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Nonlinear oscillators with rational terms: A new semi-analytical technique

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Abstract. This paper presents a new generalization of the standard homotopy analysis approach towards solving nonlinear oscillators equations with rational terms. By using this method, analytical approximations to the frequency of these oscillators and periodic solutions are calculated. Excellent agreement of the approximate frequencies and periodic solutions with exact ones is demonstrated and discussed. It is shown that this method is very simple, effective and convenient for solving nonlinear oscillator problems with rational terms.

1. Introduction

Many engineering oscillatory systems are governed by nonlinear differential equations [1-3]. In general, such problems are not acquiescent to exact treatment. Also, it is not easy to solve such nonlinear differential equations, especially analytically. In recent years, many new solution methodologies [4-21] for nonlinear oscillators problems have been developed. These nonlinear oscillator problems arise in the application of the dynamics of a particle moving in cubic potential, ship dynamics, oscillations of one dimensional structural systems with initial curvature and oscillation of the human eardrum.

In 1990s, Liao [22,23] proposed a general analytical method for nonlinear problems, namely the Homotopy Analysis Method (HAM), which stems from the basic ideas of homotopy in topology. This technique has lucratively been applied in the past few years to several nonlinear problems, such as nonlinear oscillators with discontinuities [24-26], boundary layer flow [27], heat transfer [28], delayed differential equation [29], chaotic dynamical systems [30] and fractional diffusion equations [31].

The purpose of this paper is to propose a generalization of the standard homotopy analysis method to nonlinear oscillators with rational terms. It is shown how one can control the convergence of approximate solutions and make a fast convergence by applying the present method. Also, it is revealed that the approximate solutions given by the proposed method are more accurate than the solution given by the standard Homotopy Analysis Method (HAM).

2. Oscillators with rational terms and generalization of standard HAM

In order to clarify this method, it is possible to consider the nonlinear differential equation:

$$x''(t) + \alpha x(t) + \frac{x^{2n+1}(t)}{1+\beta x^2(t)} = 0, \tag{1}$$

which corresponds to a set of nonlinear oscillators with rational terms where $x(t)$ is an unknown real function

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and α and β are known parameters. Moreover, the initial conditions for solving Eq. (1) are:

$$x(0)=A, \qquad x'(0)=0. \tag{2}$$

Now, Eq. (1) can be written in the form:

$$x''(t)+\alpha x(t)+\beta x''(t)x^2(t)+\alpha\beta x^3(t)+x^{2n+1}(t)=0. \tag{3}$$

To solve Eq. (3) by the generalization of standard HAM, a new independent variable, $\tau=\omega t$, is introduced. Substituting $\tau=\omega t$ and $x(t)=X(\tau)$ in Eqs. (3) and (2), we have:

$$\omega^2X''(\tau)+\alpha X(\tau)+\beta\omega^2X''(\tau)X^2(\tau)+\alpha\beta X^3(\tau) + X^{2n+1}(\tau)=0, \tag{4}$$

$$X(0)=A, \qquad X'(0)=0, \tag{5}$$

where prime denotes the derivative with respect to τ.

The nonlinear oscillators with rational terms are periodic motions with the frequency as ω. Thus, X can be expressed by a set of base functions, such as:

$$\{e_j(\tau)|j=1,2,3,\cdots\}=\{\cos(j\tau)|j=1,2,3,\cdots\},$$

so that:

$$X(\tau)=\sum_{j=0}^{\infty}d_j\cos(j\tau), \tag{6}$$

where d_j, $j=1,2,\cdots$ are the coefficients.

Let ω_0 and $X_0(\tau)$ denote the initial approximations of ω and $X(\tau)$, respectively. Considering the rule of solution expression Eq. (6) and initial conditions, it is obvious that the initial guess of solutions can be described as:

$$X_0(\tau)=A\cos(\tau). \tag{7}$$

Under the rule of the solution expression denoted by Eq. (6), it is obvious one should choose the auxiliary linear operator:

$$\mathrm{L}[X(\tau),\omega_0]=\omega_0^2[X''(\tau)+X(\tau)], \tag{8}$$

with the property:

$$\mathrm{L}[C_1\sin(\tau)+C_2\cos(\tau)]=0, \tag{9}$$

where C_1 and C_2 are constants. From Eq. (4), we define a nonlinear operator:

$$N[X(\tau),\omega,\alpha,\beta]=\omega^2X''(\tau)+\alpha X(\tau) + \beta\omega^2X''(\tau)X^2(\tau)+\alpha\beta X^3(\tau)+X^{2n+1}(\tau). \tag{10}$$

By means of the homotopy technique, we construct a general zero-order deformation equation as follows:

$$(1-A(q;c))\mathrm{L}[\phi(\tau;q)-X_0(\tau)] = \hbar H_1(\tau)qN[\phi(\tau;q),\Omega(q),\alpha,\beta], \tag{11}$$

where:

$$A(q;c)=(1-c)\sum_{j=1}^{\infty}c^{j-1}q^j,$$

$$|c|<1, \qquad A(1;c)=1, \qquad A(0;c)=0. \tag{12}$$

When the parameter, q, increases from 0 to 1, the solution, $\phi(\tau;q)$, varies from $X_0(\tau)$ to $X(\tau)$, so does the $\Omega(q)$ from ω_0 to ω. If this continuous variation is smooth enough, the Maclaurin's series, with respect to q, can be constructed for $\phi(\tau;q)$ and $\Omega(q)$ as follows:

$$\phi(\tau;q)=X_0(\tau)+\sum_{m=1}^{\infty}X_mq^m,$$

$$\Omega(q)=\omega_0+\sum_{m=1}^{\infty}\omega_mq^m, \tag{13}$$

where:

$$X_m(\tau)=\frac{1}{m!}\frac{\partial^m\phi(\tau;q)}{\partial q^m}|_{q=0},$$

$$\omega_m=\frac{1}{m!}\frac{\partial^m\Omega(q)}{\partial q^m}|_{q=0}. \tag{14}$$

Assume that the auxiliary parameters, $\hbar$ and c, are chosen where the power series defined in Eq. (13) are convergent at $q=1$. Then, we have the series solutions:

$$X(\tau)=X_0(\tau)+\sum_{m=1}^{\infty}X_m,$$

$$\omega=\omega_0+\sum_{m=1}^{\infty}\omega_m. \tag{15}$$

For the sake of simplicity, we introduce the following vectors

$$\overrightarrow{X}_m=\{X_0,X_1,\cdots,X_m\},$$

$$\overrightarrow{\omega}_m=\{\omega_0,\omega_1,\cdots,\omega_m\}. \tag{16}$$

By differentiating the zeroth-order deformation equation m times with respect to q, then dividing the equation by $m!$ and setting $q = 0$, the mth-order deformation equation is formulated as follows:

$$\text{Ł}[X_m(\tau) - \sum_{k=1}^{m-1}(1-c)c^{m-k-1}X_k(\tau)] = \hbar H_1(\tau)R_m(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m), \qquad m \geq 1, \tag{17}$$

$$X_m(0) = 0, \qquad X'_m(0) = 0, \tag{18}$$

where:

$$\begin{aligned} R_m\left(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m\right) &= \sum_{j=0}^{m-1} X''_{m-1-j}\sum_{i=0}^{j}\omega_i\omega_{j-i} \\ &+ \alpha X_{m-1} + \beta\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}X''_i\sum_{s=0}^{j-i}\omega_s\omega_{j-i-s}\right) \\ &\times \sum_{k=0}^{m-1-j}X_kX_{m-1-j-k} + \alpha\beta\sum_{j=0}^{m-1}X_j \\ &\times \sum_{k=0}^{m-1-j}X_kX_{m-1-j-k} + \sum_{j_1=0}^{m-1}X_{m-1-j_1} \\ &\times \sum_{j_2=0}^{j_1}X_{j_1-j_2}\cdots\sum_{j_{2n}=0}^{j_{2n-1}}X_{j_{2n-1}-j_{2n}}X_{j_{2n}}, \\ & m \geq 1. \end{aligned} \tag{19}$$

Under the rule of the solution expression denoted by Eq. (6), the auxiliary function, $H_1(\tau)$, can be chosen as $H_1(\tau) = 1$. It should be emphasized that $R_m(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m)$ is a function of X_i and ω_i, where $i = 0, 1, \cdots, m-1$.

Furthermore, $R_m(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m)$ can be expressed by:

$$R_m(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m) = \sum_{k=0}^{\psi(m)} c_{m,k}(\omega_{m-1})\cos((2k+1)\tau). \tag{20}$$

Under the rule of the solution expression, the solution $X_m(\tau)$ of Eq. (17) should not contain the constant term and the so-called secular term, $\tau\cos(\tau)$. To avoid the secular term, $R_m(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m)$ should not contain the constant term and the term $\cos(\tau)$, which leads to the additional algebraic equation for determining ω_{m-1}:

$$c_{m,0}(\omega_{m-1}) = 0. \tag{21}$$

Therefore, the general solution of Eq. (17) reads:

$$\begin{aligned} X_m(\tau) &= \sum_{k=1}^{m-1}(1-c)c^{m-k-1}X_k(\tau) \\ &+ \frac{\hbar}{\omega_0^2}\sum_{k=1}^{\psi(m)}\frac{c_{m,k}(\omega_{m-1})}{(1-(2k+1)^2)}\cos((2k+1)\tau) \\ &+ C_1\sin(\tau) + C_2\cos(\tau), \qquad m \geq 1, \end{aligned} \tag{22}$$

where C_1 and C_2 are two constants. Using the rule of the solution expression denoted by Eq. (6), we have $C_1 = 0$, and C_2 can be determined by:

$$X_m(0) = X_m(\pi), \qquad m = 1, 2, \cdots, \tag{23}$$

which ensure the amplitude equal to A.

Thus, the Nth order approximation can be given by:

$$X_N(\tau) \approx X_0(\tau) + \sum_{j=0}^{N}X_j(\tau),$$

$$\widetilde{\omega} \approx \omega_0 + \sum_{j=0}^{N}\omega_j. \tag{24}$$

We can derive the following remark instantly.

Remark 1: The value $c = 0$ reduces the present method to the standard HAM.

3. Numerical examples and discussion

The success of this homotopy lies in the fact that this technique provides a convenient way to increase the convergence region of the series solution. The $A(q;c)$ introduces a second auxiliary parameter into the zero-order deformation equation and proposes a generalization of the homotopy analysis method. The new auxiliary parameter adds a new dimension to the convergence region. The convergence the of series solution depends upon $\hbar$ and c. Thus, we need only to focus on properly choosing parameters $\hbar$ and c, so that the series solution be convergent.

For given α and β, the periodic solution, with the known amplitude A and the corresponding unknown frequency ω, can be determined by the semi-analytic technique mentioned above.

Note that the series solutions contain two unknown convergence-control parameters, which can be determined by the following procedure.

We first calculate the Averaged Residual Error (ARE) using the following formula:

$$E_N^n(\hbar, c) = \frac{1}{n+1}\sum_{j=0}^{n}(N[X_N(\tau_j), \widetilde{\omega}, \alpha, \beta])^2, \tag{25}$$

where:

$$\tau_j = \frac{j}{n},\ i = 0, 2, \cdots, n. \tag{26}$$

The best values of $\hbar$ and c are given by two nonlinear algebraic equations:

$$\frac{\partial E_N^n}{\partial \hbar} = 0, \qquad \frac{\partial E_N^n}{\partial c} = 0. \tag{27}$$

In this section, the presented technique in Section 2 is applied to solve the nonlinear oscillators with rational terms. For all examples, we choose $N = 5$ and $n = 20$.

3.1. Example 1

The governing non-dimensional equation of motion for the finite extensibility of the oscillator is [18]:

$$x''(t) + \frac{x(t)}{1 - x^2(t)} = 0, \qquad x(0) = A, \qquad (A < 1),$$

$$x'(0) = 0, \tag{28}$$

with the exact frequency:

$$\omega_{ex} = \frac{2\pi}{4\int_0^A \frac{dx}{\sqrt{[\ln(1-x^2) - \ln(1-A^2)]}}}. \tag{29}$$

In view of Eq. (19), we can construct the following equation:

$$R_m\left(\overrightarrow{X}_{m-1}, \overrightarrow{\omega}_m\right) = \sum_{j=0}^{m-1} X''_{m-1-j} \sum_{i=0}^{j} \omega_i \omega_{j-i} - \sum_{j=0}^{m-1}\left(\sum_{i=0}^{j} X''_i \sum_{s=0}^{j-i} \omega_s \omega_{j-i-s}\right) \sum_{k=0}^{m-1-j} X_k X_{m-1-j-k} + X_{m-1}, \qquad m \geq 1. \tag{30}$$

Thus, for the first-order approximation, $R_1(X_0, \omega_0)$ leads to:

$$R_1(X_0, \omega_0) = \left(-A\omega_0^2 + A + \frac{3}{4}A^3\omega_0^2\right)\cos(\tau) + \frac{1}{4}A^3\omega_0^2\cos(3\tau). \tag{31}$$

Therefore, Eq. (31) gives:

$$\omega_0 = \frac{2}{\sqrt{4 - 3A^2}}. \tag{32}$$

For $m = 2$, we have:

$$\omega_1 = -\frac{5\hbar}{32}\frac{A^4\sqrt{4 - 3A^2}}{(16 - 24A^2 + 9A^4)}. \tag{33}$$

Also, for $m = 3$, we have Eq. (34) shown in Box I. For a given value of A, we find the "optimal" values of $\hbar$ and c by solving the algebraic equations, $\frac{dE_5^{20}}{d\hbar} = 0$ and $\frac{dE_5^{20}}{dc} = 0$. In the particular cases:

(a) When $A = 0.1$, we obtained the following "optimal" values:

$$\hbar = -0.5618,\ c = -0.1168, \tag{35}$$

which give the corresponding minimum, ARE $E_5^{20} = 3.6922E - 16$.

The approximate frequency in the case $\hbar = -0.5618$ and $c = -0.1168$ is $\widetilde{\omega} = 1.003773204$ and, therefore, the absolute error between the approximate and the exact frequency is $8.9662E-9$. The absolute error function with $N = 5$ has been plotted for $\hbar = -0.6811$ and $c = -0.1101$ in Figure 1.

(b) When $A = 0.5$, we obtain the following expressions:

$$\hbar = -1.1641, \qquad c = 7.5484E - 3,$$

$$E_5^{20} = 4.2279E - 14. \tag{36}$$

The approximate frequency in the case $\hbar = -1.1641$ and $c = 7.5484E - 3$ is $\widetilde{\omega} = 1.111347529$,

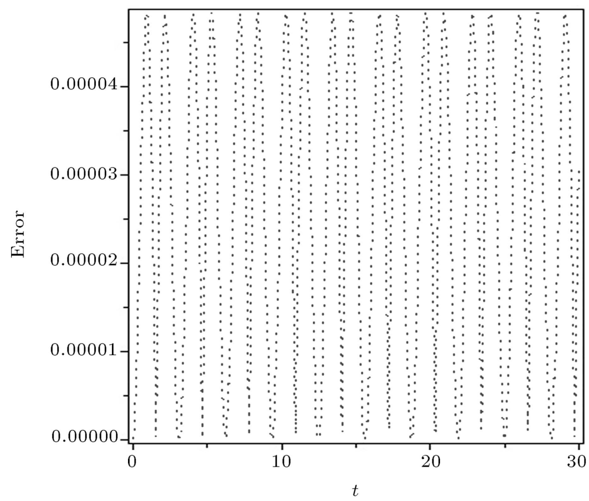

Figure 1. The absolute error with $\hbar = -0.5618$ and $c = -0.1168$ for $A = 0.1$ (Example 1).

$$\omega_2 = \frac{(881\hbar A^4 - 1920A^2 + 1920cA^2 - 3328\hbar A^2 - 2560c + 2560\hbar + 2560)\hbar A^4}{4096(27A^6 - 108A^4 + 144A^2 - 64)(4 - 3A^2)^{-\frac{1}{2}}}. \tag{34}$$

Box I.

and, therefore, the absolute error between the approximate and the exact frequency is $1.6197E-8$. Figure 2 shows the error between the exact solution and the approximate solution for $\hbar = -1.1641$ and $c = 7.5484E-3$.

(c) When $A = 0.9$, we obtain the following results:

$$\hbar = -1.7303, \qquad c = 2.7808E-2,$$

$$E_5^{20} = 1.5816E-4. \tag{37}$$

The approximate frequency in the case $\hbar = -1.7303$ and $c = 2.7808E-2$ is $\tilde{\omega} = 1.716266773$, and, therefore, the absolute error between the approximate and the exact frequency is $8.9812E-4$. Figure 3 shows the error between the exact solution and the approximate solution for $\hbar = -1.7303$ and $c = 2.7808E-2$.

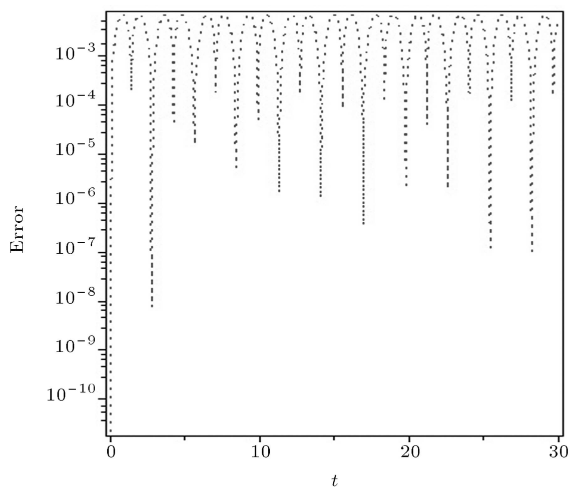

Figure 2. The absolute error with $\hbar = -1.1641$ and $c = 7.5484E-3$ for $A = 0.5$ (Example 1).

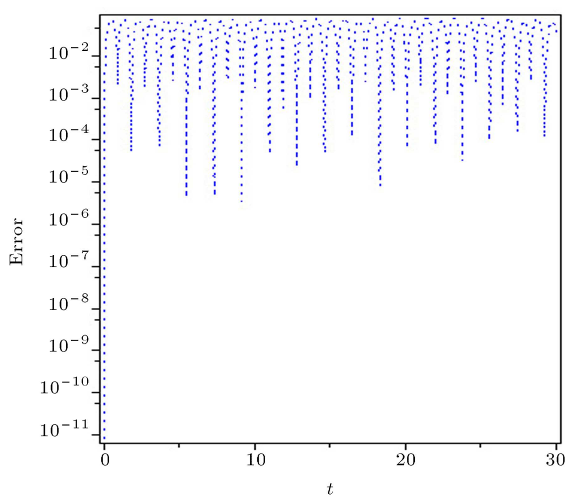

Figure 3. The absolute error with $\hbar = -1.7303$ and $c = 2.7808E-2$ for $A = 0.9$ (Example 1).

3.2. Example 2

The governing non-dimensional equation of motion for the the Duffing-harmonic oscillator is [3]:

$$x''(t)+\frac{x^3(t)}{1+x^2(t)}=0, \qquad x(0)=A, \qquad x'(0)=0, \tag{38}$$

with the exact frequency:

$$\omega_{ex} = \frac{2\pi}{4\int_0^A \frac{dx}{\sqrt{(A^2-x^2)+[\ln(1+x^2)-\ln(1+A^2)]}}}. \tag{39}$$

Proceeding in a similar manner, we have:

$$R_m\left(\vec{X}_{m-1}, \vec{\omega}_m\right) = \sum_{j=0}^{m-1} X''_{m-1-j}\sum_{i=0}^{j}\omega_i\omega_{j-i}$$

$$-\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j} X''_i \sum_{s=0}^{j-i}\omega_s\omega_{j-i-s}\right)\sum_{k=0}^{m-1-j} X_k X_{m-1-j-k}$$

$$+\sum_{j_1=0}^{m-1} X_{m-1-j_1}\sum_{j_2=0}^{j_1} X_{j_1-j_2}X_{j_2}, \qquad m \geq 1. \tag{40}$$

Thus, for the first-order approximation, $R_1(X_0, \omega_0)$ leads to:

$$R_1(X_0,\omega_0) = \left(-A\omega_0^2 - \frac{3}{4}A^3\omega_0^2 + \frac{3}{4}A^3\right)\cos(\tau)$$

$$+\left(\frac{1}{4}A^3 - \frac{1}{4}A^3\omega_0^2\right)\cos(3\tau). \tag{41}$$

Therefore, Eq. (41) gives:

$$\omega_0 = \frac{3A}{\sqrt{12+9A^2}}. \tag{42}$$

For $m = 2$, we have:

$$\omega_1 = \frac{\hbar}{12}\,\frac{A(1+2A^2)\sqrt{12+9A^2}}{16+24A^2+9A^4}. \tag{43}$$

Also, for $m = 3$, we have Eq. (44) shown in Box II. We consider different values of A and we compare our results with exact results. We obtain the "optimal" values of $\hbar$ and c by solving the algebraic equations, $\frac{dE_5^{20}}{d\hbar} = 0$ and $\frac{dE_5^{20}}{dc} = 0$. In the particular cases:

(a) For $A = 1$, we obtain the following expressions:

$$\hbar = -0.6811, \qquad c = -0.1101,$$

$$E_5^{20} = 4.3477E-10. \tag{45}$$

The approximate frequency in the case $\hbar = -0.6811$ and $c = -0.1101$ is $\tilde{\omega} = 0.6367925650$,

$$\omega_2 = \frac{(60A^6\hbar+264A^4\hbar+144A^4-144A^4c+302A^2\hbar+264A^2-264cA^2+95\hbar-96c+96)\hbar A}{288(64+144A^2+108A^4+27A^6)(12+9A^2)^{-\frac{1}{2}}}. \quad (44)$$

Box II.

and, therefore, the absolute error between the approximate and the exact frequency is 0.

For $\hbar \neq 0$ and $c = 0$, we obtained the following "optimal" value:

$$\hbar = -0.6280, \quad (46)$$

which gives the corresponding minimum, ARE $E_5^{20} = 8.2847E-9$. The approximate frequency in the case $\hbar = -0.6280$ and $c = 0$ is $\widetilde{\omega} = 0.6370742655$, and, therefore, the absolute error between the approximate and the exact frequency is $4.4237E-4$.

The absolute error function with $N = 5$ has been plotted for $\hbar = -0.6811$ and $c = -0.1101$ in Figure 4.

(b) When $A = 5$, we obtain the following results:

$$\hbar = -3.8386E-2, \qquad c = -0.2783,$$

$$E_5^{20} = 2.2359E-4. \quad (47)$$

The approximate frequency in the case $\hbar = -3.8386E-2$ and $c = -0.2783$ is $\widetilde{\omega} = 0.9693656739$, and, therefore, the absolute error between the approximate and the exact frequency is $2.4714E-3$.

For $\hbar \neq 0$ and $c = 0$, the optimal convergence occurs at $\hbar = -5.1936E-1$ and has an ARE of $E_8^{20} = 3.6572E-4$. The approximate frequency in this case is $\widetilde{\omega} = 0.9692835042$, and therefore, the absolute error between the approximate and the exact frequency is $2.5865E-4$.

Figure 5 shows the error between the exact solution and the approximate solution for $\hbar = -3.8386E-2$ and $c = -0.2783$.

(c) When $A = 20$, we obtain the following expressions:

$$\hbar = -2.5288E-3, \qquad c = -0.2971,$$

$$E_5^{20} = 1.2055E-2. \quad (48)$$

The approximate frequency in the case $\hbar = -2.5288E-3$ and $c = -0.2971$ is $\widetilde{\omega} = 0.9979759610$, and therefore, the absolute error between the approximate and the exact frequency is $3.5815E-4$.

In case of $c = 0$, the corresponding ARE E_5^{20} has the minimum $1.44839E-2$ at the "optimal" value $\hbar = -3.4336E-3$ and the approximate frequency in the case $\hbar = -3.4336E-3$ and $c = 0$ is $\widetilde{\omega} = .9979697311$. Therefore, the absolute error between the approximate and the exact frequency is $3.6791E-4$. Figure 6 shows the error between the exact solution and the approximate solution for $\hbar = -2.5288E-3$ and $c = -0.2971$.

4. Concluding remarks

In this work, approximate analytical expressions for the solution and the period of nonlinear oscillators with rational elastic terms are obtained by means of a newly

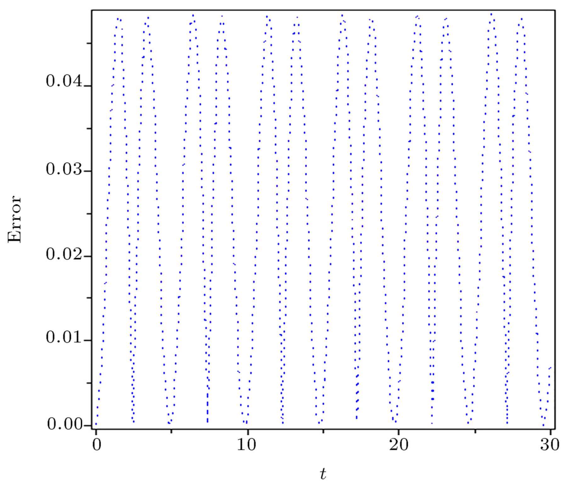

Figure 4. The absolute error with $\hbar = -0.6811$ and $c = -0.1101$ for $A = 1$ (Example 2).

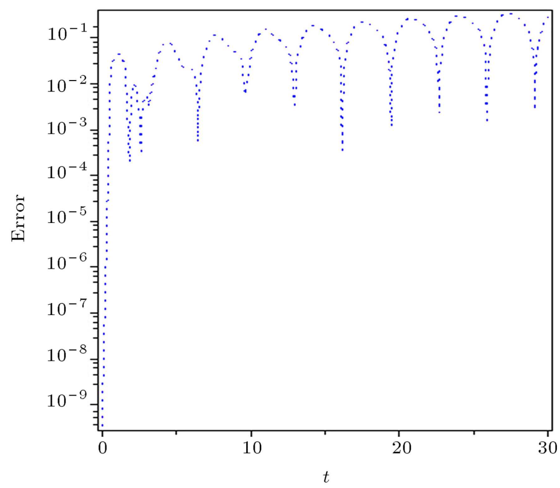

Figure 5. The absolute error with $\hbar = -3.8386E-2$ and $c = -0.2783$ for $A = 5$ (Example 2).

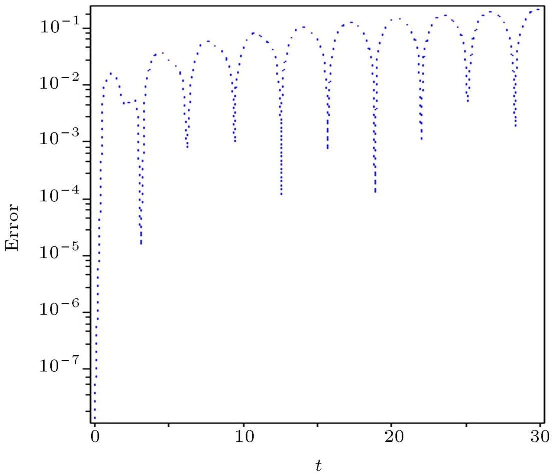

Figure 6. The absolute error with $\hbar = -2.5288E-3$, $c = -0.2971$ for $A = 20$ (Example 2).

developed method, namely, generalization of standard HAM. Numerical comparisons presented confirm the accuracy of the present method for these nonlinear oscillators. In this method we control the convergence using two unknown convergence-control parameters $\hbar$ and c which are optimally determined. Because of its accuracy, simplicity and reliability, it is believed that the present method can be further generalized for various nonlinear oscillators.

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