\[ \frac{\partial^n f}{\partial x^n}|_{x=x_i} = \sum_{j=1}^{N} w_{ij} f(x_i). \]  

where \( w_{ij} \) is weighting coefficients; and \( N \) is number of grid points. The weighting coefficients can be expressed as [26]:

\[ w_{ij} = \frac{1}{x_j - x_i}, \]  

for \( j \neq i \).  

\[ w_{ij} = \sum_{k=1}^{N} \frac{1}{x_j - x_k}, \]  

for \( j = i \).  

Substitution of \( \frac{dy}{dx} \) from Eq. (13) into the governing equation gives [1]:

\[ w(i, 1)y_1 + \sum_{j=2}^{N} w_{ij}y_j = f(x_i, y_i). \]  

(11)
In these expressions, \( y_1 \) is known from the specified boundary condition. Therefore, the governing equation becomes simplified into a set of \( N-1 \) algebraic nonlinear equations to be solved for \( y_2, y_3, \ldots, y_n \). This is accomplished by the well-known Newton’s method [1]:

\[
U_{\text{iter}+1} = U_{\text{iter}} + bU_{\text{iter}}',
\]

(12)

\[
J_{\text{iter}}bU_{\text{iter}} = -(F)_{\text{iter}}.
\]

(13)

where \( U \) is vector of unknowns; \( F \) is equation vector; and \( J \) is Jacobian matrix as given by [1]:

\[
J_{\text{mm}} = \frac{\partial F_{\text{m}}}{\partial U_{\text{n}}}. \tag{14}
\]

Choosing a correct distribution of grids could assist the accuracy of modeling. Cosine distribution of grids was adopted herein [1]:

\[
x(i-1) - x(i) = 0.5 \left[ 1 - \cos \left( \frac{(i-1) \cdot \Pi}{n} \right) \right]
\]

\[
-0.5 \left[ 1 - \cos \left( \frac{(i) \cdot \Pi}{n} \right) \right], \tag{15}
\]

where \( x \) is the position of each grid. Schematic presentation of the scheme is depicted in Figure 1.

3. Results and discussion

In this section, comparing with experimental data, results of the foregoing model are evaluated. There are several approaches concerning the formula of discharge coefficient; the most suitable one is determined in the first numerical test. On this basis, the flow over side-weir was modeled by DQM with different discharge coefficients; thereby the one producing less errors was obtained. The labyrinth side-weir was considered in the second test. The purpose of this test is to examine how one-dimensional dynamic equation can model the flow field and to assess which terms in the equation are more influential. The latter is investigated via the sensitivity analysis.

3.1. Rectangular side-weir

The flow field associated with side-weirs of different heights was numerically captured by DQM and verified against available data from literature [8]. Different weir coefficients were examined and the most appropriate one was determined.

The flow occurs in a rectangular channel with a width of \( b = 30 \) cm and a length of \( 570 \) cm (Figure 2). The height of side-weir is \( p = 15 \) cm and the channel conveys a discharge of \( Q = 0.03888 \) m\(^3\)/s before reaching the side-weir.

Several proposed formulae for discharge coefficients are summarized in Table 1, where \( F_1 \) is Froude number of the approach flow; \( z = h/H \); \( h \) is water depth at the channel; \( H \) is specific energy (assumed constant); \( \psi = 1 - KF \); \( F \) is Froude number; \( K \) is function of \( L/b \); \( L \) is length of side-weir; \( b \) is channel width for \( L/b = 0.402 \); \( \psi = 1 - 0.07F \) for \( L/b = 0.605 \) and \( K = 1 - 0.087F \) [7].

The same test, but now with \( p = 30 \) cm and \( Q = 0.0375 \) m\(^3\)/s was also considered. The measured and computed water profiles are given in Figures 3

Figure 1. Cosine distribution of grid points.

Figure 2. (a) Plan view, and (b) longitudinal cross section of the channel in the first case.

Figure 3. Water profile over side-weirs was obtained in laboratory and modeled with DQM with different weir coefficients and Mushu model.
Table 1. Existing formulae for discharge coefficient.

<table>
<thead>
<tr>
<th>Researcher</th>
<th>Weir coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subramanya and Awasty (1972)</td>
<td>( C_m = 0.861 \left( \frac{1 - F_l}{3 + F_l} \right)^{0.5} )</td>
</tr>
<tr>
<td>Nandesumoorthy and Thomson (1972)</td>
<td>( C_m = 0.288 \left( \frac{2 - F_l}{1 + 2F_l} \right)^{0.5} )</td>
</tr>
<tr>
<td>Ranga Ruja et al. (1979)</td>
<td>( C_m = 0.54 - 0.4F_l )</td>
</tr>
<tr>
<td>Singh et al. (1994)</td>
<td>( C_m = 0.33 - 0.18F_l + 0.49 \frac{P}{L} )</td>
</tr>
<tr>
<td>Jalil and Borghesi (1987)</td>
<td>( C_m = 0.71 - 0.41F_l - 0.21 \frac{P}{L} )</td>
</tr>
<tr>
<td>Borghesi et al. (1987)</td>
<td>( C_m = 0.7 \left( 0.48F_l - 0.3 \frac{P}{L} + 0.06 \right)^{0.5} )</td>
</tr>
<tr>
<td>Hager (1987)</td>
<td>( C_m = 0.485 \left( \frac{2 + 3F_l}{2 + 3F_l} \right)^{0.5} )</td>
</tr>
<tr>
<td>Mshu (2001)</td>
<td>( C_m = 0.611 \sqrt{3\Psi z - 2} )</td>
</tr>
</tbody>
</table>

Table 2. Estimating the discharge of lateral flow.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( Q ) (L/s)</th>
<th>( y_l ) (cm)</th>
<th>( V_l ) (m/s)</th>
<th>( F_l )</th>
<th>( P ) (cm)</th>
<th>( L ) (cm)</th>
<th>( Q_w ) observed (L/s)</th>
<th>( Q_w ) estimated (L/s)</th>
<th>( Q_w ) estimated (L/s)</th>
<th>Error</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>14.155</td>
<td>0.211939</td>
<td>0.179855</td>
<td>12</td>
<td>75</td>
<td>9.11</td>
<td>9.00</td>
<td>9.00</td>
<td>1.2775</td>
<td>1.2775</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>15.393</td>
<td>0.321865</td>
<td>0.361384</td>
<td>12</td>
<td>75</td>
<td>18.62</td>
<td>19.20</td>
<td>19.40</td>
<td>3.1149</td>
<td>4.1890</td>
</tr>
<tr>
<td>3</td>
<td>35</td>
<td>16.534</td>
<td>0.422879</td>
<td>0.331825</td>
<td>12</td>
<td>75</td>
<td>28.53</td>
<td>27.40</td>
<td>27.40</td>
<td>3.2827</td>
<td>3.2827</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>14.831</td>
<td>0.600637</td>
<td>0.503098</td>
<td>12</td>
<td>75</td>
<td>16.43</td>
<td>17.30</td>
<td>17.30</td>
<td>4.0729</td>
<td>4.6865</td>
</tr>
<tr>
<td>5</td>
<td>5.4</td>
<td>18.007</td>
<td>0.170145</td>
<td>0.126033</td>
<td>16</td>
<td>75</td>
<td>8.34</td>
<td>8.60</td>
<td>8.70</td>
<td>3.1175</td>
<td>4.1635</td>
</tr>
<tr>
<td>6</td>
<td>25.1</td>
<td>19.269</td>
<td>0.260322</td>
<td>0.189487</td>
<td>16</td>
<td>75</td>
<td>17.70</td>
<td>17.60</td>
<td>17.70</td>
<td>0.7600</td>
<td>0.0000</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td>18.498</td>
<td>0.378419</td>
<td>0.280916</td>
<td>16</td>
<td>75</td>
<td>12.50</td>
<td>12.80</td>
<td>13.00</td>
<td>3.4000</td>
<td>4.0000</td>
</tr>
<tr>
<td>8</td>
<td>45</td>
<td>21.542</td>
<td>0.417789</td>
<td>0.287395</td>
<td>16</td>
<td>75</td>
<td>36.37</td>
<td>37.80</td>
<td>38.00</td>
<td>3.9318</td>
<td>4.4817</td>
</tr>
</tbody>
</table>

Figure 4. Water profile over side-weirs was obtained in laboratory and modeled with DQM and Mshu model.

and 4. From Figure 3, it is evident that the most suitable weir coefficients are those suggested by Hager and Subramanya. As Figure 4 indicates, a relatively less deviation from the experiments was obtained when incorporating Mshu’s formula into DQM. Referring to Eq. (16), the sum of Relative Errors (RE) for DQM combined with Hager discharge coefficient was 1.6%, while it was reported as 5.63% by Mshu [10].

\[
RE = \sum_{i=1}^{n} \left| \frac{y_{obs} - y_{pred}}{y_{obs}} \right| \times 100 \tag{16}
\]

Figure 5. Plan of the channel and weir in Sample (2). where \( y_{obs} \) is the observed data; \( y_{pred} \) is the predicted data; and \( n \) is the number of grid.

3.2. Labyrinth side-weir

In the second sample, discharge of lateral flow over labyrinth side-weir was estimated by the one-dimensional formulation. For further verification, the experimental data recorded by Eminioglu (2010) was also utilized (Figure 5). Experimental discharge before side-weir and weir heights were shown in Table 2. Besides, the water profile over labyrinth side-weirs are estimated for two samples (\( Q = 0.02446 \) m³/s and the heights for Samples (2-1) and (2-2) are 12 and 16 cm, respectively). Figure 3 shows the water surface
In order to conduct sensitivity analysis on each term in Eq. (3), flow over labyrinth side-weir was modeled by Eqs. (5), (17), and (18). Root Mean Square Errors (RMSE) of each model were calculated by Eq. (19) as shown in Table 2.

\[
\frac{dy}{dx} = \frac{g \frac{dQ}{dx} + \left( \frac{a Q_{1}^2}{2} \frac{\partial A}{\partial x} \right) y - c t e}{1 - F_{a}^2}, \tag{17}
\]

\[
\frac{dy}{dx} = \frac{S_{0} - S_{f} + \left( \frac{a Q_{1}^2}{2} \frac{\partial A}{\partial x} \right) y - c t e}{1 - F_{a}^2}, \tag{18}
\]

\[
\text{RMSE} = \sum_{i=1}^{n} \left( \frac{y_{i}|\text{obs}}{y_{i}|\text{pred}} - 1 \right)^{2}, \tag{19}
\]

Data from experimental studies was applied for verification of the model based on one-dimensional dynamic equation and DQM (Figure 6). Comparison between numerical and experimental results shows RE=99.5% for Sample (2-1) and RE=0.16% for Sample (2-2). These negligible errors illustrate that DQM, when combined with the one-dimensional dynamic equation, provides an efficient tool to appropriately model the flow over labyrinth side-weirs and estimate the generated water surface profile.

Table 2 shows the contribution of each term in Eq. (3). In both of the samples, the term \( \frac{g Q_{1}^2}{2} \frac{\partial A}{\partial x} y - c t e \), which represents the non-prismatic effects, has the most influence on accuracy of the numerical predictions. The term \( g Q_{1} \frac{dQ}{dx} \), which accounts for spatial variability of flow discharge, also contributes significantly to estimate water profile. However, the bed slope and the friction slope appear to have the least effect (less than 3% change in RMSE). So as these samples demonstrate, it seems reasonable to ignore the effects of bed and friction slopes in favor of a simplified numerical model. Consequently, Eq. (22) is offered as an appropriate mathematical model governing the flow over labyrinth side-weir.

4. Conclusions

The numerical model carried out by DQM on weirs and labyrinth side-weirs was verified by several experimental models and other numerical models; the results show that this model has less errors. Furthermore, DQM is less intricate and needs less points to reach the appropriate results. Comparison between numerical results and experimental data shows that relative error for the model on weirs is less than 2 percent, and it is less than 3.5 percent on labyrinth side-weir.

The results of sensitivity analysis and analysis of contribution of each parameter show that the channel slope and the friction slope are not requisite parameters and they can be eliminated to simplify the mathematical representation of problem.

References


Biographies

Mohammad Nabatian received the BS degree in Civil Engineering from Persian Gulf University and the MS degree in Hydraulic Structure from Shiraz University. His research concentrates on the numerical and physical modeling of hydraulic structures. He is interested in Differential Quadrature Method and Characteristic Method. He worked in 2012-2014 as a researcher and structure designer for Tir and Seton Pans Company plus Tarh and Sazeh Katibe Company where he was in charge of the calculation department.

Mehrab Amiri received BS and MSc degrees in Hydraulic Structures in 2004 and 2007, respectively, from Shiraz University, Iran, where he is currently serving as a faculty member at the Department of Civil and Environmental Engineering. His research interests include: computational hydraulics, morphodynamics and analysis of hydrological time series. He has also published and presented various papers in journals and at conferences in these fields.

Reza Hashemi is a research fellow at the Centre for Applied Marine Sciences (CAMS), School of Ocean Sciences. He was awarded BEng (1997) and MSc (1999) in Civil Engineering from Shiraz University. His MSc thesis was about automatic calibration of a numerical model (i.e. Princeton Transport Model) using nonlinear programming techniques. His PhD (2006) concerned numerical modelling of the free surface flow and shallow water equations using Differential Quadrature Method. During his PhD studies, he went to Bangor University and collaborated in research projects in CAMS, relating to tide and surge modelling in the Bristol Channel and morphodynamic modelling in the Irish Sea. He worked as an academic...
staff member in Shiraz University (2006-2011) and continued to collaborate with CAMS in that period in areas like applications of Artificial Neural Network and Smoothed Particle Hydrodynamics in hydrodynamic and morphodynamic modelling. He was appointed to this post in December 2011.

**Nasser Talebeydokhti** is a professor in Civil and Environmental Engineering at the School of Engineering, Shiraz University. He is a member of The Academy of Sciences of Iran and also editor-in-chief of Iranian Journal of Science and Technology Transaction of Civil Engineering. His major areas of specialty are hydraulic engineering, sediment transport, river engineering, and environmental engineering. He has published more than 40 journal papers.