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Thermoelastic bending response of a laminated plate resting on elastic foundations

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Abstract. The thermoelastic bending response is presented for a simply-supported composite laminated plate subjected to a thermal field. The sinusoidal plate theory as well as classical and other shear deformation theories is used. The laminated plate may be composed of one material or a combination of two materials. The layers may be symmetric cross-ply or anti- or non-symmetric angle-ply lay-up. The numerical illustrations concerned with the thermal bending response of the presented rectangular plate are studied. The effects due to many parameters, such as shear deformation, aspect ratio, side-to-thickness ratio, thermal loading and elastic foundations, are all investigated.

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1. Introduction

A plate on an elastic foundation belongs to the problem of mutual action between two media. Early research adopted single-parameter Winkler's model to simulate the foundation. It is considered that the displacement on a foundation surface is limited only on the loaded domain, which conflicts with the practical response situation. In some analysis of plates on elastic foundations, a single-parameter is used to describe the foundation behavior [1]. In this model, it is assumed that there is a proportional interaction between the external forces and the deflection of the applied point in the foundation. A two-parameter elastic foundation model can reflect the practical deformation of a foundation, so it is widely accepted by investigators.

Plates supported by elastic foundations are commonly encountered technical problems in many engineering applications. The studies of plates resting on elastic foundations have attracted the attention of many researchers [2-15]. In the present article, the shear deformable plate theory developed by Zenkour [16-18] is used to study the thermoelastic bending response of laminated plates resting on elastic foundations. An accurate solution for the simply-supported cross-ply laminated plate is presented. Pasternak's model is used to describe the two-parameter elastic foundation. A special case of Winkler's foundation model is obtained by considering one-parameter elastic foundation. The interaction between the plate and the elastic foundations is considered and included in the equilibrium equations. Numerical results for deflections and stresses are presented. A comparison between results according to different plate theories is reported in many tables.

2. Geometrical preliminaries

The present study considers a composite rectangular plate of length a, width b and uniform thickness h as shown in Figure 1. Rectangular Cartesian coordinates (x, y, z) are used and the mid-plane is defined by $z = 0$ and its bounding planes are defined by $z = \pm\frac{1}{2}h(x)$. The plate is composed of n orthotropic layers oriented

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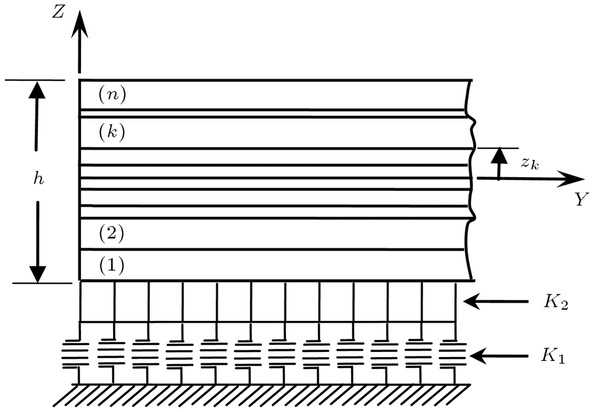

Figure 1. Schematic diagram for the laminated plate resting on elastic foundations.

at angles $\theta_1, \theta_2, ..., \theta_n$. The material of each layer is assumed to possess one plane of elastic symmetry parallel to the $x-y$ plane. Perfect bonding between the orthotropic layers and temperature-independent mechanical and thermal properties are assumed. Let the plate be subjected to a transverse load, $q(x, y)$, and temperature field, $T(x, y, z)$.

The displacements, u_i, of a material point in the plate are assumed as [16-19]:

$$u_1 = u(x, y) - z\frac{\partial w}{\partial x} + f(z)\psi(x, y),$$

$$u_2 = v(x, y) - z\frac{\partial w}{\partial y} + f(z)\varphi(x, y),$$

$$u_3 = w(x, y), \tag{1}$$

where u, v, and w are the displacements of the middle surface along the axes x, y and z, respectively, and ψ and φ are the rotations about the y and x axes and account for the effect of transverse shear. The coefficient of ψ and φ is given by f and it should be odd function of z. The displacement field of classical and shear deformation theories are given by taking suitable forms for $f(z)$, as below:

- Classical Plate Theory (CPT): $f(z) = 0$,
- First-order shear deformation Plate Theory (FPT): $f(z) = z$,
- Higher-order shear deformation Plate Theory (HPT): $f(z) = z[1 - \frac{4}{3}(\frac{z}{h})^2]$,
- Sinusoidal shear deformation Plate Theory (SPT): $f(z) = \frac{h}{\pi}\sin(\frac{\pi z}{h})$.

In addition, the applied temperature distribution, T, is assumed as:

$$T(x, y, z) = T_1(x, y) + \frac{z}{h}T_2(x, y) + \frac{f(z)}{h}T_3(x, y), \tag{2}$$

where T_i are the transverse temperature loads.

Also, the load-displacement relation between the plate and the supporting foundations is given according to the two-parameter Pasternak's model by:

$$R = K_1 w - K_2\nabla^2 w, \tag{3}$$

where R is the foundation reaction per unit area, K_1 and K_2 are Winkler's and Pasternak's foundation stiffness, respectively, and ∇^2 represents Laplace operator. Winkler's model is simply obtained when $K_2 = 0$.

The six strain components ε_{ij} compatible with the displacement field in Eq. (1) are:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

while the stress-strain relations for a linear elastic plate are given by:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & 0 & 0 & c_{26} \\ 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & 0 & 0 & c_{66} \end{bmatrix}^k \times \begin{Bmatrix} \varepsilon_1 - \alpha_1 T \\ \varepsilon_2 - \alpha_2 T \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 - \alpha_6 T \end{Bmatrix}^k, \tag{5}$$

where $\varepsilon_1 = \varepsilon_{11}$, $\varepsilon_2 = \varepsilon_{22}$, $\varepsilon_4 = 2\varepsilon_{23}$, $\varepsilon_5 = 2\varepsilon_{13}$ and $\varepsilon_6 = 2\varepsilon_{12}$. Note that, $c_{ij}^{(k)}$ are the transformed elastic coefficients and (α_1, α_2, α_6) are the thermal expansion coefficients in the plate coordinates.

The governing equilibrium equations can be derived using the principle of virtual displacements as:

$$N_{1,1} + N_{6,2} = 0, \quad N_{6,1} + N_{2,2} = 0,$$

$$M_{1,11} + 2M_{6,12} + M_{2,22} - R = 0,$$

$$S_{1,1} + S_{6,2} - Q_5 = 0, \quad S_{6,1} + S_{2,2} - Q_4 = 0, \tag{6}$$

where N_i and $M_i (i = 1, 2, 6)$ are the basic components of stress resultants and stress couples, S_i are additional stress couples associated with the transverse shear effects and $Q_j (j = 4, 5)$ are transverse shear stress resultants. They can be obtained by integrating Eq. (5) over the thickness of the composite plate.

3. Closed-form solution

The determination of thermal stresses is of fundamental importance in the design of many structural components. An exact closed-form solution to Eq. (6) can be constructed when the plate is of a rectangular geometry (Figure 1) with the following edge conditions, loading and plate constructions.

The following set of simply supported boundary conditions along the edges of the plate is considered:

$$v = w = \varphi = N_1 = M_1 = S_1 = 0 \quad \text{at} \quad x = 0, a,$$

$$u = w = \psi = N_2 = M_2 = S_2 = 0 \quad \text{at} \quad y = 0, b. \qquad (7)$$

Under a general co-ordinate transformation, an initially orthotropic material becomes generally anisotropic. However, there are three specific coordinates transformations under which an orthotropic material retains monoclinic symmetry, namely, rotations about the axes x, y or z. For example, if the material is orthotropic with respect to the old coordinate system, it follows under rotation through an angle, θ_k, about the z-axis that the transformation formulae for the stiffness, $c_{ij}^{(k)}$, are of the form [20]:

$$\left\{\begin{matrix} c_{11} \\ c_{12} \\ c_{22} \\ c_{16} \\ c_{26} \\ c_{66} \end{matrix}\right\}^k = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ -c^3s & cs(c^2-s^2) & cs^3 & 2cs(c^2-s^2) \\ -cs^3 & cs(s^2-c^2) & c^3s & 2cs(s^2-c^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \times \left\{\begin{matrix} c_{11} \\ c_{12} \\ c_{22} \\ c_{66} \end{matrix}\right\},$$

$$\left\{\begin{matrix} c_{44} \\ c_{45} \\ c_{55} \end{matrix}\right\}^k = \begin{bmatrix} c^2 & s^2 \\ cs & -cs \\ s^2 & c^2 \end{bmatrix} \left\{\begin{matrix} c_{44} \\ c_{55} \end{matrix}\right\}, \qquad (8)$$

where $c = \cos\theta_k$ and $s = \sin\theta_k$ and c_{ij} are the (plane stress-reduced) material stiffness of the lamina:

$$c_{11} = \frac{E_1}{1 - v_{12}v_{21}}, \quad c_{12} = \frac{v_{21}E_1}{1 - v_{12}v_{21}} = \frac{v_{12}E_2}{1 - v_{12}v_{21}},$$

$$c_{22} = \frac{E_2}{1 - v_{12}v_{21}}, \quad c_{44} = G_{23},$$

$$c_{55} = G_{13}, \quad c_{66} = G_{12}, \qquad (9)$$

in which E_i are Young's moduli in the material principal directions, v_{ij} are Poisson's ratios, and G_{ij} are shear moduli.

To solve this problem, Navier presented the transverse sinusoidal temperature loads and deflection in the form:

$$\{T_i, w\} = \{\bar{T}_i, W\} \sin(\lambda x) \sin(\mu y),$$

$$i = 1, 2, 3, \qquad (10)$$

where $\lambda = \pi/a$, $\mu = \pi/b$, $\bar{T}_i$ are constants, and W is arbitrary parameter. In addition, the remainder displacements take the forms:

$$\{u, \psi\} = \{U, X\} \cos(\lambda x) \sin(\mu y),$$

$$\{v, \varphi\} = \{V, Y\} \sin(\lambda x) \cos(\mu y), \qquad (11)$$

where U, V, X and Y are additional arbitrary parameters. The above solution form for $\{u, v, w, \psi, \varphi\}$ satisfies the simply supported boundary conditions, and the parameters U, V, W, X and Y will be determined subjected to the condition that the solution satisfies the differential equations given in Eq. (6).

4. Numerical examples and discussion

Numerical results for deflections and stresses are presented for symmetric cross-ply 0°/90°/0°/... laminated plates subjected to thermal loading. Additional results for anti-symmetric angle-ply composite plates are also presented. The plate is subjected to thermal loading with $T_1(x, y) = 0$ and $T_2(x, y) = T_3(x, y)$. For the sake of completeness and comparison, the results due to the present theory are compared with those due to the classical and other shear deformation plate theories.

All laminas are assumed to be of the same thickness and made of the same orthotropic material. The lamina properties of E-glass/epoxy are [21]:

$$E_1 = 15E_0, \qquad E_2 = 6E_0,$$

$$G_{12} = G_{13} = 3E_0, \qquad G_{23} = 1.5E_0,$$

$$v_{12} = 0.3, \qquad \alpha_1 = 7\alpha_0,$$

$$\alpha_2 = 2.3\alpha_0, \qquad \alpha_6 = 0,$$

where $E_0 = 1$ GPa and $\alpha_0 = 10^{-6}(1/°\text{C})$. The dimensionless deflection and stresses are given by:

$$\bar{w} = \frac{wh}{\alpha_0 \bar{T}_2 \alpha^2}, \quad \bar{\sigma}_1 = -\frac{h}{E_0\alpha_0\bar{T}_2 a}\sigma_1\left(\frac{a}{2}, \frac{b}{2}, \bar{Z}\right),$$

$$\bar{\sigma}_2 = -\frac{h}{E_0\alpha_0\bar{T}_2 a}\sigma_2\left(\frac{a}{2}, \frac{b}{2}, \bar{Z}\right),$$

$$\bar{\sigma}_4 = \frac{1}{E_0\alpha_0\bar{T}_2}\sigma_4\left(\frac{a}{2}, 0, \bar{Z}\right),$$

$$\bar{\sigma}_5 = \frac{1}{E_0\alpha_0\bar{T}_2}\sigma_5\left(0, \frac{b}{2}, \bar{Z}\right),$$

$$\bar{\sigma}_6 = \frac{h}{E_0\alpha_0\bar{T}_2 a}\sigma_6(0, 0, \bar{Z}).$$

Dimensionless deflections, $\bar{w}$, for three-layer, five-layer, and nine-layer cross-ply laminated plates are presented

Table 1. Dimensionless deflection $\bar{w}(0)$ for cross-ply rectangular plates subjected to thermal effect.

			$0^\circ/90^\circ/0^\circ$			$0^\circ/90^\circ/\ldots$ 5 layers			$0^\circ/90^\circ/\ldots$ 9 layers		
k_1	k_2	Theory	$s = 0.5$	$s = 1.0$	$s = 2.0$	$s = 0.5$	$s = 1.0$	$s = 2.0$	$s = 0.5$	$s = 1.0$	$s = 2.0$
0.0	0.0	CPT	6.10460	3.75313	1.23574	6.03507	3.55054	1.11013	5.97562	3.40040	1.03428
		FPT	12.19375	7.48004	2.46276	12.05706	7.08463	2.22255	11.94042	6.79181	2.07647
		HPT	10.93794	6.69164	2.18940	10.75790	6.33730	1.99827	10.67164	6.08616	1.86959
		SPT	10.77522	6.59008	2.15464	10.59347	6.24370	1.97083	10.51276	5.99700	1.84293
0.1	0.0	CPT	0.83641	0.87619	0.73899	0.77171	0.82889	0.69621	0.72287	0.79384	0.66830
		FPT	1.58161	1.64129	1.37727	1.46305	1.55525	1.30194	1.37351	1.49133	1.25214
		HPT	1.40865	1.46300	1.22920	1.30360	1.39014	1.17099	1.22747	1.33621	1.12714
		SPT	1.38643	1.44006	1.21010	1.28420	1.36984	1.15456	1.20946	1.31678	1.11100
0.1	0.1	CPT	0.75593	0.76104	0.61667	0.69674	0.71996	0.58802	0.65214	0.68951	0.56895
		FPT	1.42826	1.42216	1.13122	1.31997	1.34764	1.08098	1.23831	1.29226	1.04703
		HPT	1.27194	1.26751	1.01050	1.17609	1.20454	0.97234	1.10665	1.15784	0.94245
		SPT	1.25186	1.24760	0.99488	1.15859	1.18695	0.95863	1.9041	1.14101	0.92893

Table 2. Dimensionless in-plane longitudinal stress $\bar{\sigma}_1(0.5)$ for cross-ply rectangular plates subjected to thermal effect.

			$0^\circ/90^\circ/0^\circ$			$0^\circ/90^\circ/\ldots$ 5 layers			$0^\circ/90^\circ/\ldots$ 9 layers		
k_1	k_2	Theory	$s = 0.5$	$s = 1.0$	$s = 2.0$	$s = 0.5$	$s = 1.0$	$s = 2.0$	$s = 0.5$	$s = 1.0$	$s = 2.0$
0.0	0.0	CPT	0.83266	2.43307	4.25645	0.88766	2.60730	4.39920	0.93467	2.73642	4.48539
		FPT	1.64031	4.76115	8.26226	1.74893	5.09932	8.53988	1.84198	5.35258	8.71608
		HPT	0.74702	3.56101	6.70832	0.89598	3.88300	6.96411	0.96549	4.10328	7.12453
		SPT	0.70530	3.47848	6.57892	0.85676	3.79711	6.83266	0.92154	4.01269	6.99044
0.01	0.0	CPT	2.53628	3.23091	4.34490	2.82308	3.36207	4.46999	2.92364	3.45928	4.54642
		FPT	5.26867	6.31083	8.43557	5.51426	6.56926	8.68043	5.71379	6.76399	8.83843
		HPT	4.02671	4.95914	6.86411	4.28471	5.21309	7.09372	4.45736	5.38427	7.23804
		SPT	3.93751	4.85587	6.73223	4.19625	5.10909	6.96088	4.36381	5.27620	7.10261
0.01	0.01	CPT	2.83330	3.34396	4.38457	2.95828	3.46902	4.50208	3.05874	3.56170	4.57425
		FPT	5.52392	6.52535	8.51209	5.77042	6.77278	8.74316	5.97001	6.95941	8.89340
		HPT	4.25661	5.15241	6.93296	4.51508	5.39719	7.15158	4.68844	5.56163	7.28903
		SPT	4.16398	5.09624	6.79999	4.42333	5.29069	7.01812	4.59163	5.45115	7.15300

in Table 1. Similar results for the in-plane longitudinal stress $\bar{\sigma}_1$, the in-plane normal stress $\bar{\sigma}_2$, the in-plane transverse stress $\bar{\sigma}_6$, as well as the transverse shear stresses $\bar{\sigma}_4$ and $\bar{\sigma}_5$ are also presented in Tables 2-6. Different aspect ratios $s = a/b$ are taken with $a/h = 10$ and $\bar{T}_2 = \bar{T}_3$. The elastic foundation parameters $k_1 = hK_1$ and $k_2 = hK_2$ have different values. Three cases of elastic foundations are discussed:

Case 1: Plates without elastic foundations ($k_1 = k_2 = 0$);

Case 2: Plates follow Winkler's model ($k_1 = 0.01$ or 0.1, $k_2 = 0$);

Case 3: Plates follow Pasternak's model ($k_1 = 0.01$ or 0.1, $k_2 = 0.01$ or 0.1).

As shown in Table 1, the CPT gives the smallest deflections while the FPT gives the largest deflections for all cases studied. The deflections of FPT may be twice the corresponding ones of CPT. In general, the deflection is very sensitive to the inclusion of elastic foundations. It rabidly decreases as one of the foundation parameters is included and decreases again if both foundations are included. The deflection also decreases with the increase of the number of layers and the value of the aspect ratio.

The in-plane, longitudinal $\bar{\sigma}_1$, normal $\bar{\sigma}_2$, and transverse $\bar{\sigma}_6$ stresses are also presented in Tables 2-4, respectively. The FPT gives the largest in-plane stresses which may be twice that of CPT. The in-plane longitudinal stress, $\bar{\sigma}_1$, increases as the aspect ratio and the number of layers increase. However, the in-plane normal stress $\bar{\sigma}_2$ increases as the number of layers increases. They increase also with the inclusion

Table 3. Dimensionless in-plane normal stress $\bar{\sigma}_2(0.3)$ for cross-ply rectangular plates subjected to thermal effect.

			0°/90°/0°			0°/90°/... 5 layers			0°/90°/... 9 layers		
k_1	k_2	Theory	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	CPT	0.20283	-0.07757	-0.15766	0.43700	-0.36633	-0.64143	0.44713	-0.28885	-0.49747
		FPT	0.41721	-0.11466	-0.23854	0.90234	-0.63211	-1.10521	0.91979	-0.48867	-0.84793
		HPT	0.44287	-0.03392	-0.13982	0.94397	-0.43329	-0.87186	0.95775	-0.30664	-0.63976
		SPT	0.44121	-0.02854	-0.13245	0.94005	-0.41727	-0.85208	0.95289	-0.29304	-0.62216
0.01	0.0	CPT	0.44194	0.14468	-0.09599	0.85415	0.08654	-0.52318	0.87582	0.14486	-0.39554
		FPT	0.88626	0.32256	-0.11661	1.72292	0.25857	-0.87116	1.76194	0.36295	-0.64605
		HPT	0.86209	0.35772	-0.03183	1.67388	0.36101	-0.66350	1.70715	0.45196	-0.46078
		SPT	0.85406	0.35729	-0.02615	1.65834	0.36455	-0.64689	1. 69076	0.45386	-0.44603
0.01	0.01	CPT	0.45922	0.17617	-0.06833	0.88328	0.15070	-0.46959	0.90494	0.20632	-0.34904
		FPT	0.91926	0.38308	-0.06278	1.77874	0.38189	-0.76670	1.81767	0.48087	-0.55535
		HPT	0.89148	0.41186	0.01590	1.72350	0.47095	-0.57050	1.75674	0.55700	-0.38037
		SPT	0.88298	0.41061	0.02083	1.70718	0.47277	-0.55531	1.73959	0.55728	-0.36690

Table 4. Dimensionless in-plane transverse stress $\bar{\sigma}_6(-0.5)$ for cross-ply rectangular plates subjected to thermal effect.

			0°/90°/0°			0°/90°/... 5 layers			0°/90°/... 9 layers		
k_1	k_2	Theory	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	CPT	0.90375	1.11126	0.73177	0.89345	1.05127	0.65739	0.88466	1.00682	0.61248
		FPT	1.78832	2.20752	1.53965	1.77032	2.09354	1.39783	1.75455	2.00879	1.30915
		HPT	1.61005	1.98425	1.37635	1.58355	1.87624	1.24846	1.57116	1.80178	1.16765
		SPT	1.58661	1.95509	1.35589	1.55952	1.84839	1.22964	1.54796	1.77541	1.14999
0.01	0.0	CPT	0.55450	0.83657	0.68568	0.53117	0.79141	0.62050	0.51235	0.75795	0.58068
		FPT	1.10259	1.66994	1.44910	1.05980	1.58496	1.32465	1.02498	1.52150	1.24561
		HPT	0.98687	1.49589	1.29438	0.94325	1.41494	1.18075	0.91319	1.35910	1.10858
		SPT	0.97207	1.47362	1.27516	0.92872	1.39351	1.16267	0.89944	1.33883	1.09162
0.01	0.01	CPT	0.52926	0.79765	0.66501	0.50587	0.75460	0.60378	0.48706	0.72269	0.56617
		FPT	1.05435	1.59553	1.40913	1.01147	1.51455	1.29199	0.97670	1.45402	1.21706
		HPT	0.94318	1.42839	1.25815	0.89972	1.35109	1.15052	0.86965	1.29781	1.08204
		SPT	0.92901	1.40707	1.23948	0.88583	1.33055	1.13278	0.85652	1.27838	1.06539

of elastic foundations. The largest in-plane transverse stress $\bar{\sigma}_6$ occurs for the square plates. It decreases with the inclusion of the elastic foundation and as the number of layer is increasing.

In Tables 5 and 6, the transverse shear stresses are presented according to various laminated plate schemes and different values of the aspect ratio. The FPT gives the largest shear stress, $\bar{\sigma}_4$, and the smallest shear stress, $\bar{\sigma}_5$. The shear stress, $\bar{\sigma}_4$, increases with the increase of the aspect ratio for all plates with and without elastic foundations. However, the shear stress $\bar{\sigma}_5$ increases with the increase of the aspect ratio for plates without elastic foundations only. The shear stress, $\bar{\sigma}_4$, decreases with the increase of the number of layers while the shear stress, $\bar{\sigma}_5$, increases with the increase of the number of layers. Once again, the shear stresses are very sensitive to the inclusion of the elastic foundations. When one of the foundations or both are included the shear stress $\bar{\sigma}_4$ decreases, while $\bar{\sigma}_5$ increases. The inclusion of elastic foundations gives the smallest shear stresses, $\bar{\sigma}_5$, for the square plates.

Additional results for the deflection, $\bar{w}$, the in-plane longitudinal stress, $\bar{\sigma}_1$, and the transverse shear stress, $\bar{\sigma}_5$, are presented in Tables 7-9 according to different symmetric and non-symmetric angle-ply schemes. The side-to-thickness ratio is fixed to be $a/h = 10$ and the two thermal loading are equal (i.e., $\bar{T}_2 = \bar{T}_3$). The symmetric cross-ply (0°/90°/90°/0°), anti-symmetric angle-ply (0°/45°/90°/ − 45°), and semi-symmetric angle-ply (0°/45°/90°/−45°/0°) plates are used in this example. The case (0°/45°/90°/−45°) gives the smallest deflections for all aspect ratios and elastic foundations. It

Table 5. Dimensionless transverse shear stress $\bar{\sigma}_4(0)$ for cross-ply rectangular plates subjected to thermal effect.

			0°/90°/0°			**0°/90°/... 5 layers**			**0°/90°/... 9 layers**		
k_1	k_2	**Theory**	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	FPT	1.34424	2.30205	2.35137	0.59705	1.05202	1.04007	0.54536	0.97144	0.93246
		HPT	1.25992	2.16691	2.20669	0.59101	1.07064	1.08788	0.56633	1.03122	1.00829
		SPT	1.25566	2.16096	2.19907	0.59868	1.08844	1.10953	0.57416	1.04729	1.02458
0.01	0.0	FPT	0.32015	1.07915	1.86317	0.07036	0.44059	0.81861	0.00544	0.36228	0.72548
		HPT	0.28685	1.00168	1.74291	0.04549	0.42853	0.85197	-0.01615	0.36739	0.78042
		SPT	0.28358	0.99640	1.73569	0.04267	0.43288	0.86820	-0.01860	0.37125	0.79238
0.01	0.01	FPT	0.24811	0 .90986	1.64762	0.03452	0.35593	0.71976	-0.03029	0.27793	0.63248
		HPT	0.21864	0.84061	1.53796	0.00841	0.33965	0.74666	-0.05469	0.27548	0.67805
		SPT	0.21547	0.83545	1.53090	0.00468	0.34214	0.76048	-0.05783	0.27764	0.68805

Table 6. Dimensionless transverse shear stress $\bar{\sigma}_5(0)$ for cross-ply rectangular plates subjected to thermal effect.

			0°/90°/0°			**0°/90°/... 5 layers**			**0°/90°/... 9 layers**		
k_1	k_2	**Theory**	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	FTP	0.26885	0.92082	1.88109	0.52242	1.84104	3.64023	0.50641	1.80410	3.46341
		HPT	0.37671	1.22347	2.45771	0.62800	2.12626	4.19772	0.60125	2.03757	3.89820
		SPT	0.39582	1.27258	2.54881	0.63934	2.14720	4.23776	0.61806	2.07163	3.95715
0.01	0.0	FPT	1.73647	1.69613	2.01372	3.40058	3.26649	3.85624	3.31931	3.13366	3.65142
		HPT	2.29023	2.23132	2.62743	4.02328	3.82041	4.45701	3.81266	3.56621	4.11651
		SPT	2.37866	2.31662	2.72430	4.07359	3.86296	4.50103	3.87626	3.62350	4.17900
0.01	0.01	FPT	1.83972	1.80346	2.07228	3.59639	3.46385	3.95265	3.50545	3.31776	3.73589
		HPT	2.42437	2.37064	2.70243	4.25411	4.05490	4.57275	4.02517	3.77786	4.21460
		SPT	2.51760	2.46091	2.80186	4.30711	4.10045	4.61853	4.09189	3.83838	4.27867

Table 7. Dimensionless deflections $\bar{w}$ for different schemes of composite plates subjected to thermal effect.

			0°/90°/90°/0°			**0°/45°/90°/ − 45°**			**0°/45°/90°/ − 45°/0°**		
k_1	k_2	**Theory**	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	CPT	6.07021	3.64890	1.16727	5.62469	3.12891	1.07901	5.97254	3.54539	1.18655
		FPT	12.11873	7.25799	2.31611	11.28145	6.28095	2.16678	11.92895	7.06340	2.35768
		HPT	10.82906	6.47497	2.05900	10.10358	5.62424	1.93913	10.66942	6.31809	2.10693
		SPT	10.66382	6.37584	2.02722	9.95196	5.54003	1.91011	10.50826	6.22318	2.07529
0.1	0.0	CPT	0.80329	0.85185	0.71618	0.67853	0.75650	0.69023	0.79952	0.84880	0.72856
		FPT	1.51441	1.59071	1.33212	1.29009	1.41825	1.29015	1.51201	1.58737	1.35392
		HPT	1.33995	1.41048	1.18666	1.15197	1.26862	1.15744	1.34451	1.41462	1.21233
		SPT	1.31830	1.38800	1.16837	1.13449	1.24961	1.14041	1.32353	1.39285	1.19425
0.1	0.1	CPT	0.72562	0.73990	0.60148	0.61213	0.65802	0.58603	0.72234	0.73800	0.61199
		FPT	1.36686	1.37827	1.10122	1.16302	1.23025	1.07544	1.36496	1.37670	1.11886
		HPT	1.20923	1.22183	0.98146	1.03846	1.10040	0.96539	1.21365	1.22669	1.00231
		SPT	1.18967	1.20234	0.96634	1.02270	1.08392	0.95125	1.19471	1.20780	0.98739

gives also the smallest longitudinal stress, $\bar{\sigma}_1$, for square plates. However, it gives the highest shear stress, $\bar{\sigma}_5$.

In addition, some plots for deflection and stresses are given in Figures 2-8. The symmetric cross-ply (0°/90°/0°) and (0°/90°/90°/0°), anti-symmetric angle-ply (0°/45°/90°/ − 45°), and semi-symmetric angle-ply (0°/45°/90°/ − 45°/0°) plates are used in all plots. The variation of the deflection, $\bar{w}$, versus the side-to-thickness ratio, a/h, is given in Figure 2. As is well known, all theories may be acceptable when

Table 8. Dimensionless in-plane longitudinal stress $\bar{\sigma}_1(0.5)$ for different schemes of composite plates subjected to thermal effect.

			$0°/90°/90°/0°$			$0°/45°/90°/-45°$			$0°/45°/90°/-45°$		
k_1	k_2	**Theory**	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	CPT	0.85986	2.52271	4.33426	1.06423	2.04618	2.68825	1.06268	2.71564	4.35164
		FPT	1.69545	4.93551	8.40063	2.10587	4.04487	5.30773	2.09229	5.32396	8.48498
		HPT	0.82876	3.73005	6.82637	1.43759	3.17179	4.29852	1.17924	4.07448	6.90980
		SPT	0.78930	3.64627	6.69510	1.39836	3.10617	4.21549	1.13350	3.98505	6.77821
0.01	0.0	CPT	2.76110	3.29840	4.41286	2.34268	2.51502	2.74803	2.86900	3.42571	4.42907
		FPT	5.38108	6.43324	8.55269	4.60146	4.96212	5.42552	5.60293	6.70127	8.63533
		HPT	4.14336	5.07545	6.96274	3.68877	4.00313	4.40731	4.34434	5.31897	7.04645
		SPT	4.05460	4.97160	6.82941	3.61681	3.92568	4.32285	4.25246	5.21168	6.91295
0.01	0.01	CPT	2.89617	3.40831	4.44832	2.42939	2.58230	2.77524	2.99805	3.52725	4.46400
		FPT	5.63558	6.64048	8.62027	4.76611	5.09048	5.47829	5.84708	6.89367	8.70212
		HPT	4.37110	5.26117	7.02338	3.83705	4.11943	4.45608	4.56382	5.49257	7.10719
		SPT	4.27886	5.15451	6.88913	3.76293	4.04032	4.37098	4.46869	5.38276	6.97284

Table 9. Dimensionless transverse shear stress $\bar{\sigma}_5(0)$ for different schemes of composite plates subjected to thermal effect.

			$0°/90°/90°/0°$			$0°/45°/90°/-45°$			$0°/45°/90°/-45°/0°$		
k_1	k_2	**Theory**	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$	$s=0.5$	$s=1.0$	$s=2.0$
0.0	0.0	FTP	1.15567	2.01051	2.01430	5.11625	5.07104	13.40317	-0.27152	-4.73863	4.60851
		HPT	1.05343	1.86371	1.88757	6.29296	6.24473	16.38253	-0.17701	-2.70868	7.78642
		SPT	1.05138	1.86457	1.89088	6.45341	6.40491	16.78171	-0.16012	-2.54489	8.42357
0.01	0.0	FPT	0.20725	0.89310	1.59236	5.11625	5.07104	13.40317	1.74696	0.80227	4.69771
		HPT	0.16057	0.80142	1.48444	6.29296	6.24473	16.38253	2.26837	1.38910	8.54529
		SPT	0.15592	0.79771	1.48565	6.45341	6.40491	16.78171	2.34184	1.46904	9.42773
0.01	0.01	FPT	0.14176	0.73849	1.40483	5.11625	5.07104	13.40317	1.86774	1.15096	4.75231
		HPT	0.9922	0.65477	1.30519	6.29296	6.24473	16.38253	2.41793	1.73245	9.10914
		SPT	0.09442	0.65048	1.30546	6.45341	6.40491	16.78171	2.49524	1.81319	10.20907

$a/h \geq 30$. The CPT gives accurate deflections for thin plates only. The FPT failed to give accurate deflections for plates with $a/h < 20$. The HPT gives accurate deflections that are compared well with those of the SPT for all side-to-thickness ratios.

Figure 3 shows that the in-plane longitudinal stresses, $\bar{\sigma}_1$, due to CPT and FPT have the same response for symmetric or semi-symmetric laminated plates. Both CPT and FPT failed to get accurate stress compared with HPT and SPT. The FPT overestimates $\bar{\sigma}_1$ while CPT underestimates it. As a non-expected case, the CPT gives in-plane stress more accurately than that of the FPT for anti-symmetric angle-ply $(0°/45°/90°/-45°)$ when $a/h \leq 5$.

Figure 4 shows that the shear deformation theories FPT, HPT and SPT give reliable in-plane normal stress, $\bar{\sigma}_2$. The CPT failed to get accurate normal stresses except for the symmetric cross-ply $(0°/90°/0°)$ and $(0°/90°/90°/0°)$ when $8 \leq a/h \leq 18$ and $10 \leq a/h \leq 15$, respectively.

The distributions of in-plane stresses and transverse shear stresses through-the-thickness of the laminated plates are plotted in Figures 5-8. The value of the side-to-thickness ratio is taken to be $a/h = 5$. The stress $\bar{\sigma}_2$ is given for rectangular plates with $a/b = 3$ while other stresses are given for rectangular plates with $b/a = 3$. Figures 5 and 6 show that the CPT and FPT failed to get accurate in-plane stresses through the plate thickness. They give suitable in-plane stresses in some position, at the center of the plate and near its edges only. The FPT gives accurate normal stresses, $\bar{\sigma}_2$, in the core layers of the symmetric 3-layer and 4-layer cross-ply rectangular plates. Finally, Figures 7 and 8 display the distributions of the transverse shear stresses through-the-thickness of the laminated plates. The FPT may be acceptable at the interfaces and the center of the symmetric cross-ply laminated plates. The HPT gives shear stress with very few relative errors compared with the SPT for symmetric plates. These

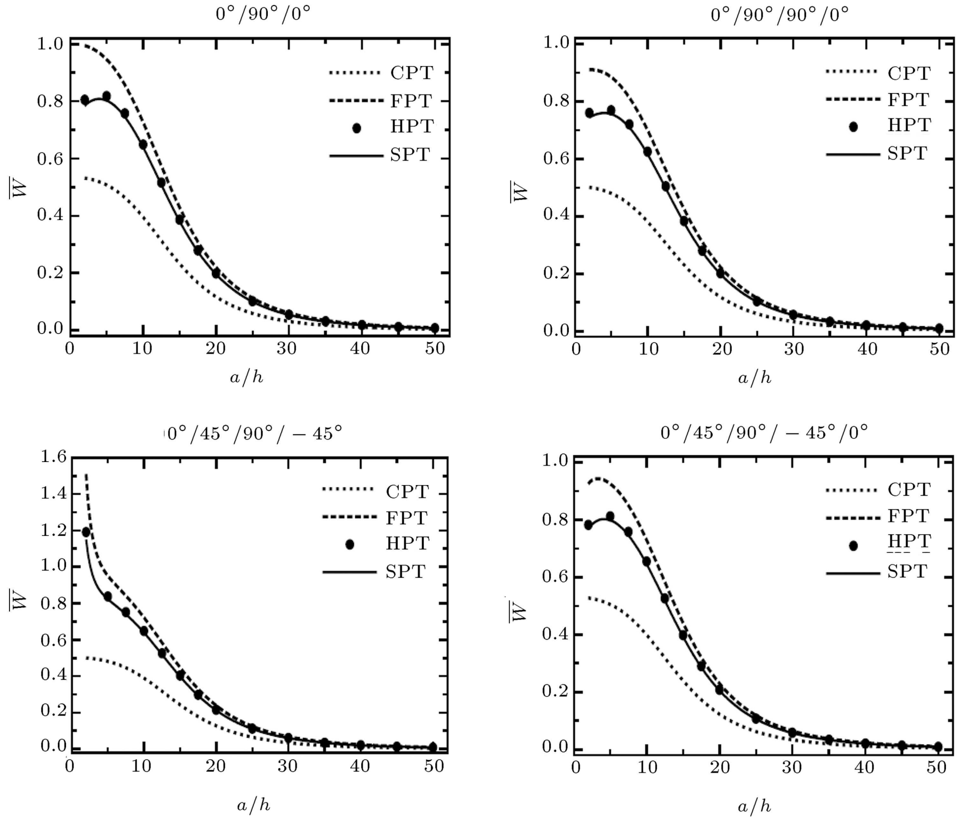

Figure 2. Dimensionless deflection $\bar{w}$ vs the side-to-thickness ratio, a/h, of laminated plates resting on elastic foundations.

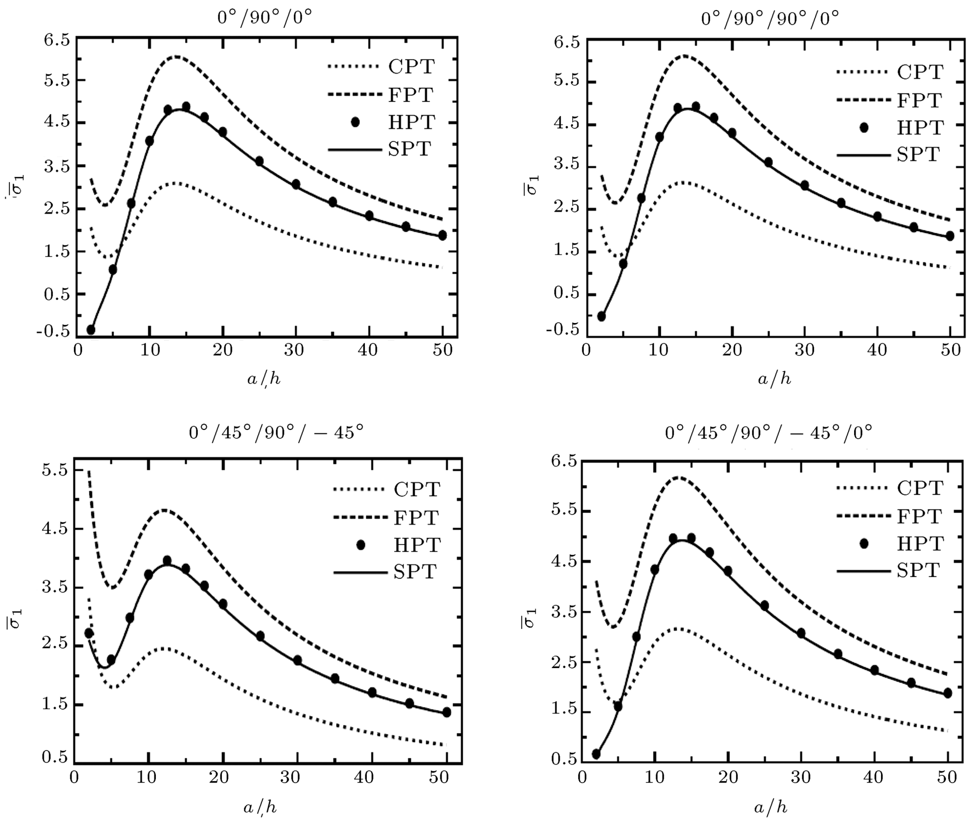

Figure 3. Dimensionless in-plane longitudinal stress, $\bar{\sigma}_1$, vs the side-to-thickness ratio, a/h, of laminated plates resting on elastic foundations.

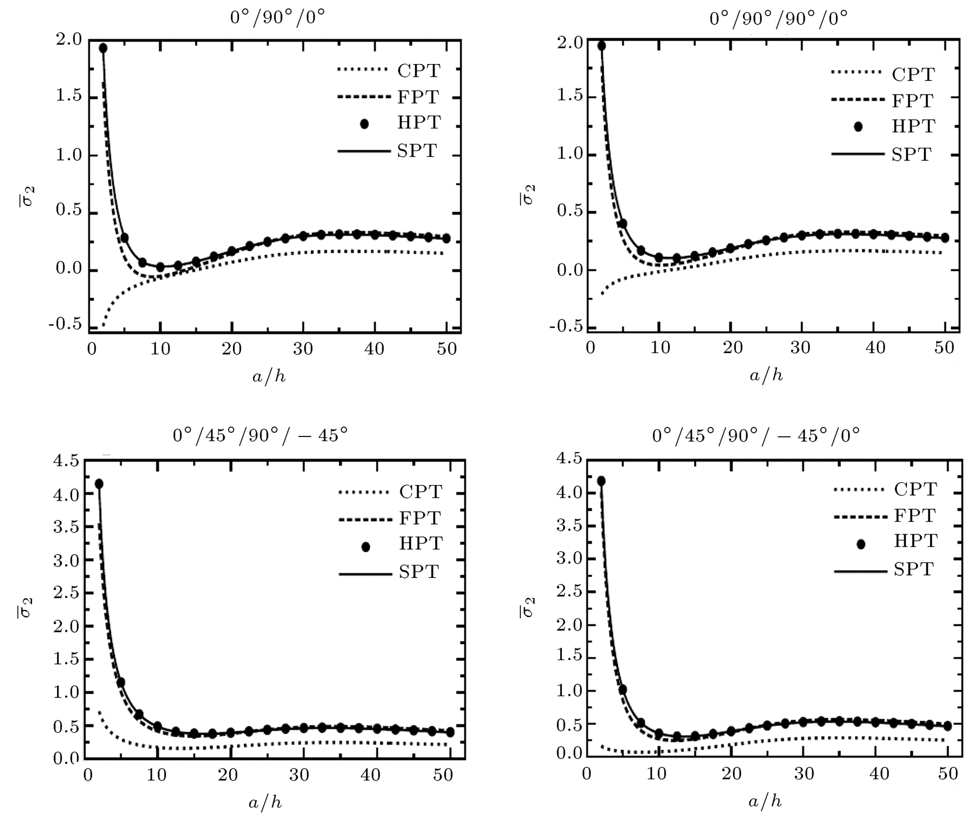

Figure 4. Dimensionless in-plane normal stress, $\bar{\sigma}_2$, vs the side-to-thickness ratio, a/h, of laminated plates resting on elastic foundations.

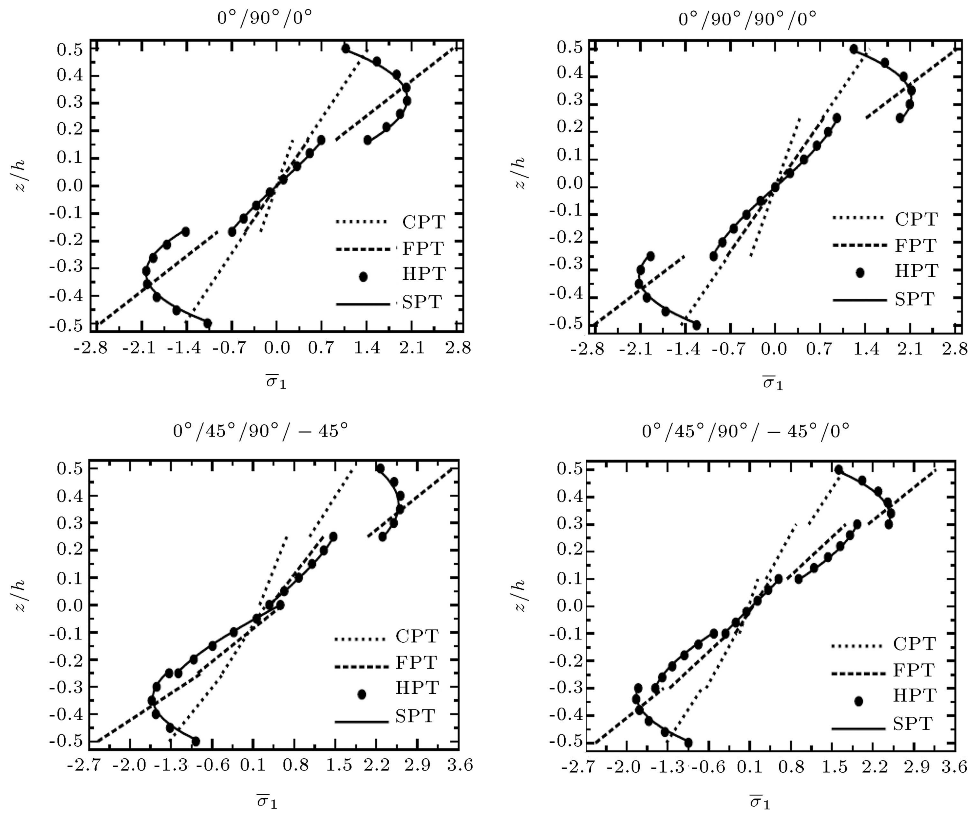

Figure 5. The through-the-thickness distribution of dimensionless in-plane longitudinal stress, $\bar{\sigma}_1$, of laminated plates resting on elastic foundations.

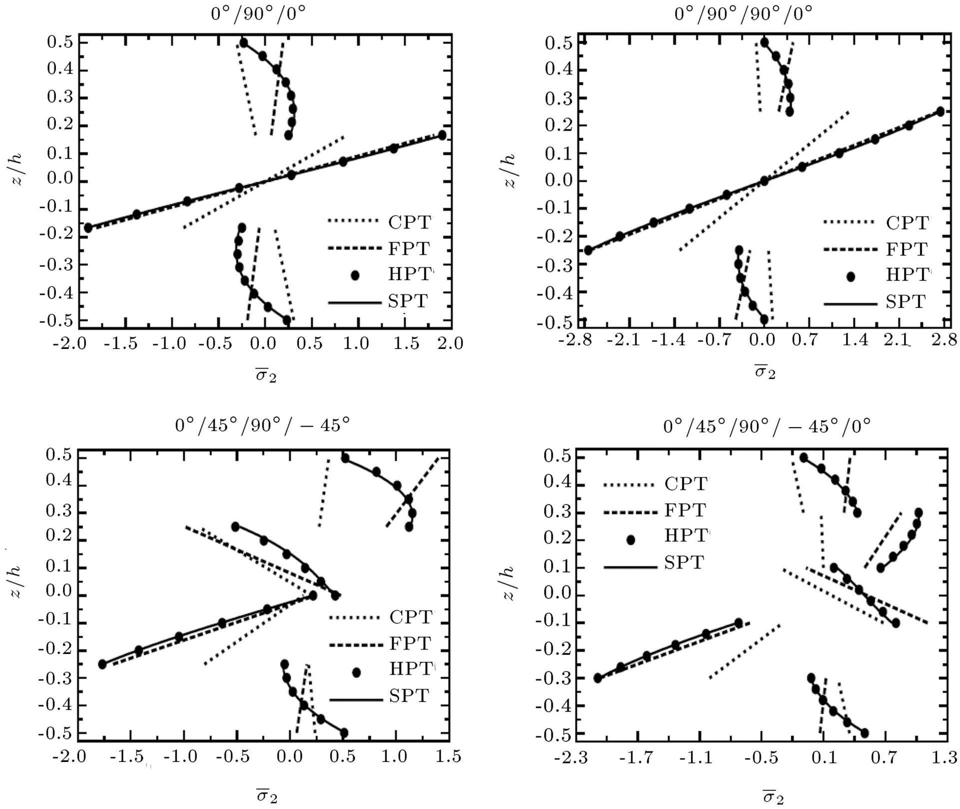

Figure 6. The through-the-thickness distribution of dimensionless in-plane normal stress, $\bar{\sigma}_2$, of laminated plates resting on elastic foundations.

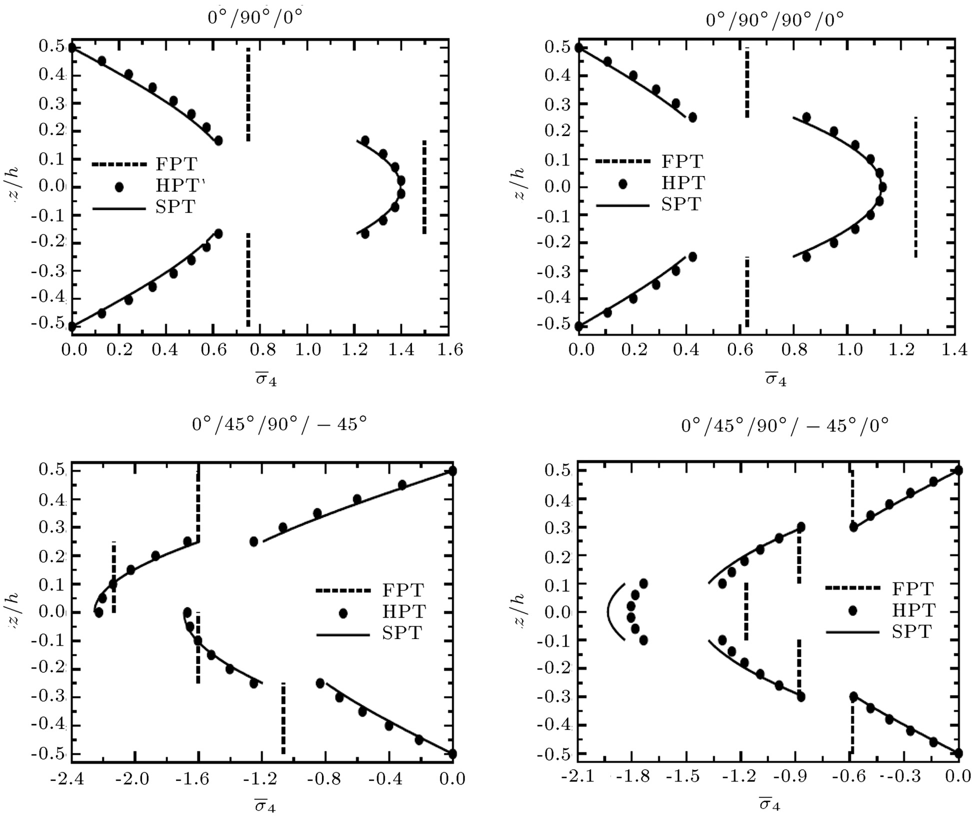

Figure 7. The through-the-thickness distribution of dimensionless transverse shear stress, $\bar{\sigma}_4$, of laminated plates resting on elastic foundations.

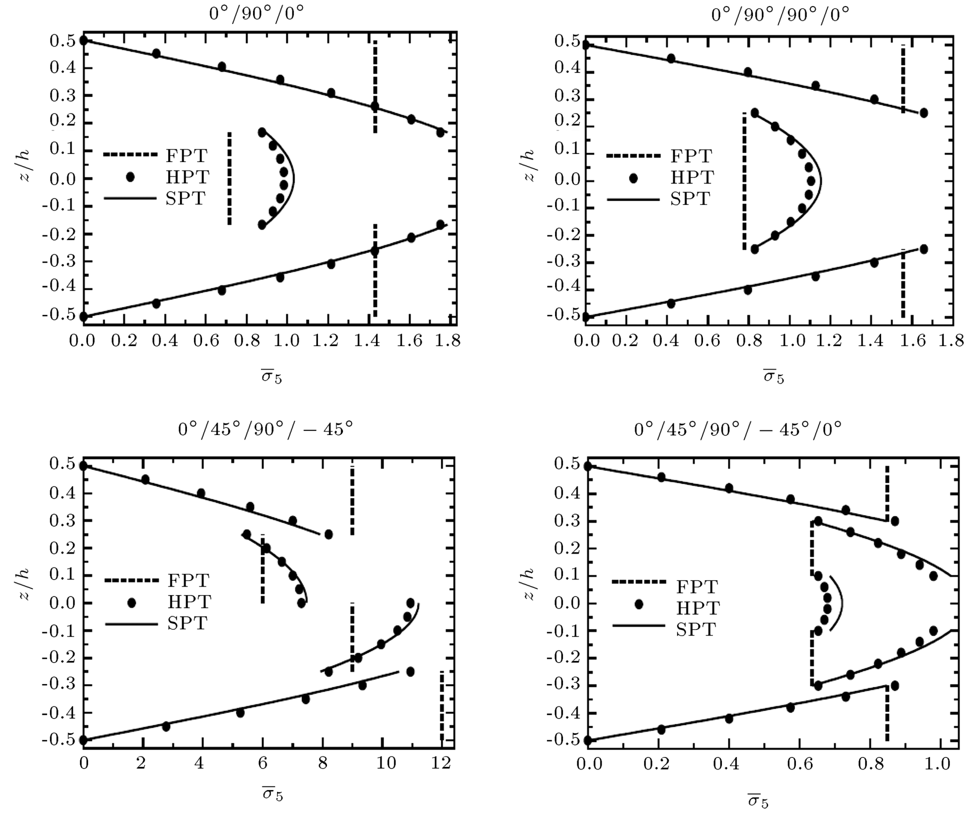

Figure 8. The through-the-thickness distribution of dimensionless transverse shear stress, $\bar{\sigma}_5$, of laminated plates resting on elastic foundations.

relative errors may increase at the core-interfaces of the (0°/45°/90°/ − 45°/0°) plate. Moreover, HPT failed to get accurate shear stresses through the core layer of right angle for the symmetric or semi-symmetric laminated plates.

5. Conclusion

Numerical results are very sensitive to the variation of elastic foundation parameters and the inclusion of shear deformation effect. The CPT is independent of nonlinear term of temperature distribution. It gives inaccurate results compared with the shear deformation theories. The FPT failed to get any accurate transverse shear stresses for plate follows Pasternak's model. The FPT gives results with higher relative errors compared with the SPT. The HPT gives results close to those due to the SPT. The relative errors between SPT and HPT may be decreased for plates follow Pasternak's model. The shear deformation theories of SPT and HPT give accurate deflections and stresses compared with FPT. The relative errors between SPT and HPT, for deflections and all stresses except $\bar{\sigma}_1$, are increased for plates follow Winkler's or Pasternak's models. Additional results for deflections and stresses are tabulated here to serve as bench marks for future comparisons.

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