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Finding an improved region of efficiency via DEA-efficient hyperplanes

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Abstract. The analysis of efficiency is conducted for two vital purposes: firstly, in order to evaluate the current level of efficiency; secondly, to provide information on how to improve the level of efficiency, which is to provide benchmarking information. The inefficient Decision Making Units (DMUs) are usually able to improve their performance, and Data Envelopment Analysis (DEA) projections provide a prescription for improvement. However, sometimes, an inefficient DMU cannot move its performance toward the best practice by either decreasing its inputs or increasing its outputs. On the other hand, it can scarcely reach its efficient benchmark. This research suggests a method to find an improved region of efficiency through DEA-efficient hyperplanes by providing an algorithm for detecting an improved efficiency path. In addition to the production of reasonable benchmarking information, the proposed algorithm provides the general requirements that satisfy the demands which every professional decision-maker should meet. Finally, we provide a more detailed description of some new issues, extending the insights from this analysis of the benchmark region from the under-evaluated inefficient DMU. Finally, numerical examples are provided to demonstrate the results of the analysis.

1. Introduction

The traditional Data Envelopment Analysis, as initially proposed by Charnes et al. [1], is a relatively new "data-oriented" approach to assessing the performance of a set of DMUs that use multiple inputs to produce multiple outputs. Most DEA approaches, such as the approach introduced by [2], focus on relative efficiencies.

Recently, some researchers have studied the performance improvement in DEA to meet the decision-maker's goals [3-6]. So many DEA-efficiency improvement papers have certain points of view such as the modified formulation of the DEA model where bounds are imposed on each factor; this model was proposed by Kao [7]. Another model was introduced by Chen et al. [8] that modified the original DEA model by decomposing the normalizing equation for being associated with the different dual variables. In addition, by means of the equation of defining hyperplanes of PPS, one can find the region of efficiency improvement. In 2011, Khoshfetrat and Daneshvar [9], in a paper, proposed improving the weak efficiency frontiers in fuzzy DEA models, where they implicitly dealt with the knowledge of the sensitivity analysis in depth.

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This may indeed show the importance of obtaining the defining hyperplanes of the PPS.

The studies of efficient frontiers and characterization of different types of hyperplanes have been less intensively investigated [10,11]. The connotation of the efficient frontier DMUs in the DEA was studied by Jahanshahloo et al. [12]; this study found the piecewise linear frontier of the production function. In the aforementioned study, the researchers developed a way to obtain efficient frontier using 0-1 integer programming. Despite the greater computational efforts required to solve the 0-1 integer programming for small n (number of DMUs), solving the 0-1 model for large n is preferable to solving n linear programs; these preference relationships can be derived as strengths. In addition, searching for the efficient frontier in the DEA was considered by Korhonen [13]. Their approach was developed to create a free search for the efficient frontier in Multiple-Objective Linear Programming (MOLP) MOLP-this can also be used in DEA. Furthermore, Jahanshahloo et al. [14] suggested an approach to finding strong defining hyperplanes of the PPS in DEA, in which the observed deficiency of an alternative optimal solution of a multiplier form has been removed. In addition, the construction of all DEA-efficient frontiers in Generalized Data Envelopment Analysis (GDEA) was discussed by Yu and Zeleny [15]. In their work, they derived a necessary and sufficient condition for a point to be nondominated. It provided a simple proof confirming that the set of all nondominated extreme points is indeed connected. They then obtained a decomposition theorem and some necessary and sufficient conditions for a face to be nondominated. Lotfi et al. [16] provided an algorithm in order to find efficient hyperplanes with the Variable Returns to Scale (VRS) assumption of technology by using the MOLP structure. They characterized the efficient hyperplanes that make up the DEA efficient frontier only by finding the efficient faces of the MOLP problem without exploring its extreme efficient points. Sensitivity and stability analysis of the specific under-evaluated DMU is one of the areas of interest in DEA. In 1985, Charnes et al. [17] initiated the sensitivity analysis of a CCR model for a specific efficient DMU with a single output. They constructed variations in the data for the DMU under consideration and initiated the change in the inverse matrix used to produce solutions in typical simplex algorithm computer codes. Then, in 2015, Forghani et al. considered sensitivity analysis in the two-stage DEA, and found that the necessary and sufficient conditions for preserving a DMU's efficiency classification were developed when various data changes were applied to all the DMUs [18]. Many more papers have studied sensitivity analysis with various data types such as integer data type [3], fuzzy data [19], etc. Another model proposed by Huang et al. [20], later generalized by Jahanshahloo et al. [21], showed that all the efficient surfaces of the BCC model passing through DMU_P were generated for the case; in this model, the largest stability region of the under-evaluated extreme efficient DMU was constructed by the omission of DMU_P from the observations set. Later, the largest stability region proposed in their paper was improved by [22]. Today, many algorithms have been proposed to find the efficient region of a specified DMU simply specified by the concept of all the possible values that remain or become efficient [23,24].

DEA also provides benchmarking information in order to improve the efficiency of the DMU. This information distinguishes DEA from other efficiency methodologies. The reference points that are used for obtaining the benchmarking information are vital not only for evaluating the potential performance of the DMU, but also for providing information on how to improve its performance, as observed by Bogetoft and Hougaard [5]. Inappropriate selection of reference points might lead to misleading guidance on how to improve efficiency. The DEA model was introduced by Farrell [25], and later developed by Charnes et al. [1]; it uses an oriented radial measure of efficiency for the identification of reference points on the boundary of the PPS with the input orientation or the output orientation of the observed unit. The main weakness of radial efficiency measures is that they disregard the possible existence of slacks associated with the projected points on the production frontier (see the following examples: [9,19,23,26]). Many practical studies have used non-oriented measures; for example, Colwell and Davis [27] used an intermediation approach that specified inputs in the form of costs and outputs in the form of revenue. Finding the closest target has been considered for both oriented models (see the following examples: [28-30]) and non-oriented models (see the following examples: [31-35]). The only question that most DEA studies focus on is "how is a company inefficient?"; the question of "how can a company become efficient?" is the second case for review. Interactive DEA (IDEA) usage was suggested by Post and Spronk [32] in order to make the benchmarking information sensible from the inefficient DMU's viewpoint. In addition, for obtaining the most relevant and easily attainable benchmarking information, the least-distance measure was proposed by Baek and Lee [36], which provides a well-defined efficiency measure, too. To answer the second question, this study focuses on the best rational way for an inefficient DMU to meet the DM's needs, whether it is still inefficient or not. On the flip side, the simplistic claim that the region of efficiency improvement is more relevant than finding their efficiency is really misleading. Applying the results of [16,20-22,27,38], the current paper aims

to obtain the improved region of efficiency by using all the defining hyperplanes of PPS. Now, there is a need to evaluate the use of DEA as a decision aid for considering the improved region of efficiency based on the overall value judgment; this is clarified in [39]. The value judgment here can, therefore, be expressed in our new proposed algorithm to describe the benchmarking region improvement concept. By means of the new algorithm, the improvement path for an inefficient DMU based on value judgment is thereafter identified. In measuring the improvement region, the incorporation of value judgment is, therefore, also very important. The remaining parts of this paper will be unfolded as follows. To explain the new benchmarking region, we start with a description of obtaining the defining hyperplanes of PPS in Section 2, which briefly reviews a mathematical basis used for this study. In Section 3, we illustrate our proposed algorithm to obtain the improvement of the benchmarking region. In Section 4, three numerical examples are documented for intuitively describing the new algorithm in order to find an improved region of efficiency. Section 5 provides our concluding remarks and future directions for research.

2. Preliminaries

In this section, a brief description of the BCC input-oriented model and the approaches to finding efficient hyperplanes are included.

2.1. The BCC input-oriented model

The DEA-BCC model was proposed for dealing with situations whose proportionality between the inputs and outputs is not constant along the efficient frontier. The BCC model [4] was initially developed by adding a convexity restriction to the CCR [1] model envelope formulation. This generated a VRS frontier. Suppose that we have n DMUs, where each DMU$_j$($j = 1, 2, \cdots, n$) produces the same s outputs in (possibly) different amounts, y_{rj} $(r = 1, 2, \cdots, s)$, using the same m inputs, and x_{ij} $(i = 1, 2, \cdots, m)$ in (possibly) different amounts. The BCC input-oriented model evaluates the efficiency of DMU$_o$, the DMU under consideration, by solving the following "multiplier linear form" program:

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{ro} - u_o, \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{io} - u_o \le 0, \\
& j = 1, \cdots, n, \qquad u_o, \text{ free}, \\
& v_i, u_r \ge \varepsilon > 0, \\
& i = 1, 2, \cdots, m, \qquad r = 1, 2, \cdots, s,
\end{aligned} \tag{1}$$

where u_i and v_i are the decision variables, respectively, that are associated with output r and input i, and ε is a positive non-Archimedean infinitesimal.

In addition to the efficiency score, the most practical information given by the DEA study is the set of Pareto-efficient projection points for the under-evaluated DMU. The coordination of a projection point can be interpreted as the "target" levels of performance of the inputs and outputs. The obtained targets provide an indication of how an assessed DMU improves its performance efficiently. Therefore, the closer the assessed DMU is to the targets, the lesser practical effort it needs to be efficient. The basic DEA model in [1], which is called the CCR model, has led to several extensions, most notably the BCC model of [4]. Assume that there are n DMU$_s$ (DMU$_j$: $j = 1, 2, \cdots, n$) that consume $X_j = (x_{1j}, \cdots, x_{mj})$ to produce $Y_j = (y_{1j}, \cdots, y_{sj})$. The BCC input-oriented (BCC-I) model evaluates the efficiency of DMU$_o$ and the considered DMU by solving the following linear program:

$$\begin{aligned}
\min \quad & \theta, \\
\text{s.t.} \quad & \sum_{j} \lambda_j x_{ij} \le \theta x_{io}; \qquad \forall\, i, \\
& \sum_{j} \lambda_j y_{rj} \ge y_{ro}; \qquad \forall\, r, \\
& \sum_{j} \lambda_j = 1; \qquad \forall\, j, \\
& \lambda_j \ge 0; \qquad \forall\, j,
\end{aligned} \tag{2}$$

where θ^* is the optimal solution obtained from Eq. (2).

Definition 1: $(\sum_{j=1}^{n} \lambda_j^* x_j, \sum_{j=1}^{n} \lambda_j^* y_j) = (\theta^* x_o, y_o)$ is the input-oriented projection point of DMU$_o$ on the efficient frontier.

2.2. Finding efficient hyperplanes

A method for generating all the efficient surfaces of the BCC model passing through DMU$_p$ was suggested by [20,38] and improved by [22], which are as follows:

$$\begin{aligned}
\min \quad & u_o, \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} = u_o,
\end{aligned}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq u_o,$$

$$j = 1, \cdots, n,$$

$$\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1,$$

$$v_i \geq \varepsilon, \quad u_r \geq \varepsilon,$$

$$i = 1, 2, \cdots, m, \qquad r = 1, 2, \cdots, s, \tag{3}$$

$$\begin{aligned}
\max \quad & u_o, \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} = u_o, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq u_o, \\
& j = 1, 2, \cdots, n, \\
& \sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1, \\
& v_i \geq \varepsilon, \quad u_r \geq \varepsilon, \\
& i = 1, 2, \cdots, m, \qquad r = 1, 2, \cdots, s.
\end{aligned} \tag{4}$$

Note that in an optimal solution to Relations (3) and (4), all of those observed efficient points that satisfy their respective constraints as equalities also lie on the efficient facet contained in the hyperplane passing through $(x_p,\ y_p)$. To conduct sensitivity analysis of each efficient unit called DMU_p, Jahanshahloo et al. [21] used defining hyperplanes that are binding at DMU_p and the new frontier, which is formed by the omission DMU_p from the observations set.

Moreover, the MOLP problem suggested by [16] for finding efficient hyperplanes with the VRS assumption of technology in DEA is as follows:

$$\begin{aligned}
\max \quad & \{UY_1 - VX_1 - u_o, UY_2 - VX_2, \\
& \qquad - u_o, \cdots, UY_n - VX_n - u_o\}, \\
\text{s.t.} \quad & UY_j - VX_j - u_o \leq 0, \qquad j = 1, \cdots, n, \\
& 1U + 1V = 1, \\
& V \geq 1\varepsilon, \quad U \geq 1\varepsilon.
\end{aligned} \tag{5}$$

Changing the inequality constraints of Model (5) to equalities and minimizing the objective function of the converted model are as follows:

$$\begin{aligned}
\min \quad & \{S_1, S_2, \cdots, S_n\}, \\
\text{s.t.} \quad & UY_j - VX_j - u_o + S_j = 0, \qquad j = 1, \cdots, n, \\
& 1U + 1V = 1, \\
& S_j \geq 0, \qquad j = 1, \cdots, n, \\
& V \geq 1\varepsilon, \quad U \geq 1\varepsilon.
\end{aligned} \tag{6}$$

Assume that $J = \{1, \cdots, n\}$ and $M = \{I | I \subseteq J\}$. The feasible region of Model (6) is denoted by W For $I \in M$; $F(I)$ represents a face of W [15]. Note that $F(\emptyset) = W$; for $I \in M$, $F(I) = \emptyset$ is possible. Model (SP_I), which is similar to the model proposed by [37], is as follows:

$$\begin{aligned}
Z_I = \max \quad & \sum_{j=1}^{n} T_j - \sum_{j=1}^{n} S_j, \\
\text{s.t.} \quad & UY_j - VX_j - u_o + S_j = 0, \\
& j = 1, \cdots, n, \\
& U'Y_j - V'X_j - u'_o + T_j = 0, \\
& j = 1, \cdots, n, \\
& S_j - T_j \leq 0, \qquad j = 1, \cdots, n, \\
& T_j = 0, \qquad j \in I, \\
& 1U + 1V = 1, \\
& 1U' + 1V' = 1, \\
& S_j \geq 0, \quad T_j \geq 0, \qquad j = 1, \cdots, n, \\
& U \geq 1\varepsilon, \qquad U' \geq 1\varepsilon, \\
& V \geq 1\varepsilon, \qquad V' \geq 1\varepsilon.
\end{aligned} \tag{7}$$

The algorithm for finding efficient hyperplanes with the VRS technology in DEA, which was proposed by Hosseinzadeh et al. [16], is as follows:

Step 1. Let $k = 1$, $F^k = \emptyset$, $EF^k = \emptyset$, and $J = \{1, 2, \cdots, n\}$. Subscripted sets F^k and EF^k are the set of DMUs on the faces of the PPS and the set of subscripts of the DMUs on efficient faces of the PPS, respectively. With the VRS assumption of technology, dictating that both sets of DMUs contain k-element sets, proceed to Step 2;

Step 2. Evaluate $M^k = \{I \subseteq J \| I| = k\}$. If $|J| < k$ and $k = n + 1$, then stop. Otherwise, go to Step 3;

Step 3. If $M^k \neq \emptyset$, select $I \in M^k$; then, proceed to Step 4. Otherwise, go to Step 5;

Step 4. Solve SP_I and set $M^k = M^k \backslash I$. If $Z_I > 0$, then $F^k = F^k \cup I$ and proceed to Step 3. If $Z_I = 0$, then $F^k = F^k \cup I$ and $EF^k = EF^k \cup I$. Suppose that $(U^*, V^*, u_o^*)^k$ is the optimal solution of SP_I as the gradients of the efficient hyperplanes. If SP_I is infeasible, proceed to Step 3;

Step 5. Let $k = k + 1$, $F^k = \emptyset$, $EF^k = \emptyset$, and $\cup_{I \in F^k} I = J$. Proceed to Step 2.

Owing to the finite number of DMUs and the feasibility of linear programming problem SP_I, the iterative proposed algorithm is guaranteed to converge.

Another approach to identifying the equations of the weak defining hyperplanes of the PPS of the BCC model was proposed by Jahanshahloo et al. [38]. The summary of this identification is as follows:

Step 1. Considering each DMU_k $(k \in J)$ evaluate the following models:

$$\begin{aligned}
\min \quad & \theta_l^k - \varepsilon \left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+ \right), \\
\text{s.t.} \quad & \sum_{j \in J-\{k\}} \lambda_j^k x_{lj} + S_l^- = \theta_l^k x_{lk}, \\
& \sum_{j \in J-\{k\}} \lambda_j^k x_{ij} + S_i^- = x_{ik}, \\
& i = 1, \cdots, m, \quad i \neq l, \\
& \sum_{j \in J-\{k\}} \lambda_j^k y_{rj} - S_r^+ = y_{rk}, \\
& r = 1, \cdots, s, \\
& \sum_{j \in J-\{k\}} \lambda_j^k = 1, \\
& \lambda_j^k \geq 0 \qquad (j \in J = \{1, \cdots, n\}) - \{k\}, \\
& S_i^- \geq 0, \quad S_l^- \geq 0, \qquad r = 1, \cdots, s, \\
& l = 1, \cdots, m, \\
& S_r^+ \geq 0 \qquad i = 1, \cdots, m, \quad \theta_l^k \text{ free.}
\end{aligned} \tag{8}$$

$$\begin{aligned}
\max \quad & \varphi_q^k + \varepsilon \left(\sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+ \right), \\
\text{s.t.} \quad & \sum_{j \in J-\{k\}} \mu_j^k x_{ij} + t_i^- = x_{ik}, \qquad i = 1, \cdots, m, \\
& \sum_{j \in J-\{k\}} \mu_j^k y_{qj} - t_q^- = \varphi_q^k y_{qk}, \\
& \sum_{j \in J-\{k\}} \mu_j^k y_{rj} - t_r^+ = y_{rk}, \\
& r = 1, \cdots, s, \quad r \neq q, \\
& \sum_{j \in J-\{k\}} \mu_j^k = 1, \\
& \mu_j^k \geq 0, \qquad j \in J - \{k\}, \\
& t_i^- \geq 0, \qquad i = 1, \cdots, m, \\
& t_r^+ \geq 0, \qquad r = 1, \cdots, s, \quad q = 1, \cdots, s, \\
& \varphi_q^k \text{ free.}
\end{aligned} \tag{9}$$

in which ε is a non-Archimedean number;

Step 2. Form two sets I_l and O_q. Corresponding with any $\text{DMU}_j \in I_l$ (or O_q), form the weak efficient virtual, $\text{DMU}_j^l(\text{DMU}_j^{m+q})$, as mentioned previously. Put the indices of the extreme efficient and weak efficient virtual DMUs in F. Let $|F| = L$;

Step 3. For each $p, q \in F$ in which $p \neq q$, evaluate $\text{DMU}_k = \frac{1}{2}\text{DMU}_p + \frac{1}{2}\text{DMU}_q$; if it is efficient, then $p \in F_q$ and $q \in F_p$;

Step 4. For each $(j = 1, \cdots, L)$, $\bar{F}_j = F - F_j$;

Step 5. Choose the arbitrary $m + s$ members of F so that none of them belongs to another $\bar{F}$. Call this set $D = \{j_1 . \cdots . j_{m+s}\}$. Using D, a hyperplane can be constructed as follows:

$$\begin{vmatrix}
x_1 - x_{1j_1} & \cdots & x_m - x_{mj_1} \\
x_{1j_2} - x_{1j_1} & \cdots & x_{mj_2} - x_{mj_1} \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
x_{1j_{m+s}} - x_{1j_1} & \cdots & x_{mj_{m+s}} - x_{mj_1}
\end{vmatrix}$$

$$\begin{vmatrix}
y_1 - y_{1j_1} & \cdots & y_s - y_{sj_1} \\
y_{1j_2} - y_{1j_1} & \cdots & y_{sj_2} - y_{sj_1} \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
y_{1j_{m+s}} - y_{1j_1} & \cdots & y_{sj_{m+s}} - y_{sj_1}
\end{vmatrix} = 0,$$

where $x_1, \cdots, x_m$, $y_1, \cdots, y_s$ are variables, x_{pj_t} $(p = 1 \cdots, m; t = 1, \cdots, m + s)$ is the pth input of the DMU_{j_t}, and $y_{qj_t} (q = 1, \cdots, s; t = 1, \cdots, m + s)$ is the qth output of DMU_{j_t}. Suppose that the equation

of the afore-mentioned hyperplane is in the form of $P^t + \alpha = 0$, where $z = (x_1, \cdots, x_m, y_1, \vdots, y_s)$, P is the gradient vector of the hyperplane, and α is a scalar;

Step 6. If P has any component that is less than or equal to zero, proceed to Step 8; otherwise, let $w = (x_1^w, \cdots, x_m^w, y_1^w, \cdots, y_s^w)$ be defined as follows:

$$x_i^w = \max\{x_{ij} | j = 1, \cdots, n\}, \qquad i = 1, \cdots, m,$$

$$y_r^w = \min\{y_{rj} | j = 1, \cdots, n\}, \qquad r = 1, \cdots, s.$$

If:

$$P^t z_j + \alpha = 0, \qquad j \in D,$$

$$P^t z_j + \alpha \leq 0, \qquad j \in F - D,$$

$$P^t w + \alpha < 0,$$

then $P^t + \alpha = 0$ is supporting; otherwise, proceed to Step 8.

Step 7. If at least one of $m + s$ members of D is a weak efficient virtual DMU, then $P^t + \alpha = 0$ is a weak defining hyperplane. Otherwise, it is a strong defining hyperplane;

Step 8. If another subset of F with $m + s$ members can be found, go to Step 5; otherwise, stop.

3. The proposed algorithm

The proposed algorithm for finding the benchmarking region of efficiency derived by α percentage increase in the pre-efficiency value of the under-evaluated inefficient DMU with the VRS technology in the DEA is as follows:

In order to incorporate such a priori information into the process of our algorithm, we first specify α percentage related to the under-evaluated inefficient DMU based on value judgment.

The benchmarking region of an efficiency improvement for an inefficient DMU, RE_A, is just defined by the following equation:

$$RE_A = \left\{ \begin{pmatrix} X_{iA}^{*T} \\ \\ Y_{rA}^{*T} \end{pmatrix}^T \Big| X_{iA}^{*T} \leq X_{iA}^T, Y_{rA}^{*T} \geq Y_{rA}^T \right\}.$$

The first step of this method is to find all the defining (efficient) hyperplanes of the PPS. These hyperplanes can be found by using the algorithm proposed in [2,3,14,18,20,22,40,41] (one of these papers can be applied to find the defining hyperplanes of the PPS). To check the accuracy of all these defining hyperplanes, however, we compared all of these approaches, which resulted in the same defining hyperplane (see Examples 1 and 2 in Section 4). Let $P = \{p_1, \cdots, p_2, p_t\}$ be the set of these defining (efficient) hyperplanes. Then, consider DMU_A as an inefficient DMU under evaluation to find the new benchmarking region by increasing the efficiency (obtained from Model (2)) of DMU_A by α percentage. Ω_1 is considered as a set of the extreme efficient DMUs from the PPS. Assume that $k = 1$, and by setting $\overline{\text{DMU}}_A^{(k)}$ as a projection point of DMU_A using Model (2), p_k represents a hyperplane in which the projection point satisfied hyperplane p_k equation.

It is, however, to be noted that if $\overline{\text{DMU}}_A^{(k)}$ satisfies multiple hyperplanes–it is laid on the intersection of hyperplanes–we should follow the rest of the process for all of them. Following this, from $\overline{\text{DMU}}_A^{(k)}$, we will consider $\widetilde{\text{DMU}}_A^{(k)}$ whose efficiency is $\beta_A^* = \theta_A^* + \alpha\%$. To calculate the input/output amounts of $\widetilde{\text{DMU}}_A^{(k)}$, use the following formula:

$$\widetilde{\text{DMU}}_A^{(k)} = \left(\tilde{x}_A^{(k)}, \tilde{y}_A^{(k)}\right) = \left(\frac{\bar{x}_A^{(k)}}{\beta_A^*}, y_A^{(k)}\right) = \left(\frac{\theta_A^* x_A^{(k)}}{\beta_A^*}, y_A^{(k)}\right). \tag{10}$$

Suppose that P_k' is the hyperplane that passes through $\widetilde{\text{DMU}}_A^{(k)}$ and has the same normal vector as hyperplane p_k. By defining Ω_1, we find the extreme efficient DMUs that lie on hyperplane p_k so that at least one of its inputs or outputs is more than $\overline{\text{DMU}}_A^{(k)}$ (in case of the existence of more than one DMU, follow the same process for all of them). Whether such a DMU exists or not, the following two case studies are considered:

Case 1. If such a DMU exists, consider the new DMU as $\overline{\text{DMU}}_A^{(k+1)}$ subscript. Now, we must consider whether $\overline{\text{DMU}}_A^{(k+1)}$ belongs to RE_A or not:

i) If $\overline{\text{DMU}}_A^{(k+1)}$ belongs to RE_A, suppose that hyperplane p_{k+1} represents a defining efficient hyperplane in which $\overline{\text{DMU}}_A^{(k+1)}$ satisfies hyperplane p_{k+1} Equation. Consider $\widetilde{\text{DMU}}_A^{(k+1)}$, whose efficiency is $\beta_A^* = \theta_A^* + \alpha\%$. To calculate the input/output amounts of $\widetilde{\text{DMU}}_A^{(k+1)}$, use the following formula:

$$\widetilde{\text{DNU}}_A^{(k+1)} = \left(\tilde{x}_A^{(k+1)}, \tilde{y}_A^{(k+1)}\right) = \left(\frac{x_A^{(k+1)}}{\beta_A^*}, y_A^{(k+1)}\right). \tag{11}$$

Here, we confront two cases as follows:

a) If $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ belongs to RE_A, suppose that hyperplane P'_{k+1} is the equation which passes through $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ and has the same normal vector as hyperplane p_{k+1}; hence, start the process from the beginning;

b) If $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ does not belong to RE_A, suppose that hyperplane P'_{k+1} is the equation which passes through $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ and has the same normal vector as hyperplane p_{k+1}; then, to find a point on hyperplane P'_{k+1}, use the following proposed model:

$$\begin{aligned}
\max \quad & \varphi+\varepsilon(1s^{+}+1s^{-}),\\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_j = x_A - s^{-},\\
& \sum_{j=1}^{n}\lambda_j y_j = \varphi y_A + s^{+},\\
& \sum_{j=1}^{n}\lambda_j = 1,\\
& \bar{U}\left(\varphi y_A+s^{+}\right)-\bar{V}\left(x_A-s^{-}\right)+\bar{u}_o=0,\\
& \lambda_j \geq 0, \qquad (12)
\end{aligned}$$

where $\bar{U}Y - \bar{V}X + \bar{u}_o = 0$ is the equation of hyperplane P'_{k+1}.

Theorem 1. The efficiency score of each transformed point of the extreme efficient DMUs is β_A^*.

Proof. Assume $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ with coordinates:

$$\left(x_{\widetilde{\mathrm{DMU}}_A^{(k+1)}}, y_{\widetilde{\mathrm{DMU}}_A^{(k+1)}}\right)=\left(\frac{x_A^{(k+1)}}{\beta_A^*}, y_A^{(k+1)}\right),$$

as an arbitrary transformed extreme efficient DMU that belongs to the virtual improved frontiers. Considering $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ with the BCC input-oriented model, we obtain:

$$\begin{aligned}
\min \quad & \theta_{\widetilde{DMU}_A^{(K+1)}},\\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_j \leq \theta_{\widetilde{\mathrm{DMU}}_A^{(K+1)}} x_{\widetilde{\mathrm{DMU}}_A^{(K+1)}}\\
& \qquad = \theta_{\widetilde{\mathrm{DMU}}_A^{(K+1)}}\left(\left(\frac{1}{\beta_A^*}\right)x_A^{(K+1)}\right),\\
& \sum_{j=1}^{n}\lambda_j y_j \geq y_{\widetilde{\mathrm{DMU}}_A^{(K+1)}} = y_A^{(K+1)},\\
& \sum_{j=1}^{n}\lambda_j = 1,\\
& \lambda_j \geq 0, \qquad (13)
\end{aligned}$$

A feasible solution of this model can be:

$(\theta_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} = \beta_A^*, \quad \lambda_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} = 1,$

$\lambda_j = 0 \quad (j \neq \widetilde{\mathrm{DMU}}_A^{(k+1)}))$.

Then, the optimal $\theta_{\widetilde{\mathrm{DMU}}_A^{(k+1)}}$, i.e., $\theta^*_{\widetilde{\mathrm{DMU}}_A^{(k+1)}}$, is not greater than β_A^*.$(\theta^*_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} \leq \beta_A^*)$.

It will be sufficient to show that $(\theta^*_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} \not< \beta_A^*)$.

By the contradictory assumption, suppose that $(\theta^*_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} < \beta_A^*)$. Hence, $\theta^*_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} = \beta_A^* - \varepsilon$ for some $\varepsilon > 0$. By evaluating Model (13) with this assumption, we get:

$$\begin{cases}
\sum_{j=1}^{n}\lambda_j x_j \leq \theta_{\widetilde{\mathrm{DMU}}_A^{(K+1)}} x_{\widetilde{\mathrm{DMU}}_A^{(K+1)}}\\
\qquad = \theta^*_{\widetilde{\mathrm{DMU}}_A^{(K+1)}}\left(\left(\frac{1}{\beta_A^*}\right)x_A^{(K+1)}\right)\\
\qquad = \left(\frac{\beta_A^*-\varepsilon}{\beta_A^*}\right)x_A^{(K+1)}\\
\qquad = \left(1-\frac{\varepsilon}{\beta_A^*}\right)x_A^{(K+1)}\\
\sum_{j=1}^{n}\lambda_j y_j \geq y_{\widetilde{\mathrm{DMU}}_A^{(K+1)}} = y_A^{(K+1)}\\
\sum_{j=1}^{n}\lambda_j = 1\\
\lambda_j \geq 0,
\end{cases}$$

It is obvious that $(1-\frac{\varepsilon}{\beta_A^*}) < 1$. In this manner, a feasible solution for $\mathrm{DMU}_A^{(k+1)}$ is:

$$\theta_{\mathrm{DMU}_A^{(k+1)}} = (1-\frac{\varepsilon}{\beta_A^*}) < 1, \qquad \lambda_{\mathrm{DMU}_A^{(k+1)}} = 1,$$

$\lambda_j = 0, \quad (j \neq \mathrm{DMU}_A^{(k+1)})$.

Consequently, $\mathrm{DMU}_A^{(k+1)}$ is inefficient and is in contrary to the assumption.

Therefore, we have $\theta_{\widetilde{\mathrm{DMU}}_A^{(k+1)}} = \beta_A^*$, thus completing the proof. □

Let an optimal solution of Model (12) be $(\varphi^*, s^{-*}, s^{+*})$. Based on the following theorems, this model gives us a point that firstly belongs to RE_A; secondly, this reference point of DMU_A, i.e., $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$, dominates DMU_A and belongs to hyperplane P'_{k+1}.

Theorem 2. Given Model (12), we have:

a) $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$ is a dominating DMU_A and, thus, $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$ also belongs to RE_A;

b) $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$ lies on hyperplane P'_{k+1};

c) Model (12) is feasible.

Proof.

(a) Since $s^{-*} \geq 0$, $s^{+*} \geq 0$, and $\varphi^* \geq 0$, then we have $(x_A - s^{-*} \leq x_A)$ and $(\varphi^* y_A + s^{+*} \geq y_A)$.

Considering DMU_A as an inefficient unit, the optimal solution of Model (12) is expressed as $(\varphi^* > 1)$ or $[(s^{-*}.s^{+*}) \neq 0]$. Then, we have:

$$\begin{pmatrix} -(x_A - s^{-*}) \\ \varphi^* y_A + s^{+*} \end{pmatrix} \gneqq \begin{pmatrix} -x_A \\ y_A \end{pmatrix},$$

which means that $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$ is a dominating DMU_A; thus, $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$ also belongs to REA;

(b) Evidently, based on the fourth constraint of Model (12), the obtained reference point from Model (12), i.e., point $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$, lies on hyperplane P'_{k+1}: $\bar{U}Y - \bar{V}Y + \bar{u}_o = 0$;

(c) Considering that the extreme points used in the proposed algorithm belong to RE_A, if the extreme DMU does not belong to RE_A in one step, the new obtained point from Model (12) will be in RE_A in the next step of the algorithm (see Section b in Case 1). Thus, each transformed point of the extreme efficient DMUs will belong to RE_A (if the transformed point of the extreme DMU does not belong to RE_A in one step, then the new obtained point from Model (12) will be in RE_A in the next step of the algorithm (see Section ii) in Case 1)).

According to the algorithm procedure, the new improved facets (the new virtual frontiers) are obtained from the convex hull of these points. Besides, PPS is convex and, thus, the convex hull of these points belongs to RE_A; consequently, the new improved facets also belong to RE_A.

This illustrates not only the fact that the fourth constraint belongs to RE_A, but also that the reference point of DMU_A, i.e., $(x_A - s^{-*}, \varphi^* y_A + s^{+*})$, belongs to RE_A (based on Part A of Theorem 2). The model is thus always feasible. □

Replace the new point with $\widetilde{\text{DMU}}_A^{(k+1)}$ by connecting DMU_A to the new point $\widetilde{\text{DMU}}_A^{(k+1)}$, which lies on hyperplane P'_{k+1} (see Figure 1); this process is finished;

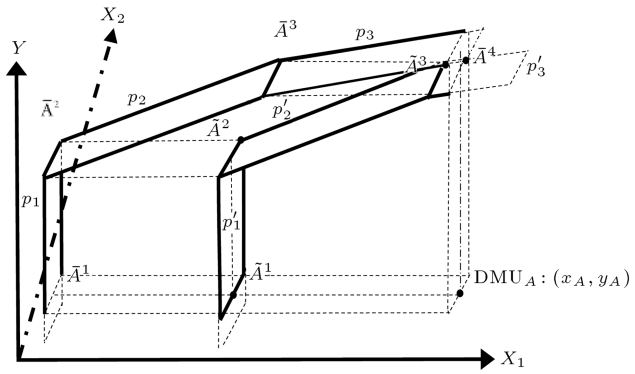

Figure 1. The benchmarking region of efficiency (Case 1).

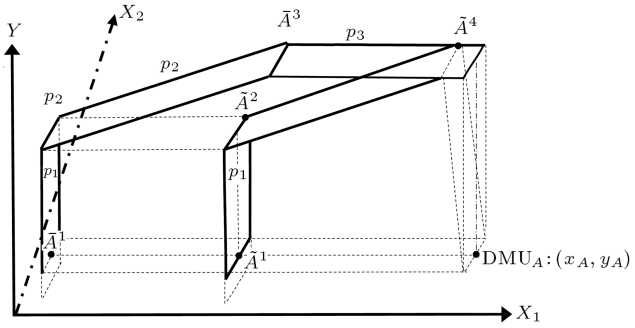

Figure 2. The benchmarking region of efficiency (Case 2. $\widetilde{\text{DMU}}_A^{(3)}$ belongs to RE_A).

ii) If $\overline{\text{DMU}}_A^{(k+1)}$ does not belong to RE_A, then the process is the same with Part b) of Section i), except that instead of the equation of hyperplane P'_{k+1} in the fourth constraint of Model (12), the fourth constraint is replaced with hyperplane P'_k.

Case 2. If such a unit does not exist, hyperplane p_{k+1} is a weak defining hyperplane. Similarly, calculate the output/input amounts of $\widetilde{\text{DMU}}_A^{(k+1)}$ by Eq. (11). If $\widetilde{\text{DMU}}_A^{(k+1)}$ belongs to RE_A, then it is sufficient to connect DMU_A to $\widetilde{\text{DMU}}_A^{(k+1)}$ (see Figure 2); otherwise, find a point from Model (12) with the equation of hyperplane P'_{k+1} in the fourth constraint.

Replace the new point with $\widetilde{\text{DMU}}_A^{(k+1)}$ and name it $\widetilde{\text{DMU}}_A^{(k+1)}$. It is then sufficient to connect DMU_A to $\widetilde{\text{DMU}}_A^{(k+1)}$s (see Figure 3).

Evidently, $\widetilde{\text{DMU}}_A^{(k+1)}$ lies on both hyperplanes, P'_k and P'_{k+1}. Throughout the algorithm,

$$\widetilde{\text{DMU}}_A^{(1)} \rightarrow \widetilde{\text{DMU}}_A^{(2)} \rightarrow \cdots \rightarrow \widetilde{\text{DMU}}_A^{(k+1)}$$

represents the benchmarking movement in which the points $\widetilde{\text{DMU}}_A^{(1)}.\widetilde{\text{DMU}}_A^{(2)}.\cdots.\widetilde{\text{DMU}}_A^{(k+1)}$ lie on hyperplanes $P'_1, P'_2, \cdots, P'_{k+1}$, respectively; the set of sub-

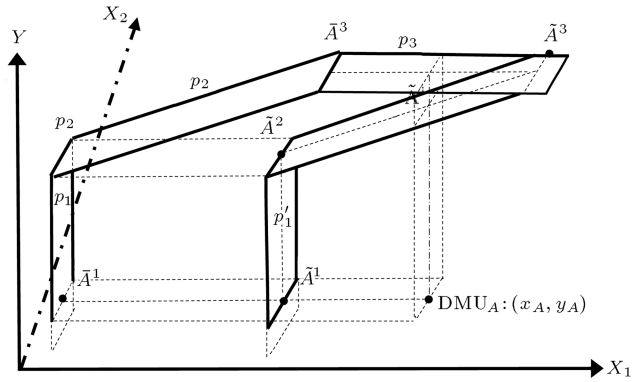

Figure 3. The benchmarking region of efficiency (Case 2. $\widetilde{\mathrm{DMU}}_A^{(3)}$ does not belong to RE_A).

scripts of the efficient improved faces of the PPS for DMU_A is considered (except the second part of Case 2 in which $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ lies on hyperplane P_k'). Note that all the new shifted units of the extreme points in the PPS have an equivalent efficiency value of $\beta_A^* = \theta_A^{*(K)} + \alpha\%$.

Hence, starting at benchmark point $\widetilde{\mathrm{DMU}}_A^{(1)}$, passing through all the obtained hyperplanes throughout the algorithm and, eventually, stopping at $\widetilde{\mathrm{DMU}}_A^{(k+1)}$, one can achieve the benchmark region of deficiency improvement.

Among the previous works, this study explores an improved benchmark and, sequentially, an improved region of efficiency for an inefficient DMU. In comparison, some papers find a benchmark turning the inefficient DMU to an efficient DMU [25]. Additionally, some other papers used the DEA sensitivity analysis just to explore a point as a virtual benchmark for an inefficient DMU [18].

Practically, an inefficient DMU sometimes cannot improve its performance to the best practice by either decreasing its inputs or increasing its outputs.

The current paper suggests an improved region that is characterized such that the efficiency value of each point in this region will be derived by the hyperplane with the same normal vector of the original PPs hyperplane. In fact, we improve all the hyperplanes that belong to RE_A. Considering that any change has implications, the process of finding a suitable benchmark is constantly faced with changes and evidently increasing costs. We, therefore, always try to remove or reduce these changes. The strength of the present approach is that, with considerably little numerical effort, one can get an insight into the improved efficiency region of a given DMU; unlike many previous works, it is based on exact results of a well-defined value judgment.

The proposed algorithm. The summary of the proposed algorithm is given below. Here, DMU_A is considered as an inefficient DMU under evaluation in order to find a new region by increasing the efficiency (obtained by Model 2) of DMU_A by $\alpha\%$.

Step 1. Find all the defining (efficient) hyperplanes of the PPS. Let $P = \{p_1, \cdots, p_2, p_t\}$ be the set of these defining (efficient) hyperplanes;

Step 2. Assume that $k = 1$ and $\Omega_1 = \{\mathrm{DMU}_j | \theta_j^* = 1$ and $\lambda_j = 1$ in assessing DMU_j by Model (2)$\}$;

Step 3. Solve Model (2) and set $\overline{\mathrm{DMU}}_A^{(k)}$ as a projection point of DMU_A;

Step 4. Look for p_k representing a hyperplane that $\overline{\mathrm{DMU}}_A^{(k)}$ satisfies in hyperplane p_k equation;

Step 5. From $\overline{\mathrm{DMU}}_A^{(k)}$, consider $\widetilde{\mathrm{DMU}}_A^{(k)}$ whose efficiency is $\beta_A^* = \theta_A^* + \alpha\%$. To calculate the input/output amounts of $\widetilde{\mathrm{DMU}}_A^{(k)}$, use the following formulas:

For $k = 1$;

$$\widetilde{\mathrm{DMU}}_A^{(k)} = \left(\tilde{x}_A^{(k)}, \tilde{y}_A^{(k)}\right) = \left(\frac{\bar{x}_A^{(k)}}{\beta_A^*}, y_A^{(k)}\right) = \left(\frac{\theta_A^* x_A^{(k)}}{\beta_A^*}, y_A^{(k)}\right).$$

Step 6. Find the equation of hyperplane P_k' which passes through $\widetilde{\mathrm{DMU}}_A^{(k)}$ and has the same normal vector as hyperplane p_k;

Step 7. Using the definition of Ω_1, look for extreme efficient DMUs that lie on hyperplane p_k so that at least one of its inputs or outputs is more than $\overline{\mathrm{DMU}}_A^{(k)}$. (In the case of the existence of more than one DMU for all of them, follow the process from Step 8.) If such a DMU exists, go to Step 8; otherwise, go to Step 9;

Step 8. Subscript the new DMU as $\overline{\mathrm{DMU}}_A^{(k+1)}$. If $\overline{\mathrm{DMU}}_A^{(k+1)}$ belongs to RE_A, then proceed to Step 10; otherwise, proceed to Step 11;

Step 9. If such a unit does not exist, i.e., hyperplane p_{k+1} is a weak defining hyperplane, calculate the output/input amounts of $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ from Eq. (11). If $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ belongs to RE_A, then proceed to Step 14; otherwise, proceed to Step 15;

Step 10. If $\overline{\mathrm{DMU}}_A^{(k+1)}$ belongs to RE_A, hyperplane p_{k+1} represents a defining efficient hyperplane in which $\overline{\mathrm{DMU}}_A^{(k+1)}$ satisfies hyperplane p_{k+1} equation. Consider $\widetilde{\mathrm{DMU}}_A^{(k+1)}$ whose efficiency is $\beta_A^* = \theta_A^* + \alpha\%$.

To calculate the input/output amounts of $\widetilde{\text{DMU}}_A^{(k+1)}$, use the following formula:

$$\widetilde{DMU}_A^{(k+1)} = \left(\tilde{x}_A^{(k+1)}, \tilde{y}_A^{(k+1)}\right) = \left(\frac{x_A^{(k+1)}}{\beta_A^*}, y_A^{(k+1)}\right). \quad (14)$$

Here, we confront two cases: either $\widetilde{\text{DMU}}_A^{(k+1)}$ belongs to RE_A, then proceed to Step 12; or it does not belong to RE_A, then proceed to Step 13;

Step 11. If $\overline{\text{DMU}}_A^{(k+1)}$ does not belong to RE_A, then, by using Model (12), except that instead of the equation of hyperplane P'_{k+1} in the fourth constraint of Model (12), the fourth constraint is replaced with the equation of hyperplane P'_k.

Replace the new point with $\widetilde{\text{DMU}}_A^{(k+1)}$ and name it $\widetilde{\text{DMU}}_A^{(k+1)}$. By connecting DMU_A to the new point, $\widetilde{\text{DMU}}_A^{(k+1)}$, which lies on hyperplane P'_k and has at least one input/output value of DMU_A, this process is finished. Proceed to Step 16;

Step 12. If $\widetilde{\text{DMU}}_A^{(k+1)}$ belongs to RE_A, suppose that hyperplane P'_{k+1} is the equation that passes through $\widetilde{\text{DMU}}_A^{(k+1)}$ and has the same normal vector as hyperplane p_{k+1}; hence, proceed to Step 7;

Step 13. If $\widetilde{\text{DMU}}_A^{(k+1)}$ does not belong to RE_A, suppose that hyperplane P'_{k+1} is the equation which passes through $\widetilde{\text{DMU}}_A^{(k+1)}$ and has the same normal vector as hyperplane p_{k+1}; then, find a point from Model (12) in which the fourth constraint is the equation of hyperplane P'_{k+1}. Replace the new point with $\widetilde{\text{DMU}}_A^{(k+1)}$. By connecting DMU_A to the new point, $\widetilde{\text{DMU}}_A^{(k+1)}$, which lies on hyperplane P'_{k+1}, this process is finished;

Step 14. If $\widetilde{\text{DMU}}_A^{(k+1)}$ belongs to RE_A, then it is sufficient to connect DMU_A to $\widetilde{\text{DMU}}_A^{(k+1)}$. Proceed to Step 16;

Step 15. If $\widetilde{\text{DMU}}_A^{(k+1)}$ does not belong to RE_A, find a point from Model (12) with the equation of hyperplane P'_{k+1} in the fourth constraint. Replace the new point with $\widetilde{\text{DMU}}_A^{(k+1)}$ and name it $\widetilde{\text{DMU}}_A^{(k+1)}$. Then, it is sufficient to connect DMU_A to $\widetilde{\text{DMU}}_A^{(k+1)}$;

Step 16. Stop.

The PPS is a polyhedral convex set whose vertices correspond to the efficient DMUs. Besides, a polyhedron is defined as the intersection of a finite collection of generalized halfspaces. More generally, a half space is either of the two parts into which a hyperplane divides an affine space. Since the number of the defining hyperplane of a PPS is finite, the number of the new improved hyperplanes is less than that of the defining hyperplanes. The validity of the algorithm is, therefore, obvious owing to the finite number of defining hyperplanes.

4. Numerical example

This section provides a numerical example that helps clarify the concept of the improved region of the DMU's efficiency and its comparative advantage over conventional DEA models.

In light of the growing strategic importance of the function of managing a company's most important asset–its reputation–there is a need of fundamentally evaluating the improved region of the DMU's efficiency to obtain a guideline for future planning.

Example 1: Let us assume that we have eight sample DMUs, each using one input to produce one output with the data given in Table 1. All the virtual and real DMUs are depicted in Figure 4. By the foregone algorithm.

For finding the improved region by increasing the efficiency (obtained from Model (2)) of DMU_G by $\alpha\% = 13.75\%$, we first find all the maker hyperplanes of PPS. These hyperplanes' set is as follows:

$$\{-x+2=0,\ y-4x+6=0,\ 2y-x-9=0,\ y-8=0\}.$$

Table 1. Sample DMUs.

DMU	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Input	2	3	7	4.4	3.3	6	4	8
Output	2	6	8	6	4.5	5	3	4

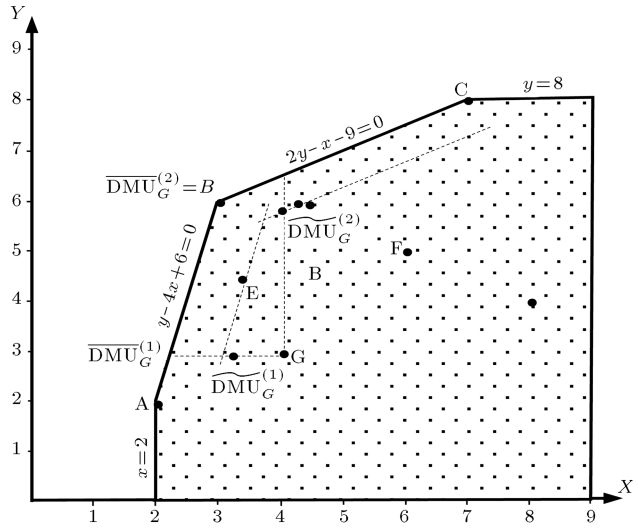

Figure 4. Efficient frontiers of PPS.

$\Omega_1 = \{\text{DMU}_A, \text{DMU}_B, \text{DMU}_C\}$ is the set of the extreme efficient DMUs of the data of Table 1.

By solving Model (2) and setting $\overline{\text{DMU}}_G^{(1)}$ as the projection point of DMU_G, we have:

$$\overline{\text{DMU}}_G^{(1)} = (\bar{x}, \bar{y}) = (2.25, 3).$$

From $\overline{\text{DMU}}_G^{(1)}$, which lies on $p_1 : y - 4x + 6 = 0$, we are considering $\widetilde{\text{DMU}}_G^{(1)}$, whose efficiency is:

$$\theta_G^* = 0.5685 \xrightarrow{\alpha\%=13.75\%;} \beta_G^* = \theta_G^* + \alpha\% = 0.7.$$

Using Eq. (10), we calculate the input/output amounts of $\widetilde{\text{DMU}}_G^{(1)}$:

$$\widetilde{\text{DMU}}_G^{(1)} = \left(\tilde{x}_G^{(1)}, \tilde{y}_G^{(1)}\right) = (3.2143, 3).$$

Now, we are looking for hyperplane p_1', which passes through $\widetilde{\text{DMU}}_G^{(1)}$ and has the same normal vector as hyperplane p_1; this is shown as follows:

$$y - 4x + 12.857 = 0.$$

DMU_B is an extreme efficient DMU that lies on hyperplane p_1; its inputs and outputs are more than $\overline{\text{DMU}}_G^{(1)}$.

Since DMU_B belongs to RE_G, we subscript it as $\overline{\text{DMU}}_G^2$.

A defining efficient hyperplane in which $\overline{\text{DMU}}_A^2$ satisfies hyperplane p_2 equation, is represented by $p_2 = 2y - x - 9 = 0$. By Eq. (11), we calculate $\widetilde{\text{DMU}}_G^2$, whose efficiency is $\beta = 0.7$, as follows:

$$\widetilde{\text{DMU}}_G^2 = \left(\tilde{x}_G^2, \tilde{y}_G^2\right) = (4.2857, 6).$$

The equation of the hyperplane which passes through $\widetilde{\text{DMU}}_G^{(2)}$ and has the same normal vector as hyperplane p_2 is as follows:

$$P_2' = 2y - x - 7.7143 = 0,$$

since $\widetilde{\text{DMU}}_G^2$ does not belong to RE_G, based on Step 13 of the algorithm, we replace $\widetilde{\text{DMU}}_G^2$ with the new point (4, 5/85715) obtained from Model (12), which lies on hyperplane P_2' and has at least one input/output value of DMU_G (see Figure 4).

Since there is no other DMU in RE_G, there are also no other improved hyperplanes.

Example 2: The data of 9 DMUs from Hosseinzadeh Lotfi's paper (2011) [19] are used as a numerical example and consist of two inputs and one output, as shown in Table 2.

We intend to find the improved region of the DMU's efficiency using the data in Table 2. We, therefore, use the proposed algorithm to obtain the improved efficiency region of DMU_2 by increasing its current efficiency value ($\theta_2^* = 0.6885246$ to $\alpha\% = 11/1475\%$).

After finding all the defining hyperplanes as shown in Figure 5, the set of extreme efficient points of PPS is as follows:

$$\Omega_1 = \{\text{DMU}_3, \text{DMU}_4, \text{DMU}_5, \text{DMU}_7\}.$$

By determining the radial projection reference point of DMU_2 by Model (2) $\overline{\text{DMU}}_2^{(1)}$, we look for hyperplane p_1 in which $\overline{\text{DMU}}_2^{(1)}$ satisfies hyperplane p_1 equation:

$$\begin{aligned}\text{DMU}_2 = (x_1, x_2, y) = (7, 3, 4) &\rightarrow \overline{\text{DMU}}_2^{(1)} \\ &= (0.6885246(7), 0.6885246(3), 4), \rightarrow \overline{\text{DMU}}_2^{(1)} \\ &= (4.8196722, 2.0655738, 4),\end{aligned}$$

$$p_1 : 0.352941y - 0.411765x_1 - 0.235294x_2 + 1.06 = 0.$$

Now, we intend to find $\widetilde{\text{DMU}}_2^{(1)}$, whose efficiency is $\theta_2^* + \alpha\% = 0.8$. The input/output amounts of $\widetilde{\text{DMU}}_2^{(1)}$ are calculated as follows:

$$\begin{aligned}\widetilde{\text{DMU}}_2^{(1)} &= \left(\tilde{x}_{12}^{(1)}, \tilde{x}_{22}^{(1)}, \tilde{y}_2^{(1)}\right) \\ &= \left(\frac{0.6885246(7)}{0.8}, \frac{0.6885246(3)}{0.8}, 4\right) \\ &= (6.02459025, 2.58196725, 4).\end{aligned}$$

In Step 6, we find the equation of hyperplane P_1' which passes through $\widetilde{\text{DMU}}_2^{(1)}$ and has the same normal vector as hyperplane p_1 which is as follows:

$$p_1 = (\alpha, \beta, \gamma) = (0.352941, -0.411765, -0.235294),$$

Table 2. The data of the eight DMUs.

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9
Input 1	4	7	8	4	2	10	12	10	13
Input 2	3	3	1	2	4	1	1	1.5	2
Output	2	4	7	3	2	5	8	7	7.5

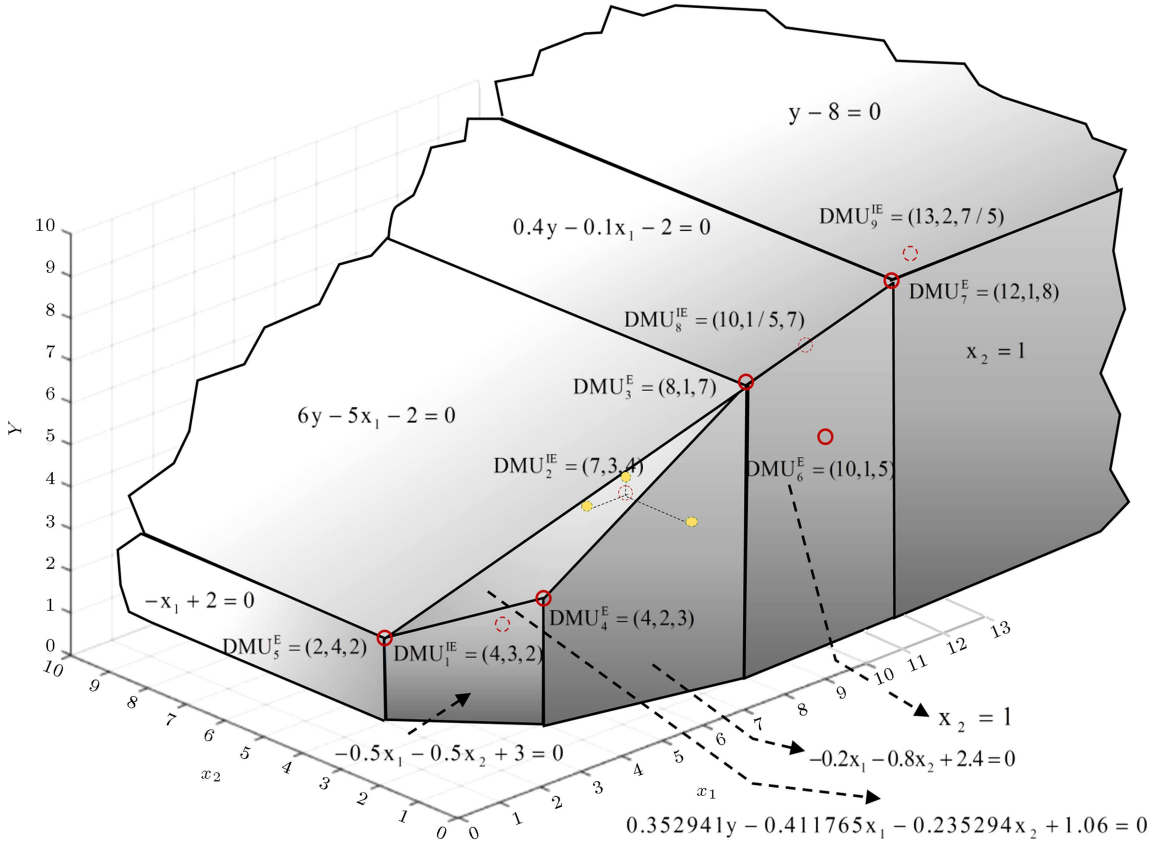

Figure 5. The visual description scheme.

$$\alpha(y-y_A)+\beta(x_1-x_{1A})+\gamma(x_2-x_{2A})=0 \rightarrow$$

$$0.352941(y-4)-0.411765(y-4)$$

$$-0.411765(x_1-6.02459025)$$

$$-0.235294(x_2-2.58196725)=0,$$

$$P_1': \ 0.352941y-0.411765x_1-0.235294x_2$$

$$+1.676472806=0.$$

DMU_3 and DMU_5 are the extreme efficient DMUs lying on hyperplane p_1 where at least one of their inputs or outputs is more than $\overline{DMU}_2^{(1)}$. Based on Step 11 of the algorithm, since $\overline{DMU}_2^{(2)}$ does not belong to RE_2, we find point $\widetilde{DMU}_A^{(2)} = (7, 2.4687495, 5.062504)$ by Model (12) in which the fourth constraint is the equation of hyperplane P_1'.

At this point, to clarify the strategy of the proposed algorithm, we follow the algorithm for $DMU_9 = (13, 2, 7.5)$ as follows.

We solve Model (2) and find that $\overline{DMU}_9^{(1)}$ is a projection point of DMU_9; $\overline{DMU}_9^{(1)} = (10, 1.5, 7.5)$.

The equation of a hyperplane is $p_1 : 0.4y-0.1x1-2=0$, such that $\overline{DMU}_9^{(1)}$ satisfies its equation.

Now, we consider $\widetilde{DMU}_9^{(1)}$ whose efficiency is $\beta_9^* = \theta_9^* + \alpha\% = 0.769230769 + 0.130769231 = 0.9(\alpha\% = 13.0769231\%)$.

By Eq. (10), the following unit is calculated:

$$\widetilde{DMU}_9^{(1)} = (11.\bar{1}, 1.709401709, 7.5)$$

The equation of hyperplane P_1' is $0.4y-0.1x_1-1.\bar{8}=0$

The extreme efficient DMU that lies on hyperplane p_1 is $DMU_7 = \overline{DMU}_9^{(2)}$ and its first input is greater than $\overline{DMU}_9^{(1)}$.

As $\overline{DMU}_9^{(2)}$ belongs to RE_9, based on Step 10 of the proposed algorithm, we find two defining efficient hyperplanes, p_2 and p_3, in which $\overline{DMU}_9^{(2)}$ satisfies their equations as follows:

$$\begin{cases} p_2 : y-8=0 \\ p_3 : x_2-1=0 \end{cases}$$

Now, by using Eq. (11), $\widetilde{DMU}_9^{(2)}$ is considered, whose efficiency is $\beta_9^* = 0.9 : \widetilde{DMU}_9^{(2)} = (13.\bar{3}, 1.\bar{1}, 8)$.

As $\widetilde{DMU}_9^{(2)}$ does not belong to RE_9, suppose that $P_2' : y-8=0, P_3' : x_2-1.\bar{1}=0$ are the equations of the hyperplanes that pass through $\widetilde{DMU}_9^{(2)}$ and have

the same normal vector as hyperplanes p_2 and p_3, respectively.

In addition, there is no other extreme efficient unit that passes through equations of hyperplanes p_2 and p_3. Based on Step 13, to understand the improved region better, Model (12) is employed. In the equation of $y-8=0$, the obtained point is $(9.7\bar{3}, 1.2\bar{6}, 8.0000325)$; in the equation of $x_2 - 1.\bar{1} = 0$, the obtained point is $(13, 1.2, 8.6750025)$. We replace the new points with $\widetilde{\text{DMU}}_9^{(2)}$ and $\widetilde{\text{DMU}}_9^{(3)}$. By connecting DMU_9 to the new points, $\widetilde{\text{DMU}}_9^{(2)}$ and $\widetilde{\text{DMU}}_9^{(3)}$, which lie on hyperplanes P_2' and P_3', the process is finished. The part of improved obtained hyperplanes P_1', P_2', and P_3', included in the points $\widetilde{\text{DMU}}_9^{(1)}$, $\widetilde{\text{DMU}}_9^{(2)}$, and $\widetilde{\text{DMU}}_9^{(3)}$, is the improvement benchmarking region.

Note that, in Figure 5, notations DMU_j^E and DMU_j^{IE}, $j = 1, \cdots, n$, respectively, represent the efficient and inefficient units.

5. Conclusions

As it is an issue of considerable importance, this study sought to make the standards more flexible for choosing the benchmarks. This standard can, however, differ from that of decision-makers. If a company has a specific corporate strategy or future vision, the standards for choosing benchmarks might be more complex. There is a central question in the evaluation of an inefficient DMU: instead of having a benchmark with an efficiency value of one, how can an inefficient DMU achieve a more realistic benchmark in which the efficiency is less than one? This means that a wide range of knowledge on this study becomes available. A framework for defining a benchmark region in evaluating an inefficient DMU was derived from the value judgment by increasing the current efficiency. This paper developed a procedure for shifting the efficient frontier toward the inefficient DMU. The procedure yielded an exact improved region that is derived by increasing the efficiency score of a specific inefficient DMU by $\alpha\%$, and the efficiency score of each point belonging to the new improved frontiers is different. It is indeed a new virtual frontier that is recognized as a realistic benchmarking region for the under-evaluated inefficient DMU. One of the advantages of the current analysis is the incorporation of value judgment into the current situation of hyperplanes through specific efficiency improvement for the under-evaluated inefficient DMU. The value judgment here can, therefore, be represented by the improved region concept. In addition, we used the sensitivity analysis of efficiency to find the new region. Further research will concentrate on the region of the MPSS (Most Productive Scale Size) by using hyperplanes to evaluate DMUs.

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