An imperfect multi-item single-machine production system with shortage, rework, and scrap considering inspection, dissimilar deficiency levels, and non-zero setup times

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Abstract. In this paper, we consider a deficient production system with permissible shortages. The production system consists in a unique machine that manufactures a number of products with a part being imperfect in the form of rework or scrap. These defective products are identified by 100% inspection during production; then, they are whether reworked or disposed of after normal production process. Like real-world production systems, there are diverse types of errors creating dissimilar breakdown severity and rework. Moreover, rework has non-zero setup time, which makes the problem closer to real-world instances, and the machines require some preparations before starting a new production cycle. Thus, we introduce an Economic Production Quantity (EPQ) problem for an imperfect manufacturing system with non-zero setup times for rework items. The rework items are classified into several categories based on their types of failure and rework rates. The aim of this study is to obtain optimum production time and shortage in each period minimizing total inventory system costs. Convexity of the objective function and exact solution procedure for the current nonlinear optimization problem are also proposed. Finally, a numerical example is presented to assess efficiency and validity of the proposed algorithm.

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1. Introduction

For any manufacturing organization, supply chain, inventory, and production management is vital to achieve success. In attaining the objectives of operation management, each company should effectively utilize resources and try to minimize costs as much as possible. In today's competitive environment, every organization wants to make sure that their customers are satisfied with the quality of their products. On the other hand, no customer wants to compromise on the quality of the product. In the classic EPQ model, it is assumed that during a production run, 100% perfect products are manufactured, which is far from the reality. There are so many reasons for which a smaller percent the items produced will be imperfect, e.g., situations such as breakdown of the machine, interruption in

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manufacturing process, etc. Therefore, one cannot ignore the issue of imperfect production as it has direct side effects on the reputation and good-will of the organization. Employing mathematical optimizations for inventory management dates back to more than a century ago when Ford Whitman Harris introduced classical Economic Order Quantity (EOQ) inventory model in 1913 [1]. This model is the foundation of lot-sizing models and one can consider Harris as the founder of the modern inventory studies [2].

In 1918, Taft [3] proposed Economic Production Quantity (EPQ) by extending the aforementioned inventory model to a manufacturer that produced an item at a constant rate. The main drawback of such inventory models is their impractical assumptions. Therefore, several studies have relaxed these assumptions to make the models more applicable to real-world instances. For example, Hadley and Whitin [4] presented a summary of the EOQ inventory model and extended this model by taking shortage into account. Moreover, Parker [5] investigated reorder point and reorder quantities for an inventory system with stochastic demand and permissible backorder. On the other hand, Yao and Lee [6], Lee and Yao [7], and Björk [8] among others developed the EOQ model for instances of employing fuzzy numbers to approximate model parameters. Moreover, Silver [9], Maddah and Jaber [10], and Khan et al. [11] considered stochastic uncertainties in parameters under several assumptions.

Flawless products and perfect quality is another impractical assumption in the basic EOQ/EPQ inventory models. In organizations of the modern world, deteriorating items have drawn attention in different forms, i.e., scrapped, defective, reworked, obsolescent, etc. One of the first instances of inventory models with deteriorating items was proposed by Whitin [12], in which items became obsolete by time. Aggarwal and Jaggi [13] studied permissible delay in payments with deteriorating products. Wu et al. [14] studied an inventory system consisting in non-instantaneous deteriorating items with partial backordering. Chang et al. [15] developed the aforementioned model by maximizing total profit, instead of minimizing cost per time unit, considering maximum inventory level constraint. Wu et al. [16] addressed deteriorating items with expiration dates and trade credit when trade credit increased revenue as well as opportunity cost and default risk. Some recent studies that have investigated inventory management with deteriorating items under different policies are Sett et al. [17], Wu and Zhao [18], Mokhtari et al. [19], Wu et al. [20], Shah and Cárdenas-Barrón [21], and Teng et al. [22]. Recently, Dobson et al. [23] proposed an EOQ inventory model for deteriorating items with a single product in which demand rate of consumers linearly decreased as a function of the age of the product.

Furthermore, EPQ inventory model has been extended by relaxing perfect production systems in different studies. Salameh and Jaber [24] considered an imperfect production system in which poor quality products could be sold at a lower price after 100% inspection. Afterwards, Goyal and Cárdenas-Barrón [25] proposed a simpler approach to solving the aforementioned problem with near optimal solutions. Wee et al. [26] generalized Salameh and Jaber [24] by modelling an imperfect production system in an uncompetitive market where shortage was permitted and fully backordered. Haji et al. [27] developed a model and its optimization for an imperfect production system with rework where the machine required setup before starting the rework procedure. Recently, Nobil et al. [28] indicated the possibility of shortage occurrence during setup time in the former models and proposed a modified algorithm to overcome this deficiency. Cárdenas-Barrón [29] studied an imperfect production system with rework and planned backorders. Hsu and Hsu [30], Farhangi et al. [31], Cárdenas-Barrón et al. [32], Shah et al. [33], Jaggi et al. [34], and Jaggi et al. [35] have presented some recent inventory management instances with imperfect products.

Producing several items on a machine is another realistic extension of EPQ inventory model. Rogers [36] proposed the first instance of this kind of inventory model. Taleizadeh et al. [37] developed the aforementioned model with respect to shortage and imperfect production system with interruptions, scrapped items, and rework. Pasandideh et al. [38] proposed an EPQ inventory model for an imperfect production system with permissible shortage. They considered a machine that produced various quality items, i.e., perfect, scrap, and reworked. They considered diverse failure levels with their own rework procedures. This classification of rework made the inventory model more realistic by considering various types of production error. Nobil et al. [39] considered utilization-allocation policy in an imperfect manufacturing system producing multiple products on several machines using single-machine properties and common cycle length to reduce calculation efforts in a hybrid genetic algorithm. Nobil and Taleizadeh [40], Nobil et al. [41], and Nobil et al. [42] have recently presented some inventory management studies with single-machine production system.

In this study, an imperfect production system with permissible shortages in backorder form is considered. A single machine produces a number of items including unconfirmed products in the form of rework or scrap. Like real-world production systems, there are various kinds of faults leading to various breakdown severity. As a result of diverse breakdown severity, products require different kinds of rework. Moreover, we extend Pasandideh et al. [38] research by considering non-zero setup times for rework as in Haji

et al. [27]. By relaxing zero setup time, the mentioned problems would be a sub-problem of the current study and the problem becomes closer to real-world instances, in which machines require some preparations before starting a new production cycle. After proving convexity, employing an exact method developed based on the derivatives, the solution procedure for the current nonlinear optimization problem is proposed.

The remainder of this paper is as follows: The problem is defined in Section 2. Section 3 proposes an exact algorithm for this problem. A numerical example is studied in Section 4 to assess efficiency of the proposed algorithm. Finally, the paper is concluded by proposing some future research directions in Section 5.

2. Problem definition

In this paper, we address a multi-item single-machine production system with imperfect products. The imperfect products may be categorized into two kinds of deficiency, namely scrap and reworked, i.e., the products that are diagnosed as repairable after inspection process. The expected proportion of produced defective items is known and constant. Moreover, repairable products are categorized in a number of classes based on the severity of defects, which directly influences the rework rate. Item i $(i = 1, 2, \cdots, m)$ is produced at the rate of (P_i) during regular production periods and λ_i percent of the products are defective. The deficiency rate (λ_i) is divided into α_i and β_i for scrap and reworked items, respectively. In other words, $\lambda_i = \alpha_i + \beta_i$, where β_i consists in n kinds of rework, i.e., v_i^j percent of item i products require the rework of type j and $j = 1, 2, \cdots, n$. Therefore, the total proportion of products i that require rework (β_i) can be stated as the sum of v_i^j, i.e., $\beta_i = v_i^1 + v_i^2 + \cdots + v_i^n$ or $\beta_i = \sum_{j=1}^{n} v_i^j$. Thus, we have:

$$\lambda_i = \alpha_i + \beta_i = \alpha_i + v_i^1 + v_i^2 + \cdots + v_i^n. \quad (1)$$

Furthermore, in the proposed inventory system, all items are produced by one machine; this property lead to the following equation:

$$T_1 = T_2 = T_3 = \cdots = T_m = T. \quad (2)$$

The following notations are employed to model the proposed non-linear single-machine inventory control problem with m kinds of items and n types of rework.

c_i Production cost of manufacturing each item kind i $(i = 1, 2, \cdots, m)$;

o_i Inspection costs per item kind i $(i = 1, 2, \cdots, m)$;

d_i Disposal costs per item kind i $(i = 1, 2, \cdots, m)$;

r_i^j Rework costs for item kind i with fault type j $(i = 1, 2, \cdots, m)$ & $(j = 1, 2, \cdots, n)$;

h_i Holding costs per year per item kind i $(i = 1, 2, \cdots, m)$;

L_i Shortage costs per year per item kind i $(i = 1, 2, \cdots, m)$;

A_i Setup costs per regular production cycle to produce item kind i $(i = 1, 2, \cdots, m)$;

B_i Setup costs per rework cycle for all rework types associated with item kind i $(i = 1, 2, \cdots, m)$;

ts_i Setup time per regular production cycle to produce item kind i $(i = 1, 2, \cdots, m)$;

G_i Setup time per rework cycle for all rework types associated with item kind i $(i = 1, 2, \cdots, m)$;

P_i Production rate for item kind i $(i = 1, 2, \cdots, m)$;

D_i Demand rate for item kind i $(i = 1, 2, \cdots, m)$;

α_i Disposal rate of produced item kind i $(i = 1, 2, \cdots, m)$;

v_i^j Percentage of produced item kind i that requires rework type j $(i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n)$;

λ_i Percentage of produced imperfect product kind i $(i = 1, 2, \cdots, m)$;

θ_i^j Rework rate for rework type j associated with item kind i $(i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n)$;

T Cycle length;

N Number of cycles in a year;

Q_i Quantity of produced item kind i per cycle $(i = 1, 2, \cdots, m)$;

S_i Quantity of shortage for item kind i per cycle $(i = 1, 2, \cdots, m)$.

A schematic diagram of inventory position for this model is represented in Figure 1. As it is demonstrated in Figure 1, production rate of item i during regular cycle is (P_i) in which λ_i percent of the products are defective; consequently, production rate of non-defective items of kind i per cycle equals $(1 - \lambda_i)P_i$. On the other hand, production rate of non-defective items should be more than or equal to the demand rate (D_i). In other words, we have:

$$(1 - \lambda_i)P_i \geq D_i \quad \text{or} \quad e_i = (1 - \lambda_i)P_i - D_i \geq 0. \quad (3)$$

After accomplishing the regular production cycle and inspection, scrap items are disposed of and rework
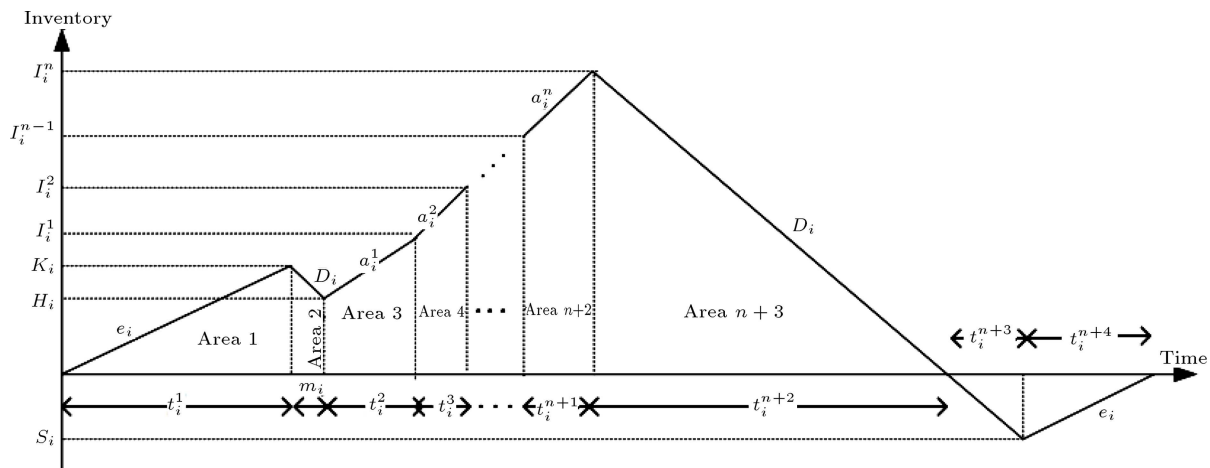
Figure 1. A schematic diagram of inventory level of item i.

process should be initiated. In this study, we assume that rework process has a non-zero setup time equal to regular production setup time. Moreover, for the sake of simplicity and as a consequence of independency of setup times, we consider one setup time, namely G_i, before starting all types of rework for item kind i. During setup time, no item is produced and item i is consumed at the rate of D_i. The rework processes initiate at rates of $\theta_i^1, \theta_i^2, \cdots$, and θ_i^n during the periods of $t_i^2, t_i^3, \cdots$ and t_i^{n+1}, respectively; during these periods, no scrap item is manufactured. Moreover, it is logical to assume that the rework process of products does not require more time than its corresponding regular production time does. In other words, the rework rate for a product is greater than or equal to its regular production rate, i.e., $\theta_i^j \geq 1$. Consequently, the demand rate (D_i) for product i is less than or equal to its rework production rate $\theta_i^j P_i$. In other words:

$$\theta_i^j P_i \geq D_i \qquad \text{or} \qquad a_i^j = \theta_i^j P_{i_i} - D_i \geq 0. \tag{4}$$

Based on Figure 1 and the aforementioned conditions, inventory level at the end of each rework process can be stated as:

$$K_i = \frac{e_i Q_i}{P_i} - S_i, \tag{5}$$

$$H_i = K_i - G_i D_i = \frac{e_i Q_i}{P_i} - S_i - G_i D_i, \tag{6}$$

$$I_i^1 = H_i + \frac{a_i^1 v_i^1 Q_i}{\theta_i^1 P_i} = \frac{e_i Q_i}{P_i} - S_i - G_i D_i + \frac{a_i^1 v_i^1 Q_i}{\theta_i^1 P_i}, \tag{7}$$

$$I_i^2 = I_i^1 + \frac{a_i^2 v_i^2 Q_i}{\theta_i^2 P_i} = \frac{e_i Q_i}{P_i} - S_i - G_i D_i + \frac{a_i^1 v_i^1 Q_i}{\theta_i^1 P_i}$$

$$\qquad + \frac{a_i^2 v_i^2 Q_i}{\theta_i^2 P_i}. \tag{8}$$

Therefore, the following is obtained:

$$I_i^j = \frac{e_i Q_i}{P_i} - S_i - G_i D_i + \left(\sum_{r=1}^{j} \frac{a_i^j v_i^j Q_i}{\theta_i^j P_i} \right),$$

$$j = 1, 2, \cdots, n. \tag{9}$$

Moreover, based on the inventory level and Figure 1, each period length can be stated as:

$$t_i^1 = \frac{K_i}{e_i} = \frac{Q_i}{P_i} - \frac{S_i}{e_i}, \tag{10}$$

$$t_i^2 = \frac{I_i^1 - H_i}{a_i^1} = \frac{v_i^1 Q_i}{\theta_i^1 P_i}, \tag{11}$$

$$t_i^3 = \frac{I_i^2 - I_i^1}{a_i^2} = \frac{v_i^2 Q_i}{\theta_i^2 P_i}. \tag{12}$$

Consequently, we have:

$$t_i^{j+1} = \frac{I_i^j - I_i^{j-1}}{a_i^j} = \frac{v_i^j Q_i}{\theta_i^j P_i}, \qquad j = 1, 2, \cdots, n. \tag{13}$$

Thus:

$$t_i^{n+1} = \frac{v_i^n Q_i}{\theta_i^n P_i}. \tag{14}$$

On the other hand, we know that:

$$t_i^{n+2} = \frac{I_i^n}{D_i} = \frac{e_i Q_i}{D_i P_i} - \frac{S_i}{D_i} - m_i + \left(\sum_{r=1}^{j} \frac{a_i^j v_i^j Q_i}{\theta_i^j D_i P_i} \right), \tag{15}$$

$$t_i^{n+3} = \frac{S_i}{D_i}, \tag{16}$$

$$t_i^{n+4} = \frac{S_i}{e_i}. \tag{17}$$

Based on the above-mentioned conditions, a period length can be obtained as:

$$T = T_i = t_i^1 + m_i + t_i^2 + t_i^3 + \cdots + t_i^{n+1} + t_i^{n+2}$$

$$+ t_i^{n+3} + t_i^{n+4}. \tag{18}$$

Moreover, based on Eq. (A.4) stated in Appendix A, a cycle length can be calculated as:

$$T = \frac{(1 - \alpha_i)Q_i}{D_i}, \tag{19}$$

and:

$$Q_i = \frac{D_i T}{(1 - \alpha_i)}. \tag{20}$$

2.1. Objective function

The objective function of the proposed problem is to optimize total inventory costs, including annual setup costs for regular production cycles, annual setup costs for rework, annual production costs, annual inspection costs, annual disposal costs, annual rework costs, annual shortage costs, and annual holding costs. As it was mentioned before, setup cost of regular production cycle for item kind i is A_i. Moreover, each year consists in N cycles and the sum of annual setup costs (SC) for regular production of all items is stated as:

$$SC = \sum_{i=1}^{m} NA_i = \sum_{i=1}^{m} \frac{A_i}{T}. \tag{21}$$

On the other hand, setup cost of rework associated with item kind i per cycle is B_i and a year consists in N cycles. Therefore, the sum of annual setup costs (UC) of rework for all items is as follows:

$$UC = \sum_{i=1}^{m} NB_i = \sum_{i=1}^{m} \frac{B_i}{T}. \tag{22}$$

Production cost per item kind i equals Q_i. As a result, production cost per cycle can be stated as $c_i Q_i$ and the total production cost per year (PC) is:

$$PC = \sum_{i=1}^{m} Nc_i Q_i = \sum_{i=1}^{m} \frac{c_i Q_i}{T}. \tag{23}$$

Substituting Q_i from Eq. (20) in Eq. (23), we have:

$$PC = \sum_{i=1}^{m} Nc_i Q_i = \sum_{i=1}^{m} \frac{D_i c_i}{(1 - \alpha_i)}. \tag{24}$$

Inspection cost per item kind i is o_i and production quantity of this kind of item per cycle is Q_i. Therefore, inspection cost per cycle is $o_i Q_i$ and annual inspection cost (IC) equals:

$$IC = \sum_{i=1}^{m} No_i Q_i = \sum_{i=1}^{m} \frac{D_i o_i}{(1 - \alpha_i)}. \tag{25}$$

Disposal cost per item kind i is d_i and quantity of produced items requiring disposal per cycle is $\alpha_i Q_i$. Thus, disposal cost per cycle is $d_i \alpha_i Q_i$ and annual disposal cost (DC) can be stated as:

$$DC = \sum_{i=1}^{m} Nd_i \alpha_i Q_i = \sum_{i=1}^{m} \frac{D_i d_i \alpha_i}{(1 - \alpha_i)}. \tag{26}$$

Rework cost of type j per item kind i is r_i^j and quantity of the produced items requiring this type of rework per cycle is $v_i^j Q_i$. Thus, rework cost type j per cycle is $r_i^j v_i^j Q_i$ and annual rework cost (RC) can be stated as:

$$RC = \sum_{i=1}^{m} \sum_{j=1}^{n} Nr_i^j v_i^j Q_i = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{D_i r_i^j v_i^j}{(1 - \alpha_i)}. \tag{27}$$

As it can be seen in Figure 1, the backordered demands for item kind i can be stated as $S_i(t_i^{n+3} + t_i^{n+4})/2$. Knowing that annual shortage cost for product kind i is L_i, annual shortage cost (BC) is:

$$BC = \sum_{i=1}^{m} \frac{NL_i S_i(t_i^{n+3} + t_i^{n+4})}{2}. \tag{28}$$

Substituting t_i^{n+3} and t_i^{n+4} with Eqs. (16) and (17), respectively, we have:

$$BC = \sum_{i=1}^{m} \frac{L_i(e_i + D_i)}{2D_i e_i} \left(\frac{(S_i)^2}{T} \right). \tag{29}$$

Annual holding cost for item kind i is h_i and with respect to the area under the curve in Figure 1, annual holding costs (HC) can be stated as:

$$HC = \sum_{i=1}^{m} Nh_i \left\{ \begin{array}{l} \dfrac{K_i \times t_i^1}{2} + \dfrac{(K_i + H_i)G_i}{2} \\[2mm] + \dfrac{(H_i + I_i^1)t_i^2}{2} + \dfrac{(I_i^1 + I_i^2)t_i^3}{2} \\[2mm] + \cdots + \dfrac{(I_i^{n-1} + I_i^n)t_i^{n+1}}{2} \\[2mm] + \dfrac{I_i^n \times t_i^{n+2}}{2} \end{array} \right\}. \tag{30}$$

This equation can be stated as follows. For calculations, see Eq. (B.10) in Appendix B:

$$HC = \frac{h_i e_i G_i D_i}{P_i(1 - \alpha_i)} + \frac{h_i e_i(1 - \lambda_i)}{2P_i(1 - \alpha_i)^2}(T)$$

$$+ \frac{h_i P_i(1 - \lambda_i)}{2e_i D_i} \left(\frac{(S_i)^2}{T} \right) - \frac{h_i G_i}{2} \left(\frac{S_i}{T} \right)$$

$$- \frac{h_i e_i}{2P_i(1 - \alpha_i)}(S_i)$$

$$+ \frac{h_i(D_i)^2}{(P_i)^2(1 - \alpha_i)^2} \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \left(\sum_{y=0}^{j-1} \frac{a_i^y v_i^y}{\theta_i^y} \right)(T)$$

$$
-\frac{h_i G_i D_i}{P_i(1-\alpha_i)}\left(D_i\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}+\sum_{j=1}^{n}\frac{a_i^j v_i^j}{2\theta_i^j}\right)
$$

$$
-\frac{h_i D_i}{P_i(1-\alpha_i)}\left(\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}+\frac{1}{2D_i}\sum_{j=1}^{n}\frac{a_i^j v_i^j}{\theta_i^j}\right)(S_i)
$$

$$
+\frac{h_i e_i(D_i)^2}{(P_i)^2(1-\alpha_i)^2}\left(\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}\right.
$$

$$
+\frac{1}{2D_i}\sum_{j=1}^{n}\frac{a_i^j v_i^j}{\theta_i^j}\right)(T)
$$

$$
+\frac{h_i(D_i)^2}{2(P_i)^2(1-\alpha_i)^2}\left(\sum_{j=1}^{n}\frac{a_i^j(v_i^j)^2}{(\theta_i^j)^2}\right.
$$

$$
+\frac{1}{D_i}\sum_{j=1}^{n}\left(\frac{a_i^j v_i^j}{\theta_i^j}\right)^2\right)(T). \tag{31}
$$

Finally, based on Eqs. (21), (22), (24), (25), (26), (27), (29), and (31), the annual total cost can be calculated as follows:

$$
TC=\sum_{i=1}^{m}\beta_i^1+\sum_{i=1}^{m}\beta_i^2(T)+\sum_{i=1}^{m}\frac{\beta_i^3}{T}-\sum_{i=1}^{m}\beta_i^4\left(\frac{S_i}{T}\right)
$$

$$
-\sum_{i=1}^{m}\beta_i^5 S_i+\sum_{i=1}^{m}\beta_i^6\left(\frac{(S_i)^2}{T}\right), \tag{32}
$$

where:

$$
\beta_i^1=\frac{h_i G_i D_i}{P_i(1-\alpha_i)}\left(e_i-D_i\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}-\sum_{j=1}^{n}\frac{a_i^j v_i^j}{2\theta_i^j}\right)
$$

$$
+\frac{D_i\left(c_i+o_i+d_i\alpha_i+\sum_{j=1}^{n}r_i^j v_i^j\right)}{(1-\alpha_i)}, \tag{33}
$$

$$
\beta_i^2=\frac{h_i e_i(D_i)^2}{(P_i)^2(1-\alpha_i)^2}\left(\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}+\frac{1}{2D_i}\sum_{j=1}^{n}\frac{a_i^j v_i^j}{\theta_i^j}\right)
$$

$$
+\frac{h_i(D_i)^2}{2(P_i)^2(1-\alpha_i)^2}\left(\sum_{j=1}^{n}\frac{a_i^j(v_i^j)^2}{(\theta_i^j)^2}\right.
$$

$$
+\frac{1}{D_i}\sum_{j=1}^{n}\left(\frac{a_i^j v_i^j}{\theta_i^j}\right)^2\right)
$$

$$
+\frac{h_i(D_i)^2}{(P_i)^2(1-\alpha_i)^2}\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}\left(\sum_{y=0}^{j-1}\frac{a_i^y v_i^y}{\theta_i^y}\right)
$$

$$
+\frac{h_i e_i(1-\lambda_i)}{2P_i(1-\alpha_i)^2}\geq 0, \qquad i=1,2,\cdots,m, \tag{34}
$$

$$
\beta_i^3=A_i+B_i\geq 0, \qquad i=1,2,\cdots,m, \tag{35}
$$

$$
\beta_i^4=\frac{h_i G_i}{2}\geq 0, \qquad i=1,2,\cdots,m, \tag{36}
$$

$$
\beta_i^5=\frac{h_i D_i}{P_i(1-\alpha_i)}\left(\sum_{j=1}^{n}\frac{v_i^j}{\theta_i^j}+\frac{1}{2D_i}\sum_{j=1}^{n}\frac{a_i^j v_i^j}{\theta_i^j}\right)
$$

$$
+\frac{h_i e_i}{2P_i(1-\alpha_i)}\geq 0, \qquad i=1,2,\cdots,m, \tag{37}
$$

$$
\beta_i^6=\frac{L_i(e_i+D_i)}{2D_i e_i}+\frac{h_i P_i(1-\lambda_i)}{2e_i D_i}\geq 0,
$$

$$
i=1,2,\cdots,m. \tag{38}
$$

2.2. Constraints

In the proposed problem, the main constraint addresses the production capacity of the machine. This constraint expresses that summation of production times for all items and their setup times should be less than or equal to the permissible production time of the machine. In other words, we have:

$$
\sum_{i=1}^{m}\left(\sum_{j=1}^{n+1}t_i^j+t_i^{n+4}+ts_i+G_i\right)\leq T. \tag{39}
$$

This constraint can be stated as follows. For detailed calculations, see Inequality (C.4) in Appendix C:

$$
T\geq\frac{\sum\limits_{i=1}^{m}(ts_i+G_i)}{1-\sum\limits_{i=1}^{m}\left(\frac{D_i}{(1-\alpha_i)P_i}\left(1+\sum\limits_{j=1}^{n}\frac{v_i^j}{\theta_i^j}\right)\right)}. \tag{40}
$$

Other constraints restrict shortage in rework setup time. In other words, we have m constraints to prepare the machine for rework equal to the number of items. With respect to Figure 1, we have:

$$
H_i\geq 0, \qquad i=1,2,\cdots,m. \tag{41}
$$

By employing Eq. (6), the following holds:

$$
\frac{e_i Q_i}{P_i}-S_i-G_i D_i\geq 0, \qquad i=1,2,\cdots,m. \tag{42}
$$

Substituting Q_i with Eq. (20), we have:

$$
\frac{e_i D_i T}{(1-\alpha_i)P_i}-S_i-G_i D_i\geq 0, \qquad i=1,2,\cdots,m. \tag{43}
$$

Therefore:

$$
T\geq\frac{(1-\alpha_i)(S_i+G_i D_i)P_i}{e_i D_i}\geq 0, \qquad i=1,2,\cdots,m. \tag{44}
$$

2.3. Final model

Based on the objective function proposed in Eq. (32) and constraints proposed in Inequalities (40) and (44), the final model of the proposed problem is as follows:

$$\text{Min} \quad TC = \sum_{i=1}^{m} \beta_i^1 + \sum_{i=1}^{m} \beta_i^2(T) + \sum_{i=1}^{m} \frac{\beta_i^3}{T}$$

$$- \sum_{i=1}^{m} \beta_i^4 \left(\frac{S_i}{T} \right) - \sum_{i=1}^{m} \beta_i^5 S_i$$

$$+ \sum_{i=1}^{m} \beta_i^6 \left(\frac{(S_i)^2}{T} \right),$$

$$\text{s.t.:} \quad T \geq \frac{\sum_{i=1}^{m}(ts_i + G_i)}{1 - \sum_{i=1}^{m} \left(\frac{D_i}{(1-\alpha_i)P_i} \left(1 + \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \right) \right)},$$

$$T \geq \frac{(1-\alpha_i)(S_i + G_i D_i) P_i}{e_i D_i} \geq 0,$$

$$i = 1, 2, \cdots, m, \qquad T > 0,$$

$$S_i \geq 0; \qquad i = 1, 2, \cdots, m. \qquad (45)$$

3. Solution method

In this section, we propose a solution method to optimize the proposed non-linear programming model. To do so, first, we prove the convexity of model (45) by investigating the objective function. It is worth mentioning that all decision variables are continuous and constraints are in linear form. As it can be seen in Appendix D, the quadratic form of objective function (45) is semi-positive. Thus, Model (45) is convex and using partial differentials, we can obtain optimum values for decision variables as follows:

$$\frac{\partial TC}{\partial S_i} = -\frac{\beta_i^4}{T} - \beta_i^5 + \frac{2\beta_i^6 S_i}{T} = 0, \qquad i = 1, 2, \cdots, m. \qquad (46)$$

Therefore:

$$S_i = \frac{\beta_i^5 T + \beta_i^4}{2\beta_i^6}, \qquad i = 1, 2, \cdots, m. \qquad (47)$$

Moreover, by differentiating objective function (45) according to T, we have:

$$\frac{\partial TC}{\partial T} = \sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \frac{\beta_i^3}{T^2} + \sum_{i=1}^{m} \beta_i^4 \left(\frac{S_i}{T^2} \right)$$

$$- \sum_{i=1}^{m} \beta_i^6 \left(\frac{(S_i)^2}{T^2} \right) = 0. \qquad (48)$$

Then:

$$T^2 = \frac{\sum_{i=1}^{m} \beta_i^3 - \sum_{i=1}^{m} \beta_i^4 S_i + \sum_{i=1}^{m} \beta_i^6 (S_i)^2}{\sum_{i=1}^{m} \beta_i^2}. \qquad (49)$$

Substituting S_i with Eq. (47), we have:

$$T = \sqrt{\frac{\sum_{i=1}^{m} \beta_i^3 - \sum_{i=1}^{m} \left(\frac{(\beta_i^4)^2}{4\beta_i^6} \right)}{\sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \left(\frac{(\beta_i^5)^2}{4\beta_i^6} \right)}}. \qquad (50)$$

Based on the obtained solutions from Eqs. (47) and (50), optimum values associated with the model can be achieved using the following algorithm:

Step 1: If $\sum_{i=1}^{m} \left(\frac{D_i}{(1-\alpha_i)P_i} \left(1 + \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \right) \right) < 0$, go to Step 2. Otherwise, the problem is infeasible and go to Step 14;

Step 2: If $\sum_{i=1}^{m} \beta_i^3 - \sum_{i=1}^{m} \left(\frac{(\beta_i^4)^2}{4\beta_i^6} \right)$ and $\sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \left(\frac{(\beta_i^5)^2}{4\beta_i^6} \right)$ are simultaneously positive or negative, go to Step 3. Otherwise, the problem is infeasible and go to Step 14;

Step 3: Calculate T from Eq. (50) and go to Step 4;

Step 4: Based on the obtained values for T, calculate shortage quantities associated with items using Eq. (47) and go to Step 5;

Step 5: If $S_i \geq 0$ for all $i = 1, 2, \cdots, m$, then go to Step 9; else go to Step 6;

Step 6: If $S_i < 0$ $(i = 1, 2, \cdots, m)$, calculate $TL_i = \frac{-\beta_i^4}{\beta_i^5}$; else for $S_i \geq 0$, let $TL_i = 0$ and go to Step 7;

Step 7: Calculate T using:

$$T = \max\{TL_1, TL_2, \cdots, TL_m\},$$

then, go to Step 8;

Step 8: Calculate S_i for all $i = 1, 2, \cdots, m$ employing Eq. (47) with respect to T; then, go to Step 9;

Step 9: Calculate values of T_G and T_i^X $(i = 1, 2, \cdots, m)$, employing the following equations:

$$T_G = \frac{\sum_{i=1}^{m}(ts_i + G_i)}{1 - \sum_{i=1}^{m} \left(\frac{D_i}{(1-\alpha_i)P_i} \left(1 + \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \right) \right)}, \qquad (51)$$

$$T_i^X = \frac{(1-\alpha_i)(S_i + G_i D_i) P_i}{e_i D_i}, \qquad i = 1, 2, \cdots, m. \qquad (52)$$

Step 10: Obtain $T_{\min}$ employing:

$$T_{\min} = \max\{T_G, T_1^X, T_2^X, \cdots, T_m^X\},$$

then, go to Step 11;

Step 11: If $T \geq T_{\min}$, then $T^* = T$; else $T^* = T_{\min}$ and go to Step 12;

Step 12: Calculate S_i for all $i = 1, 2, \cdots, m$ using Eq. (47) and go to Step 13;

Step 13: According to the obtained values for T^* and S_i^*, calculate Q_i^* and TC^* using Eqs. (20) and (32);

Step 14: Terminate the algorithm; if it is feasible, show solutions; else show infeasible.

4. Numerical experiment

In this section, we propose a numerical example to investigate efficiency of the proposed solution procedure for this model. The numerical example considers a production system consisting of two items produced by a single machine. Table 1 indicates the information associated with parameters of the proposed numerical example and costs of the system are proposed in Table 2.

The percentage of deficient items produced in the

Table 1. Values associated with parameters of the numerical example.

Parameter	Item 1	Item 2
P_i	8000	10000
D_i	3000	4200
α_i	0.010	0.013
v_i^1	0.014	0.010
v_i^2	0.012	0.012
θ_i^1	1	1.5
θ_i^2	2	2
ts_i	0.0013	0.0012
G_i	0.0010	0.0010

Table 2. Costs associated with the proposed numerical example.

Parameter	Item 1	Item 2
c_i	400	420
o_i	10	12
d_i	30	25
r_i^1	45	50
r_i^2	55	60
L_i	8	9
A_i	2000	2200
B_i	500	400
h_i	5	6

system can be calculated as follows:

$$\lambda_1 = \alpha_1 + \beta_1 = \alpha_1 + v_1^1 + v_1^2$$
$$= 0.010 + 0.014 + 0.012 = 0.036,$$

$$\lambda_2 = \alpha_2 + \beta_2 = \alpha_2 + v_2^1 + v_2^2$$
$$= 0.013 + 0.010 + 0.012 = 0.035.$$

The rest of the model parameters can be calculated as:

$$e_1 = (1 - \lambda_1)P_1 - D_1 = (1 - 0.036)8000 - 3000$$
$$= 4712 > 0,$$

$$e_2 = (1 - \lambda_2)P_2 - D_2 = (1 - 0.035)10000 - 42000$$
$$= 5450 > 0,$$

$$a_1^1 = \theta_1^1 P_1 - D_1 = 1 \times 8000 - 3000 = 5000 > 0,$$

$$a_1^2 = \theta_1^2 P_1 - D_1 = 2 \times 8000 - 3000 = 13000 > 0,$$

$$a_2^1 = \theta_2^1 P_2 - D_2 = 1.5 \times 10000 - 4200 = 10800 > 0,$$

$$a_2^2 = \theta_2^2 P_2 - D_2 = 2 \times 10000 - 4200 = 15800 > 0.$$

Moreover, β_i^j $(j = 1, \cdots, 6)$ can be calculated employing Eqs. (33)-(38). The calculated values for β_i^j are proposed in Table 3.

Now, we calculate the optimum solution for the numerical example using the proposed algorithm in Section 3 as follows:

Step 1: Since:

$$\left\{ \sum_{i=1}^{2} \left(\frac{D_i}{(1 - \alpha_i)P_i} \left(1 + \sum_{j=1}^{2} \frac{v_i^j}{\theta_i^j}\right)\right) = 0.8173 \right\} < 1,$$

go to Step 2;

Step 2: Since both:

$$\left\{ \sum_{i=1}^{m} \beta_i^3 - \sum_{i=1}^{m} \left(\frac{(\beta_i^4)^2}{4\beta_i^6}\right) \cong 5099.9 \right\},$$

and:

$$\left\{ \sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \left(\frac{(\beta_i^5)^2}{4\beta_i^6}\right) \cong 351.67 \right\},$$

are positive, go to Step 3;

Step 3: Calculate T as follows:

$$T = \sqrt{\frac{\sum_{i=1}^{m} \beta_i^3 - \sum_{i=1}^{m} \left(\frac{(\beta_i^4)^2}{4\beta_i^6}\right)}{\sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \left(\frac{(\beta_i^5)^2}{4\beta_i^6}\right)}} = 3.80821661358787,$$

and go to Step 4.

Table 3. Values associated with β_i^j for $i = 1, 2$ and $j = 1, \cdots, 6$.

Item	β_i^1	β_i^2	β_i^3	β_i^4	β_i^5	β_i^6
1	1247251.09469697	154.583355014794	2500	0.0025	1.57196969696970	40000.0021312960
2	1844885.90655319	197.079845283519	2600	0.0030	1.73957446808511	60000.0018512451

Step 4: Shortage values associated with each item are as follows:

$$S_1 = \frac{\beta_1^5 T + \beta_1^4}{2\beta_1^6} \cong 0, \qquad S_2 = \frac{\beta_2^5 T + \beta_2^4}{2\beta_2^6} \cong 0,$$

go to Step 5;

Step 5: Since $S_1 = S_2 = 0$, go to Step 9;

Step 9: Obtain the values of T_G, T_1^X, and T_2^X as follows:

$$T_G = \frac{\sum_{i=1}^m (ts_i + G_i)}{1 - \sum_{i=1}^m \left(\frac{D_i}{(1-\alpha_i) P_i} \left(1 + \sum_{j=1}^n \frac{v_i^j}{\theta_i^j} \right) \right)}$$

$$= 0.0246286036910265,$$

$$T_1^X = \frac{(1 - \alpha_1)(S_1 + G_1 D_1) P_1}{e_1 D_1}$$

$$= 0.00168085688321865,$$

$$T_2^X = \frac{(1 - \alpha_2)(S_2 + G_2 D_2) P_2}{e_2 D_2}$$

$$= 0.00181103298935632,$$

go to Step 10;

Step 10: $T_{\min}$ is obtained as:

$$T_{\min} = \max\{T_G, T_1^X, T_2^X\} = 0.0246286036910265,$$

go to Step 11;

Step 11: Since $T \geq T_{\min}$, we have $T^* = T = 3.80821661358787$; go to Step 12;

Step 12: Obtain S_i^* with respect to calculated T^* as follows:

$$S_1^* = \frac{\beta_1^5 T^* + \beta_1^4}{2\beta_1^6} \cong 0, \qquad S_2^* = \frac{\beta_2^5 T^* + \beta_2^4}{2\beta_2^6} \cong 0,$$

go to Step 13;

Step 13: Calculate Q_i^* and TC^* based on the T^* and S_i^* obtained in the former steps as follows:

$$Q_1^* = \frac{D_1 T^*}{(1 - \alpha_1)} = 11540.0503442057,$$

$$Q_2^* = \frac{D_2 T^*}{(1 - \alpha_2)} = 16205.1770790973,$$

$$TC^* = \sum_{i=1}^m \beta_i^1 + \sum_{i=1}^m \beta_i^2 (T^*) + \sum_{i=1}^m \frac{\beta_i^3}{T^*}$$

$$- \sum_{i=1}^m \beta_i^4 \left(\frac{S_i^*}{T^*} \right) - \sum_{i=1}^m \beta_i^5 S_i^*$$

$$+ \sum_{i=1}^m \beta_i^6 \left(\frac{(S_i^*)^2}{T^*} \right) = \$3094815.42042686.$$

5. Conclusion and suggestions

In this study, we investigated a defective single-machine manufacturing system for several commodities. The manufactured products may have problems with operator error, raw materials quality, and machine failure. Therefore, a percentage of the production may have a percentage of unacceptable quality. These defective products are identified by 100% inspection during production, and they are reworked or disposed of after normal production process. Goods that required rework were classified into several categories based on their types of failure, each with a different rework rate. In this system, there was a setup time to manufacture each item on the machine in a normal cycle; in the rework cycle, a new setup time was incurred. This study presented a single-machine lot-sizing problem for an imperfect manufacturing system with scrap and rework under non-zero setup times for rework items. The rework was classified into several categories based on the types of failure and rework rates. Considering the aforementioned conditions and permissible shortage backordered, the proposed problem was modelled as a non-linear programming model. The convexity of this model was proved and an exact algorithm was proposed. Finally, a numerical example was given to assess the effectiveness of the proposed algorithm under different conditions.

5.1. Managerial insights

Taking into account the behavioral factors, this study investigates a fairly practical problem. Therefore, this research can provide production managers and planners with insights in their decision-making. The nonlinear lot-sizing model of this research integrates the production decisions, inventory, and rework. Thus, the model provides a good framework for the im-

perfect manufacturing system with scrap and rework.

5.2. Limitations and future directions

This research considers only a deficient manufacturing system with permissible shortages. The production system consists in a single machine that manufactures items with a certain percentage of defective items that can be reworked or thrown away as scrap. For future work, one can consider different marketing and pricing policies with budget and storage space constraints. One can also consider the impact of information sharing between the manufacturer and the end user.

References

1. Harris, F.W. "How many parts to make at once", *Factory, The Magazine of Management*, **10**(2), pp. 135-136, 152 (1913).

2. Cárdenas-Barrón, L.E., Chung, K.J., and Treviño-Garza, G. "Celebrating a century of the economic order quantity model in honor of Ford Whitman Harris", *International Journal of Production Economics*, **155**, pp. 1-7 (2014).

3. Taft, E.W. "The most economical production lot", *Iron Age*, **101**(18), pp. 1410-1412 (1918).

4. Hadley, G. and Whitin, T.M., *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, NJ (1963).

5. Parker, L.L. "Economical reorder quantities and re-order points with uncertain demand", *Naval Research Logistics Quarterly*, **11**(3-4), pp. 351-358 (1964).

6. Yao, J.S. and Lee, H.M. "Fuzzy inventory with back-order for fuzzy order quantity", *Information Sciences*, **93**(3), pp. 283-319 (1996).

7. Lee, H.M. and Yao, J.S. "Economic order quantity in fuzzy sense for inventory without backorder model", *Fuzzy Sets and Systems*, **105**(1), pp. 13-31 (1999).

8. Björk, K.M. "An analytical solution to a fuzzy economic order quantity problem", *International Journal of Approximate Reasoning*, **50**(3), pp. 485-493 (2009).

9. Silver, E. "Establishing the order quantity when the amount received is uncertain", *INFOR: Information Systems and Operational Research*, **14**(1), pp. 32-39 (1976).

10. Maddah, B., and Jaber, M.Y. "Economic order quantity for items with imperfect quality: revisited", *International Journal of Production Economics*, **112**(2), pp. 808-815 (2008).

11. Khan, M., Jaber, M.Y., and Bonney, M. "An economic order quantity (EOQ) for items with imperfect quality and inspection errors", *International Journal of Production Economics*, **133**(1), pp. 113-118 (2011).

12. Whitin, T.M., *Theory of Inventory Management*, Princeton University Press, Princeton, NJ, USA (1957).

13. Aggarwal, S.P. and Jaggi, C.K. "Ordering policies of deteriorating items under permissible delay in payments", *Journal of the Operational Research Society*, **46**(5), pp. 658-662 (1995).

14. Wu, K.S., Ouyang, L.Y., and Yang, C.T. "An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging", *International Journal of Production Economics*, **101**(2), pp. 369-384 (2006).

15. Chang, C.T., Teng, J.T., and Goyal, S.K. "Optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand", *International Journal of Production Economics*, **123**(1), pp. 62-68 (2010).

16. Wu, J., Ouyang, L.Y., Cárdenas-Barrón, L.E., and Goyal, S.K. "Optimal credit period and slot size for deteriorating items with expiration dates under two-level trade credit financing", *European Journal of Operational Research*, **237**(3), pp. 898-908 (2014).

17. Sett, B.K., Sarkar, S., Sarkar, B., and Yun, W.Y. "Optimal replenishment policy with variable deterioration for fixed-lifetime products", *Scientia Iranica, Transactions E, Industrial Engineering*, **23**(5), pp. 2318-2329 (2016).

18. Wu, C.F. and Zhao, Q.H. "An inventory model for deteriorating items with inventory-dependent and linear trend demand under trade credit", *Scientia Iranica, Transactions E, Industrial Engineering*, **22**(6), pp. 2558 -2570 (2015).

19. Mokhtari, H., Naimi-Sadigh, A., and Salmasnia, A. "A computational approach to economic production quantity model for perishable products with backordering shortage and stock-dependent demand", *Scientia Iranica, Transactions E, Industrial Engineering*, **24**(4), pp. 2138-2151 (2017).

20. Wu, J., Al-Khateeb, F.B., Teng, J.T., and Cárdenas-Barrón, L.E. "Inventory models for deteriorating items with maximum lifetime under downstream partial trade credits to credit-risk customers by discounted cash-flow analysis", *International Journal of Production Economics*, **171**, pp. 105-115 (2016).

21. Shah N.H. and Cárdenas-Barrón, L.E. "Retailer's decision for ordering and credit policies for deteriorating items when a supplier offers order-linked credit period or cash discount", *Applied Mathematics and Computation*, **259**, pp. 569-578 (2015).

22. Teng, J.T., Cárdenas-Barrón, L.E., Chang, H.J., Wu, J., and Hu, Y. "Inventory lot-size policies for deteriorating items with expiration dates and advance payments", *Applied Mathematical Modelling*, **40**(19), pp. 8605-8616 (2016).

23. Dobson, G., Pinker, E.J., and Yildiz, O. "An EOQ model for perishable goods with age-dependent demand rate", *European Journal of Operational Research*, **257**(1), pp. 84-88 (2017).

24. Salameh, M.K. and Jaber, M.Y. "Economic production quantity model for items with imperfect quality",

International Journal of Production Economics, **64**(1), pp. 59-64 (2000).

25. Goyal, S.K. and Cárdenas-Barrón, L.E. "Note on: economic production quantity model for items with imperfect quality-a practical approach", *International Journal of Production Economics*, **77**(1), pp. 85-87 (2002).

26. Wee, H.M., Yu, J., and Chen, M.C. "Optimal inventory model for items with imperfect quality and shortage backordering", *Omega*, **35**(1), pp. 7-11 (2007).

27. Haji, R., Haji, A., Sajadifar, M., and Zolfaghari, S. "Lot sizing with non-zero setup times for rework", *Journal of Systems Science and Systems Engineering*, **17**(2), pp. 230-240 (2008).

28. Nobil, A.H., Nobil, E., and Cárdenas-Barrón, L.E. "Some observations to: lot sizing with non-zero setup times for rework", *International Journal of Applied and Computational Mathematics*, **3**(1), pp. 1511-1517 (2017).

29. Cárdenas-Barrón, L.E. "Economic production quantity with rework process at a single-stage manufacturing system with planned backorders", *Computers & Industrial Engineering*, **57**(3), pp. 1105-1113 (2009).

30. Hsu, J.T. and Hsu, L.F. "Two EPQ models with imperfect production processes, inspection errors, planned backorders, and sales returns", *Computers & Industrial Engineering*, **64**(1), pp. 389-402 (2013).

31. Farhangi, M., Niaki, S.T.A., and Vishkaei, B.M. "Closed-form equations for optimal lot sizing in deterministic EOQ models with exchangeable imperfect quality items", *Scientia Iranica, Transactions E, Industrial Engineering*, **22**(6), pp. 2621-2633 (2015).

32. Cárdenas-Barrón, L.E., Sarkar, B., and Treviño-Garza, G. "Easy and improved algorithms to joint determination of the replenishment lot size and number of shipments for an EPQ model with rework", *Mathematical and Computational Applications*, **18**(2), pp. 132-138 (2013).

33. Shah, N.H., Patel, D.G., and Shah, D.B. "EPQ model for returned/reworked inventories during imperfect production process under price-sensitive stock-dependent demand", *Operational Research*, pp. 1-17 (2016).

34. Jaggi, C.K., Tiwari, S., and Shafi, A. "Effect of deterioration on two-warehouse inventory model with imperfect quality", *Computers & Industrial Engineering*, **88**, pp. 378-385 (2015).

35. Jaggi, C.K., Cárdenas-Barrón, L.E., Tiwari, S., and Shafi, A.A. "Two-warehouse inventory model for deteriorating items with imperfect quality under the conditions of permissible delay in payments", *Scientia Iranica, Transactions E, Industrial Engineering*, **24**(1), pp. 390-412 (2017).

36. Rogers, J. "A computational approach to the economic lot-scheduling problem", *Management Science*, **4**(3), pp. 264-291 (1958).

37. Taleizadeh, A.A., Cárdenas-Barrón, L.E., and Mohammadi, B. "A deterministic multi product single machine EPQ model with backordering, scraped products, rework and interruption in manufacturing process", *International Journal of Production Economics*, **150**, pp. 9-27 (2014).

38. Pasandideh, S.H.R., Niaki, S.T.A., Nobil, A.H., and Cárdenas-Barrón, L.E. "A multiproduct single machine economic production quantity model for an imperfect production system under warehouse construction cost", *International Journal of Production Economics*, **169**, pp. 203-214 (2015).

39. Nobil, A.H., Sedigh, A.H.A., and Cárdenas-Barrón, L.E. "A multi-machine multi-product EMQ problem for an imperfect manufacturing system considering utilization and allocation decisions", *Expert Systems with Applications*, **56**, pp. 310-319 (2016).

40. Nobil, A.H. and Taleizadeh, A.A. "A single machine EPQ inventory model for a multi-product imperfect production system with rework process and auction", *International Journal of Advanced Logistics*, **5**(3-4), pp. 141-152 (2016).

41. Nobil, A.H., Sedigh, A.H.A., and Cárdenas-Barrón, L.E. "A multiproduct single machine economic production quantity (EPQ) inventory model with discrete delivery order, joint production policy and budget constraints", *Annals of Operations Research*, pp. 1-37 (2017). doi: 10.1007/s10479-017-2650-9

42. Nobil, A.H., Sedigh, A.H.A., and Cárdenas-Barrón, L.E. "Multi-machine economic production quantity for items with scrapped and rework with shortages and allocation decisions", *Scientia Iranica, Transactions E, Industrial Engineering*, **25**(4), pp. 2331-2346 (2018). doi:10.24200/sci.2017.4453

Appendix A

Calculating cycle length

Based on Eq. (18), the production cycle length is:

$$T = T_i = t_i^1 + G_i + t_i^2 + t_i^3 + \cdots + t_i^{n+1} + t_i^{n+2}$$

$$+ t_i^{n+3} + t_i^{n+4}. \tag{A.1}$$

Moreover, based on Eqs. (10) to (17), we have:

$$T = \frac{Q_i}{P_i} - \frac{S_i}{e_i} + G_i + \frac{v_i^1 Q_i}{\theta_i^1 P_i} + \frac{v_i^2 Q_i}{\theta_i^2 P_i} + \cdots + \frac{v_i^n Q_i}{\theta_i^n P_i}$$

$$+ \frac{e_i Q_i}{D_i P_i} - \frac{S_i}{D_i} - G_i + \left(\sum_{r=1}^{j} \frac{a_i^j v_i^j Q_i}{\theta_i^j D_i P_i} \right) + \frac{S_i}{D_i} + \frac{S_i}{e_i}$$

$$= \frac{Q_i}{P_i} + \frac{e_i Q_i}{D_i P_i} + \left(\sum_{r=1}^{j} \frac{v_i^j Q_i}{\theta_i^j P_i} \right) + \left(\sum_{r=1}^{j} \frac{a_i^j v_i^j Q_i}{\theta_i^j D_i P_i} \right)$$

$$= \frac{Q_i}{P_i} \left(1 + \frac{e_i}{D_i} \right) + \frac{Q_i}{P_i} \left(\sum_{r=1}^{j} \frac{v_i^j}{\theta_i^j} + \frac{a_i^j v_i^j}{\theta_i^j D_i} \right). \tag{A.2}$$

Employing Eqs. (3) and (4), the following holds:

$$T = \frac{Q_i}{P_i}\left(\frac{(1-\lambda_i)P_i}{D_i}\right) + \frac{Q_i}{P_i}\left(\sum_{r=1}^{j}\frac{v_i^j P_i}{D_i}\right)$$

$$= \frac{Q_i}{P_i}\left(\frac{(1-\lambda_i)P_i}{D_i} + \sum_{r=1}^{j}\frac{v_i^j P_i}{D_i}\right). \quad (A.3)$$

Substituting λ_i from Eq. (1) in Eq. (A.3), we have:

$$T = \frac{Q_i}{P_i}\left(\frac{(1-\lambda_i)P_i}{D_i} + \sum_{r=1}^{j}\frac{v_i^j P_i}{D_i}\right)$$

$$= \frac{Q_i}{P_i}\left(\frac{P_i}{D_i} - \frac{\alpha_i P_i}{D_i} - \sum_{r=1}^{j}\frac{v_i^j P_i}{D_i} + \sum_{r=1}^{j}\frac{v_i^j P_i}{D_i}\right)$$

$$= \frac{(1-\alpha_i)Q_i}{D_i}. \quad (A.4)$$

Appendix B

Calculating the holding cost

Based on Eq. (30), the production cycle length is:

$$HC = \sum_{i=1}^{m} Nh_i \left\{ \overbrace{\frac{K_i \times t_i^1}{2}}^{\text{Area 1}} + \overbrace{\frac{(K_i+H_i)G_i}{2}}^{\text{Area 2}} \right.$$
$$+ \overbrace{\frac{(H_i+I_i^1)t_i^2}{2}}^{\text{Area 3}} + \overbrace{\frac{(I_i^1+I_i^2)t_i^3}{2}}^{\text{Area 4}}$$
$$+ \overbrace{\frac{(I_i^2+I_i^3)t_i^4}{2}}^{\text{Area 5}} + \cdots$$
$$\left. + \overbrace{\frac{(I_i^{n-1}+I_i^n)t_i^{n+1}}{2}}^{\text{Area } n+2} + \overbrace{\frac{I_i^n \times t_i^{n+2}}{2}}^{\text{Area } n+3} \right\}. \quad (B.1)$$

Now, we calculate the specified areas in Eq. (B.1) to obtain a general relation for holding costs as follows:

$$\text{Area 1} = \frac{K_i \times t_i^1}{2} = \frac{1}{2}\left(\frac{e_i Q_i}{P_i} - S_i\right)\left(\frac{Q_i}{P_i} - \frac{S_i}{e_i}\right)$$

$$= \frac{e_i(Q_i)^2}{2(P_i)^2} - \frac{Q_i S_i}{P_i} + \frac{(S_i)^2}{2e_i}, \quad (B.2)$$

$$\text{Area 2} = \frac{(K_i+H_i)G_i}{2} = \frac{e_i G_i Q_i}{P_i} - G_i S_i - \frac{D_i(G_i)^2}{2}, \quad (B.3)$$

$$\text{Area 3} = \frac{(H_i+I_i^1)t_i^2}{2} = \frac{e_i(Q_i)^2}{2(P_i)^2}\left(\frac{v_i^1}{\theta_i^1}\right) - \frac{Q_i S_i}{P_i}\left(\frac{v_i^1}{\theta_i^1}\right)$$

$$-\frac{D_i G_i Q_i}{P_i}\left(\frac{v_i^1}{\theta_i^1}\right) + \frac{(Q_i)^2}{2(P_i)^2}\left(\frac{a_i^1(v_i^1)^2}{(\theta_i^1)^2}\right), \quad (B.4)$$

$$\text{Area 4} = \frac{(I_i^1+I_i^2)t_i^3}{2} = \frac{e_i(Q_i)^2}{2(P_i)^2}\left(\frac{v_i^2}{\theta_i^2}\right) - \frac{Q_i S_i}{P_i}\left(\frac{v_i^2}{\theta_i^2}\right)$$

$$-\frac{D_i G_i Q_i}{P_i}\left(\frac{v_i^2}{\theta_i^2}\right) + \frac{(Q_i)^2}{2(P_i)^2}\left(\frac{a_i^2(v_i^2)^2}{(\theta_i^2)^2}\right)$$

$$+ \frac{(Q_i)^2}{(P_i)^2}\left(\frac{a_i^1 v_i^1 v_i^2}{\theta_i^1 \theta_i^2}\right), \quad (B.5)$$

$$\text{Area 5} = \frac{(I_i^2+I_i^3)t_i^4}{2} = \frac{e_i(Q_i)^2}{2(P_i)^2}\left(\frac{v_i^3}{\theta_i^3}\right) - \frac{Q_i S_i}{P_i}\left(\frac{v_i^3}{\theta_i^3}\right)$$

$$-\frac{D_i G_i Q_i}{P_i}\left(\frac{v_i^3}{\theta_i^3}\right) + \frac{(Q_i)^2}{2(P_i)^2}\left(\frac{a_i^3(v_i^3)^2}{(\theta_i^3)^2}\right)$$

$$+ \frac{(Q_i)^2}{(P_i)^2}\left(\frac{a_i^1 v_i^1 v_i^3}{\theta_i^1 \theta_i^3}\right) + \frac{(Q_i)^2}{(P_i)^2}\left(\frac{a_i^2 v_i^2 v_i^3}{\theta_i^2 \theta_i^3}\right). \quad (B.6)$$

The areas (5) to $(n+2)$ can be stated as follows:

$$\text{Area } j+2 = \frac{(I_i^{j-1}+I_i^j)t_i^{j+1}}{2} = \frac{e_i(Q_i)^2}{2(P_i)^2}\left(\frac{v_i^j}{\theta_i^j}\right)$$

$$-\frac{Q_i S_i}{P_i}\left(\frac{v_i^j}{\theta_i^j}\right) - \frac{D_i G_i Q_i}{P_i}\left(\frac{v_i^j}{\theta_i^j}\right)$$

$$+ \frac{(Q_i)^2}{2(P_i)^2}\left(\frac{a_i^j(v_i^j)^2}{(\theta_i^j)^2}\right)$$

$$+ \frac{(Q_i)^2}{(P_i)^2}\left(\frac{v_i^j}{\theta_i^j}\right)\sum_{y=0}^{j-1}\left(\frac{a_i^y v_i^y}{\theta_i^y}\right),$$

$$j = 1, 2, \cdots, n. \quad (B.7)$$

On the other hand, we have:

$$\text{Area } n+3 = \frac{I_i^n \times t_i^{n+2}}{2} = \frac{I_i^n \times I_i^n}{2D_i}$$

$$= \frac{1}{2D_i}\left[\frac{e_i Q_i}{P_i} - S_i - G_i D_i \right.$$

$$\left. + \left(\sum_{r=1}^{j}\frac{a_i^j v_i^j Q_i}{\theta_i^j P_i}\right)\right]^2. \quad (B.8)$$

Finally, annual holding costs with respect to Eqs. (B.2) to (B.8) are as follows:

$$HC = \sum_{i=1}^{m}\frac{h_i}{T}\left\{\frac{e_i G_i Q_i}{P_i} + \frac{e_i(1-\lambda_i)(Q_i)^2}{2P_i D_i}\right.$$

$$+ \frac{P_i(1-\lambda_i)(S_i)^2}{2e_iD_i} - \frac{G_iS_i}{2} - \frac{e_iS_iQ_i}{2P_iD_i}$$

$$+ \frac{(Q_i)^2}{(P_i)^2} \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \left(\sum_{y=0}^{j-1} \frac{a_i^y v_i^y}{\theta_i^y} \right)$$

$$- \frac{G_iQ_i}{P_i} \left(D_i \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \sum_{j=1}^{n} \frac{a_i^j v_i^j}{2\theta_i^j} \right)$$

$$- \frac{S_iQ_i}{P_i} \left(\sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \frac{1}{2D_i} \sum_{j=1}^{n} \frac{a_i^j v_i^j}{\theta_i^j} \right)$$

$$+ \frac{e_i(Q_i)^2}{(P_i)^2} \left(\sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \frac{1}{2D_i} \sum_{j=1}^{n} \frac{a_i^j v_i^j}{\theta_i^j} \right)$$

$$+ \frac{(Q_i)^2}{2(P_i)^2} \left(\sum_{j=1}^{n} \frac{a_i^j (v_i^j)^2}{(\theta_i^j)^2} + \frac{1}{D_i} \sum_{j=1}^{n} \left(\frac{a_i^j v_i^j}{\theta_i^j} \right)^2 \right) \bigg\}. \tag{B.9}$$

Moreover, substituting Q_i from Eq. (20) in Eq. (B.9), we have:

$$HC = \frac{h_ie_iG_iD_i}{P_i(1-\alpha_i)} + \frac{h_ie_i(1-\lambda_i)}{2P_i(1-\alpha_i)^2}(T)$$

$$+ \frac{h_iP_i(1-\lambda_i)}{2e_iD_i} \left(\frac{(S_i)^2}{T} \right) - \frac{h_iG_i}{2} \left(\frac{S_i}{T} \right)$$

$$- \frac{h_ie_i}{2P_i(1-\alpha_i)}(S_i)$$

$$+ \frac{h_i(D_i)^2}{(P_i)^2(1-\alpha_i)^2} \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \left(\sum_{y=0}^{j-1} \frac{a_i^y v_i^y}{\theta_i^y} \right) (T)$$

$$- \frac{h_iG_iD_i}{P_i(1-\alpha_i)} \left(D_i \sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \sum_{j=1}^{n} \frac{a_i^j v_i^j}{2\theta_i^j} \right)$$

$$- \frac{h_iD_i}{P_i(1-\alpha_i)} \left(\sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \frac{1}{2D_i} \sum_{j=1}^{n} \frac{a_i^j v_i^j}{\theta_i^j} \right) (S_i)$$

$$+ \frac{h_ie_i(D_i)^2}{(P_i)^2(1-\alpha_i)^2} \left(\sum_{j=1}^{n} \frac{v_i^j}{\theta_i^j} + \frac{1}{2D_i} \sum_{j=1}^{n} \frac{a_i^j v_i^j}{\theta_i^j} \right) (T)$$

$$+ \frac{h_i(D_i)^2}{2(P_i)^2(1-\alpha_i)^2} \left(\sum_{j=1}^{n} \frac{a_i^j (v_i^j)^2}{(\theta_i^j)^2} \right.$$

$$\left. + \frac{1}{D_i} \sum_{j=1}^{n} \left(\frac{a_i^j v_i^j}{\theta_i^j} \right)^2 \right) (T). \tag{B.10}$$

Appendix C

Calculating production capacity

Based on Inequality (39), the production capacity is:

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n+1} t_i^j + t_i^{n+4} + ts_i + G_i \right) \leq T. \tag{C.1}$$

By employing Eqs. (10)-(14) and (17), the following holds:

$$\sum_{i=1}^{m} \left(\frac{Q_i}{P_i} - \frac{S_i}{e_i} + \sum_{j=1}^{n+1} \frac{v_i^j Q_i}{\theta_i^j P_i} + \frac{S_i}{e_i} + ts_i + G_i \right) \leq T. \tag{C.2}$$

Moreover, substituting Q_i from Eq. (20) in Eq. (C.2), we have:

$$\sum_{i=1}^{m} \left(\frac{D_iT}{(1-\alpha_i)P_i} + \sum_{j=1}^{n+1} \frac{v_i^j D_iT}{\theta_i^j(1-\alpha_i)P_i} + ts_i + G_i \right) \leq T. \tag{C.3}$$

Therefore:

$$T \geq \frac{\sum\limits_{i=1}^{m}(ts_i + G_i)}{1 - \sum\limits_{i=1}^{m} \left(\frac{D_i}{(1-\alpha_i)P_i} \left(1 + \sum\limits_{j=1}^{n} \frac{v_i^j}{\theta_i^j} \right) \right)}. \tag{C.4}$$

Appendix D

Calculating quadratic form of objective function (45)

Based on objective function (45):

$$TC = \sum_{i=1}^{m} \beta_i^1 + \sum_{i=1}^{m} \beta_i^2(T) + \sum_{i=1}^{m} \frac{\beta_i^3}{T}$$

$$- \sum_{i=1}^{m} \beta_i^4 \left(\frac{S_i}{T} \right) - \sum_{i=1}^{m} \beta_i^5 S_i + \sum_{i=1}^{m} \beta_i^6 \left(\frac{(S_i)^2}{T} \right). \tag{D.1}$$

The partial derivatives of the objective function are equal to:

$$\frac{\partial TC}{\partial T} = \sum_{i=1}^{m} \beta_i^2 - \sum_{i=1}^{m} \frac{\beta_i^3}{T^2} + \sum_{i=1}^{m} \beta_i^4 \left(\frac{S_i}{T^2} \right)$$

$$- \sum_{i=1}^{m} \beta_i^6 \left(\frac{(S_i)^2}{T^2} \right), \tag{D.2}$$

$$\frac{\partial^2 TC}{\partial^2 T} = \frac{2\sum\limits_{i=1}^{m} \beta_i^3 - 2\sum\limits_{i=1}^{m} \beta_i^4 S_i + 2\sum\limits_{i=1}^{m} \beta_i^6 (S_i)^2}{T^3}, \tag{D.3}$$

$$\frac{\partial TC}{\partial S_i} = -\frac{\beta_i^4}{T} + \beta_i^5 + \frac{2\beta_i^6 S_i}{T}, \tag{D.4}$$

$$\frac{\partial^2 TC}{\partial^2 S_i} = \frac{2\beta_i^6}{T}, \tag{D.5}$$

$$\frac{\partial^2 TC}{\partial T \partial S_i} = \frac{\partial^2 TC}{\partial S_i \partial T} = \frac{\beta_i^4 - 2\beta_i^6 S_i}{T^2}, \qquad (D.6)$$

$$\frac{\partial^2 TC}{\partial S_k \partial S_i} = \frac{\partial^2 TC}{\partial S_i \partial S_k} = 0. \qquad (D.7)$$

The Hessian matrix of the objective function equals:

$$\text{Hessian} = \begin{bmatrix} \frac{\partial^2 TC}{\partial^2 T} & \frac{\partial^2 TC}{\partial S_1 \partial T} & \frac{\partial^2 TC}{\partial S_2 \partial T} & \frac{\partial^2 TC}{\partial S_m \partial T} \\ \frac{\partial^2 TC}{\partial T \partial S_1} & \frac{\partial^2 TC}{\partial^2 S_1} & \frac{\partial^2 TC}{\partial S_2 \partial S_1} & \frac{\partial^2 TC}{\partial S_m \partial S_1} \\ \frac{\partial^2 TC}{\partial T \partial S_2} & \frac{\partial^2 TC}{\partial S_1 \partial S_2} & \frac{\partial^2 TC}{\partial^2 S_2} & \frac{\partial^2 TC}{\partial S_m \partial S_2} \\ \vdots & & & \vdots \\ \frac{\partial^2 TC}{\partial T \partial S_m} & \frac{\partial^2 TC}{\partial S_1 \partial S_m} & \frac{\partial^2 TC}{\partial S_2 \partial S_m} & \frac{\partial^2 TC}{\partial^2 S_m} \end{bmatrix}. \qquad (D.8)$$

Using (D.2) to (D.7), we have:

$$\text{Hessian} = \begin{bmatrix} \frac{2\sum_{i=1}^m \beta_i^3 - 2\sum_{i=1}^m \beta_i^4 S_i + 2\sum_{i=1}^m \beta_i^6 (S_i)^2}{T^3} & \frac{\beta_1^4 - 2\beta_1^6 S_1}{T^2} \\ \frac{\beta_1^4 - 2\beta_1^6 S_1}{T^2} & \frac{2\beta_1^6}{T} \\ \frac{\beta_2^4 - 2\beta_2^6 S_2}{T^2} & 0 \\ \vdots & \vdots \\ \frac{\beta_m^4 - 2\beta_m^6 S_m}{T^2} & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{\beta_2^4 - 2\beta_2^6 S_2}{T^2} & \cdots & \frac{\beta_m^4 - 2\beta_m^6 S_m}{T^2} \\ 0 & \cdots & 0 \\ \frac{2\beta_2^6}{T} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{2\beta_m^6}{T} \end{matrix} \Bigg]. \qquad (D.9)$$

The quadratic form of the objective function is equal to:

$$\text{Quadratic} = [T, S_1, S_2, \cdots, S_m] \, (\text{Hessian}) \begin{bmatrix} T \\ S_1 \\ S_2 \\ \vdots \\ S_m \end{bmatrix}. \qquad (D.10)$$

Substituting Hessian from Eq. (D.9) in Eq. (D.10), we have:

$$\text{Quadratic} = \frac{2\sum_{i=1}^m \beta_i^3}{T} \geq 0. \qquad (D.11)$$

Since $\beta_i^3 \geq 0$ for $i = 1, 2, \cdots, m$, the quadratic form of the objective function is semi-positive.

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