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A new generalized approach for implementing any homogeneous and non-homogeneous boundary conditions in the generalized differential quadrature analysis of beams

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Abstract. In this paper, a new way of implementing any homogeneous and non-homogeneous boundary conditions in the Generalized Differential Quadrature (GDQ) analysis of beams is presented. Like analytical methods in the solution of a differential equation, this approach governs the general solution of GDQ discrete equations for the differential equation of beams by assuming some unknown constants, and satisfies the boundary conditions in the general solution. Then, unknown constants are evaluated by solving the resultant algebraic equation system. Thus, the particular solution for the beam equilibrium differential equation is obtained by the GDQ method. As described, this approach satisfies the boundary conditions in the general solution, so, it is referred to as SBCGS (Satisfying the Boundary Conditions in the General Solution). The SBCGS approach can satisfy any type of boundary condition exactly at boundary points with high accuracy and can easily be implemented for each type of boundary condition. So, this approach overcomes the drawbacks of previous approaches by its generality and simplicity. At the end of this paper, a comparison of the SBCGS approach, using the method of substitution of boundary conditions into governing equations (the SBCGE approach), is made by their accuracy with the analysis of beam equilibrium under lateral loading with combinations of simply supported and clamped boundary conditions. Other boundary conditions and different numbers of mesh point results are also discussed for the SBCGS approach only. The results indicate that although the SBCGS approach is essentially very similar to some other approaches, like SBCGE, it is an easy and powerful method for implementation of any boundary condition to the GDQ governing equations, and provides highly accurate results.

1. Introduction

Most engineering problems are governed by a set of Partial Differential Equations (PDEs) with proper boundary conditions [1]. Currently, there are many available numerical discretization techniques, like conventional Finite Element (FE) and Finite Difference (FD), to solve engineering problems. These methods need a large number of grid points to achieve an acceptable degree of accuracy, and so, lots of virtual storage and computational effort are required. In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Bellman et al. (1971, 1972) [2,3] introduced

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the method of Differential Quadrature (DQ), where a partial derivative of a function, with respect to a coordinate direction, is expressed as a linear weighted sum of all functional values at all mesh points along that direction [1,4,5]. The basic idea of DQ comes from Gauss quadrature or Integral Quadrature (IQ), which is a simple and useful numerical integration method [1,4]. In order to apply the DQ method to solve PDEs and achieve more accurate results, two extremely important issues show how to determine weighting coefficients and implement boundary conditions.

Bellman et al. (1972) [3] suggested two methods to determine the weighting coefficients of the first order derivative. The first method solves an algebraic equation system, with the drawback that when the order of the algebraic equation system is large, its matrix is ill-conditioned. The second uses a simple algebraic formulation with this restriction; that the coordinates of grid points have to be chosen as the roots of the shifted Legendre polynomial [1]. To overcome the drawbacks of the above methods, Quan and Chang (1989) [6], and Wen and Yu (1993) [7] used Lagrange interpolation polynomials as test functions and then obtained explicit formulations to determine the weighting coefficient for the first and second order derivatives discretization. More generally, Shu and Richards (1990) [8] and Shu (1991) [9] presented the Generalized Differential Quadrature (GDQ) in which all current methods for determination of weighting coefficients are generalized under the analysis of a high order polynomial approximation and the analysis of a linear vector space. In GDQ, the weighting coefficients of the first order derivative are determined by a simple algebraic formulation without any restriction on the choice of grid points, and the weighting coefficients of the second and higher order derivatives are determined by a recurrence relationship [1,5,10]. In recent years, the DQ method has become increasingly popular in numerical solutions of initial and boundary value problems. As described, the DQ method can yield accurate solutions with relatively few grid points [1,11]. The pioneering work for application of the DQ method to the general area of structural mechanics was carried out by Bert et al. [12-14], Wang and Bert [15], and Wang et al. [16,17]. Like some other numerical methods, the GDQ method discretizes the spatial derivates and, therefore, reduces the partial differential equations into a set of algebraic equations [18,19].

A differential equation is undetermined if boundary conditions are not provided, and for their GDQ solution, the boundary conditions have to be implemented appropriately to the resultant algebraic equations. For cases where there is only one boundary condition at each boundary, the implementation is very simple and can be done in a straightforward manner. One just needs to replace the discretized governing equations of GDQ by boundary condition equations at all boundary points. Difficulty, however, arises when applying multi-boundary conditions at each boundary, which could result in difficulties in the numerical implementation of the boundary conditions. For example, to solve fourth-order differential equations, where two boundary conditions are present at each end, Bert et al. (1988) [12], and Jang et al. (1989) [20] proposed the δ-technique in which a δ-point is introduced apart from the boundary point by a small distance, as an additional boundary point, and the other boundary condition is applied at that point. It is found, however, that solution accuracy may not be assured since δ is problem-dependent, and to obtain an accurate numerical solution, the δ should be chosen to be very small (possibly no greater than 0.0001l, where l is the length of the beam or the plate) [1,5,21]. Moreover there are some difficulties in applying the multi-boundary conditions accurately at the corner points for two-dimensional problems. There are several approaches available in the literature for implementing multi-boundary conditions. One is the replaced equation approach [22], where, instead of δ separation, two DQ equations at the inner grid points are replaced by the second boundary conditions. It is found, however, that solution accuracy may vary, depending on which DQ equations at the inner grid points are replaced by the boundary conditions [23]. Wang and Bert (1993) [15] introduced the Modifying Weighting Coefficient Matrices (MWCM) method. For this approach, only one boundary condition is numerically implemented and the other boundary conditions (derivative conditions) are built into the derivative weighting coefficient matrices. This method is very simple to use, but there are some difficulties in the application of this approach for non-homogeneous derivate conditions and some limitations in its application to implementation of some combinations of boundary condition, like Clamped-Clamped (C-C) boundary conditions. The other method, introduced by Shu and Du (1997) [5], can be referred to as direct Substitution of Boundary Conditions into discrete Governing Equations (SBCGE) [24]. This method substitutes the boundary conditions directly into the governing equation, and was proposed in order to implement simply supported, clamped conditions and their combinations [18,23].

In this paper, the proposed SBCGS approach will be explained, applied to a variety of beams with different boundary conditions and validated by the accuracy of the result. Also, the general solution for GDQ analysis of beam equilibrium and a particular solution for a cantilever beam will be obtained by the SBCGS method.

2. Generalized differential quadrature

In this section, we shall adopt the GDQ method developed by Shu [1,5,21]. Following the concept of classical integral quadrature, the first-order derivative of a smooth function, $f(x)$, with respect to x at x_i, can be approximated by DQ as:

$$\frac{df(x_i)}{dx}=\sum_{j=1}^{N}C_{ij}^{(1)}f(x_j),$$

$$\text{for}\qquad i=1,2,3,\cdots,N, \tag{1}$$

where N is the number of grid points in the whole computational domain and $C_{ij}^{(1)}$ indicates the weighting coefficients of the first order derivative. Following the same procedure as in discretization of the first order derivate, a recurrence relationship may be developed to calculate the weighting coefficients for higher-order derivatives. For example, the GDQ approximation of the nth-order derivative of a one-dimensional function, $f(x)$, with respect to x, is given by [21]:

$$\frac{d^nf(x_i)}{dx^n}=\sum_{j=1}^{N}C_{ij}^{(n)}f(x_j),$$

$$\text{for } i=1,2,3,\cdots,N;\qquad n=2,3,\cdots,N-1, \tag{2}$$

where the weighting coefficients for the first and nth-order derivatives can be computed by Shu's general approach as [21,25]:

$$\begin{cases} C_{ij}^{(1)}=\dfrac{\prod_{k=1,k\neq i}^{N}(x_i-x_k)}{(x_i-x_j).\prod_{k=1,k\neq j}^{N}(x_j-x_k)} \\ \qquad\text{for}\qquad i,j=1,2,\cdots,N;\qquad j\neq i, \\ C_{ii}^{(1)}=-\displaystyle\sum_{j=1,j\neq i}^{N}C_{ij}^{(1)} \\ \qquad\text{for}\qquad i=1,2,\cdots,N, \end{cases} \tag{3}$$

$$\begin{cases} C_{ij}^{(n)}=n\left[C_{ij}^{(1)}C_{ii}^{(n-1)}-\dfrac{C_{ij}^{(n-1)}}{x_i-x_j}\right] \\ \qquad\text{for}\qquad i,j=1,2,\cdots,N;\qquad j\neq i; \\ \qquad\qquad n=2,3,\cdots,N-1, \\ C_{ii}^{(n)}=-\displaystyle\sum_{j=1,j\neq i}^{N}C_{ij}^{(n)} \\ \qquad\text{for}\qquad i=1,2,\cdots,N; \\ \qquad\qquad n=2,3,\cdots,N-1. \end{cases} \tag{4}$$

It is noted that in multi-dimensional cases, it has been shown [1,5] that each direction can be treated using the same method as in the one-dimensional case.

3. Explanation of the SBCGS approach

To compute the nth-order derivative of a known smooth function, the GDQ method can be easily applied and written in the form of a matrix equation by discretization of its domain and calculation of the weighting coefficients matrix. For example, for a one-dimensional function, y, the procedure is as follows:

$$\left\{\frac{d^ny}{dx^n}\right\}_{N\times 1}=\left[C^{(n)}\right]_{N\times N}\{y\}_{N\times 1},$$

or:

$$\left\{y^{(n)}\right\}=\left[C^{(n)}\right]\{y\}, \tag{5}$$

where N, n and $[C^{(n)}]$ indicate the number of grid points, the order of the derivative and the weighting coefficients of the nth-order derivative, respectively. Also, $\{y\}$ and $\{y^{(n)}\}$ are the discretized functional values and their nth-order derivative values in the form of column matrices.

But, in cases where $\{y\}$ is undetermined and $\{y^{(n)}\}$ is determined in the above equation (a differential equation), direct application of the above matrix form is impossible because the determinant of the weighting coefficients matrix, $[C^{(n)}]$, is zero and its inverse does not exist. This is because a very important property of the weighting coefficient matrices implies that the rank of the weighting coefficient matrix for the nth-order derivative, $[C^{(n)}]$, is $(N-n)$. According to this property, the DQ approximation should only be applied at $(N-n)$ grid points. Otherwise, the resulting discretization matrix will be singular. This is a very interesting result, which is in good agreement with the well-post problem (the number of equations is equal to the number of unknowns, because the nth-order differential equation also requires n initial or boundary conditions, which provide n equations) [1].

Note that in the present paper, A, B, C and D indicate the weighting coefficients of 1st, 2nd, 3rd and 4th-order derivatives, respectively.

As a simple instance:

$$\frac{dy}{dx}=f,$$

where:

$$0\leq x\geq L,\qquad\text{and}\qquad y(0)=\alpha.$$

Here:

(f) is a known function,

(y) is an unknown function, (6)

$$\frac{dy}{dx}=f\Rightarrow[A]\{y\}=\{f\},$$

or:

$$[A]\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix},$$

$$\text{Rank of } [A] = (N-1) \text{ so } [A]^{-1}\text{: Does not exist.} \tag{7}$$

The SBCGS approach overcomes this drawback by deriving n rows and their rival columns out of $[C^{(n)}]$ and, thereupon, converts it to a full-rank matrix, whose inverse does exist. For this purpose, it supposes that the functional values at n points are known and are indicated by some constants (it is better to select boundary points for this purpose). Thus, n equations must be eliminated, and to change the coefficients matrix to square form, it derives the rival columns of those n rows and transfers them to the other side of the equation. For example, in Eq. (6), which is a first-order differential equation discretized by GDQ and written in the matrix form as Eq. (7), the procedure is as follows:

$$[A]\{y\} = \{f\},$$

or:

$$[A]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{Bmatrix}.$$

Supposing: $y_1 = c_1$:

$$[A]\begin{Bmatrix} c_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{Bmatrix}, \tag{8}$$

Elimination of the extra equation:

$$\begin{bmatrix} - & - & - & \cdots & - \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3N} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} c_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} - \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{Bmatrix}. \tag{9}$$

Deriving out the rival column:

$$\begin{bmatrix} - & - & - & \cdots & - \\ | & a_{22} & a_{23} & \cdots & a_{2N} \\ | & a_{32} & a_{33} & \cdots & a_{3N} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ | & a_{N2} & a_{N3} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} - \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} - \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{Bmatrix} - c_1 \begin{Bmatrix} - \\ a_{21} \\ a_{31} \\ \vdots \\ a_{N1} \end{Bmatrix},$$

or:

$$[|\overline{A}]\,\{\overline{y}\} = \{\overline{f}\} - c_1\{\overline{A}_1\}. \tag{10}$$

It is noted that for simplicity in formulation throughout this paper, the elimination of a row or column in a matrix has been indicated by dashed rows or dashed columns, respectively. For example, $[|\overline{A}]$ is matrix $[A]$, whose first row and first column are eliminated. Also, $\{A_k\}$ and $\langle A_k\rangle$ indicate the kth column and kth row of matrix $[A]$.

Now, by multiplying the inverse of the coefficients matrix to both sides of the equation and returning the eliminated constant to their places in the achieved solution, the general solution of the differential equation will be governed (like Eq. (11), which is the general solution for Eq. (7)). Note that this solution is not the final and particular one, because it includes n unknown constants that must be evaluated by n initial or boundary conditions. So, it is referred to as the general solution of the differential equation.

$$\{\overline{y}\}_{(N-1)\times 1} = [|\overline{A}]^{-1}\left\{\{\overline{f}\} - c_1\{\overline{A}_1\}\right\}.$$

Returning the unknown constant to its place:

$$\{y\}_{N\times 1} = \begin{Bmatrix} c_1 \\ \{\overline{y}\} \end{Bmatrix}: \text{ General solution.} \tag{11}$$

Finally, by implementing the discretized boundary condition equations exactly at the boundary points to the general solution, particular solution of the differential equation will be obtained, which satisfies both discretized differential and boundary condition equations (like Eq. (12), which governs the particular solution for Eq. (7)).

$$\text{Eq. (1)} \Rightarrow \text{BC.}: y(0) = \alpha.$$

Discretization:

$$y_1 = \alpha.$$

Eq. (11)

$$c_1 = \alpha \Rightarrow \{y\} = \begin{Bmatrix} \alpha \\ \{\overline{y}\} \end{Bmatrix},$$

where:

$$\{\overline{y}\}=[|\overline{A}]^{-1}\{\{\overline{f}\}-\alpha\{\overline{A}_1\}\}: \text{ Particular solution.} \quad (12)$$

Note that this methodology of solution is general for any type of linear differential equation and boundary condition and there is no difference in the procedure of using this approach to solve them. Also, it is very important to know that choosing the unknown constants at each point except from those of the boundary (choosing from inner points), does not mean that implementation of boundary conditions is out of the boundary points, because this part of the operation is done to govern the general solution and implementation of boundary conditions will be done, finally, exactly at the boundary points. It does, however, provide more accurate results to choose unknown constants from the boundary points and extra ones from boundary adjacent points (for cases in which the order of the differential equation is more than two, because the number of boundary points in each direction is two).

4. Discretization of equations by GDQ

In this section, the GDQ method is applied to discrete the differential equation of a Euler-Bernoulli beam under lateral loading with various boundary conditions, which is governed by the following fourth-order differential equation:

$$\frac{d^4y}{dx^4}=\frac{f(x)}{\mathrm{EI}}, \qquad 0<x>L,$$

or:

$$\frac{d^4y}{dx^4}=q(x), \qquad 0<x>L,$$

where:

$$q(x)=\frac{f(x)}{\mathrm{EI}}, \quad (13)$$

where EI is the flexural rigidity of the beam, $f(x)$ is the lateral distributed load, and L is the length of the beam. Eq. (13) is a 4th order ordinary differential equation, which requires four boundary conditions to be a well-posed problem. These can be given by specifying two boundary conditions at the end, $x=0$, and the other two at the end, $x=L$. In this paper, the following four non-homogeneous types of boundary condition are considered:
At $x=0$ or $x=L$
Simply Supported end (SS):

$$y=\alpha_1, \qquad \text{and} \qquad M=-\mathrm{EI}\frac{d^2y}{dx^2}=\alpha_2, \quad (14)$$

where α_1 and α_2 are non-zero displacement and non-zero concentrated moment, respectively.
Clamped end (C):

$$y=\beta_1, \qquad \text{and} \qquad \theta=\frac{dy}{dx}=\beta_2, \quad (15)$$

where β_1 and β_2 are non-zero displacement and non-zero slope, respectively.
Free end (F):

$$V=-\mathrm{EI}\frac{d^3y}{dx^3}=\gamma_1, \quad \text{and} \quad M=-\mathrm{EI}\frac{d^2y}{dx^2}=\gamma_2, \quad (16)$$

where γ_1 and γ_2 are non-zero concentrated load and non-zero concentrated moment, respectively.
Guided end (G) (or sliding support which can freely slide perpendicular to the beam direction):

$$V=-\mathrm{EI}\frac{d^3y}{dx^3}=\delta_1, \qquad \text{and} \qquad \theta=\frac{dy}{dx}=\delta_2, \quad (17)$$

where δ_1 and δ_2 are non-zero concentrated load and non-zero slope, respectively.

Note that these boundary conditions are homogeneous when the constants defined in the above equations are zero.

For numerical computation, the continuous solution is approximated by the function values at discrete points. Now, the computational domain, $0<x>L$, is divided by $(N-1)$ intervals with coordinates of grid points as $x_1, x_2, \cdots, x_N$. With these coordinates of the grid points, the GDQ method can be applied to compute the weighting coefficient through Eqs. (3) and (4). Then, by applying the GDQ method to discrete the spatial derivatives, Eq. (13) yields:

$$\sum_{j=1}^{N} C_{ij}^{(4)} y_j = q_i, \quad (18)$$

where $C_{ij}^{(4)}$, $i,j=1,2,\cdots,N$, are the weighting coefficients of the 4th-order derivative, and y_i and q_i are the values of $y(x)$ and $q(x)$ at the grid points, x_i. Similarly, the derivatives in the boundary conditions can be discretized by the GDQ method. As a result, the numerical boundary conditions can be written as:
Simply Supported end (SS):

for the end of $x=0$:

$$y_1=\alpha_1, \qquad \text{and} \qquad M_1=-\mathrm{EI}_1\sum_{j=1}^{N} C_{1j}^{(2)} y_j=\alpha_2,$$

for the end of $x=L$:

$$y_N=\alpha_3, \qquad \text{and} \qquad M_N=-\mathrm{EI}_N\sum_{j=1}^{N} C_{Nj}^{(2)} y_j=\alpha_4. \quad (19)$$

Clamped end (C):

for the end of $x = 0$:

$$y_1 = \beta_1, \qquad \text{and} \qquad \theta_1 = \sum_{j=1}^{N} C_{1j}^{(1)} y_j = \beta_2,$$

for the end of $x = L$:

$$y_N = \beta_3, \qquad \text{and} \qquad \theta_N = \sum_{j=1}^{N} C_{Nj}^{(1)} y_j = \beta_4. \tag{20}$$

Free end (F):

for the end of $x = 0$:

$$V_1 = -\text{EI}_1 \sum_{j=1}^{N} C_{1j}^{(3)} y_j = \gamma_1,$$

$$M_1 = -\text{EI}_1 \sum_{j=1}^{N} C_{1j}^{(2)} y_j = \gamma_2,$$

for the end of $x = L$:

$$V_N = -\text{EI}_N \sum_{j=1}^{N} C_{Nj}^{(3)} y_j = \gamma_3,$$

$$M_N = -\text{EI}_N \sum_{j=1}^{N} C_{Nj}^{(2)} y_j = \gamma_4. \tag{21}$$

Guided end (G):

for the end of $x = 0$:

$$V_1 = -\text{EI}_1 \sum_{j=1}^{N} C_{1j}^{(3)} y_j = \delta_1,$$

$$\theta_1 = \sum_{j=1}^{N} C_{1j}^{(1)} y_j = \delta_2,$$

for the end of $x = L$:

$$V_N = -\text{EI}_N \sum_{j=1}^{N} C_{Nj}^{(3)} y_j = \delta_3,$$

$$\theta_N = \sum_{j=1}^{N} C_{Nj}^{(1)} y_j = \delta_4, \tag{22}$$

where $\text{EI}_1, \text{EI}_2, \cdots, \text{EI}_N$ are constant and equal to the flexural rigidity of beam (EI)

5. Governing the general solution of the Bernoulli-beam by the SBCGS approach

In the present section, the SBCGS approach will be used to obtain the general solution of Eq. (18). Then, an explicit formulation will be governed as the general solution for the GDQ analysis of the Bernoulli-beam. Finally, any boundary conditions can be implemented easily by satisfying them into the general solution, and, so, the particular solution will be obtained in the next section.

Eq. (18) can be written in matrix form as follows:

$$[D]\{y\} = \{q\}. \tag{23}$$

As described in Section 3, the rank of matrix D is $(N-4)$, so, we need to suppose four unknown constants in the solution progress. Now, in order to apply the SBCGS approach to Eq. (23), all the steps described in Section 3 are followed exactly:
Supposing:

$$y_1 = c_1, \qquad y_2 = c_2,$$

$$y_{N-1} = c_3, \qquad y_N = c_4,$$

$$[D] \begin{Bmatrix} c_1 \\ c_2 \\ y_3 \\ \vdots \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{N-1} \\ q_N \end{Bmatrix}. \tag{24}$$

Elimination of the extra equations:

$$\left[\underline{\underline{\overline{\overline{D}}}}\right] \begin{Bmatrix} c_1 \\ c_2 \\ y_3 \\ \vdots \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} - \\ - \\ q_3 \\ \vdots \\ - \\ - \end{Bmatrix}. \tag{25}$$

Deriving out the rival columns:

$$\left[\|\underline{\underline{\overline{\overline{D}}}}\|\right] \left\{\underline{\underline{\overline{\overline{y}}}}\right\} = \left\{\underline{\underline{\overline{\overline{q}}}}\right\} - c_1 \left\{\underline{\underline{\overline{\overline{D}}}}_1\right\} - c_2 \left\{\underline{\underline{\overline{\overline{D}}}}_2\right\} - c_3 \left\{\underline{\underline{\overline{\overline{D}}}}_{N-1}\right\} - c_4 \left\{\underline{\underline{\overline{\overline{D}}}}_N\right\}. \tag{26}$$

Multiplying the inverse of coefficient matrix:

$$\left\{\underline{\underline{\overline{\overline{y}}}}\right\}_{(N-4)\times 1} = \left[\|\underline{\underline{\overline{\overline{D}}}}\|\right]^{-1} \left\{ \left\{\underline{\underline{\overline{\overline{q}}}}\right\} - c_1 \left\{\underline{\underline{\overline{\overline{D}}}}_1\right\} - c_2 \left\{\underline{\underline{\overline{\overline{D}}}}_2\right\} - c_3 \left\{\underline{\underline{\overline{\overline{D}}}}_{N-1}\right\} - c_4 \left\{\underline{\underline{\overline{\overline{D}}}}_N\right\} \right\}. \tag{27}$$

Returning the unknown constants to their places:

$$\{y\}_{N\times 1} = \begin{Bmatrix} c_1 \\ c_2 \\ \{\overline{\underline{\overline{y}}}\} \\ c_3 \\ c_4 \end{Bmatrix} : \text{ General solution.} \tag{28}$$

For more simplification and to obtain an explicit and usable formulation for the general solution, it can be written in separated matrix form as below:

$$\{y\}_{N\times 1} = \{Q\}_{N\times 1} + [T]_{N\times 4}\{S\}_{4\times 1}, \tag{29}$$

where:

$$[T]_{N\times 4} = [\{T_1\}, \{T_2\}, \{T_3\}, \{T_4\}],$$

and:

$$\{Q\}_{N\times 1} = \begin{Bmatrix} 0 \\ 0 \\ \left[\|\overline{\underline{\overline{D}}}\|\right]^{-1} \{\overline{\underline{\overline{q}}}\} \\ 0 \\ 0 \end{Bmatrix},$$

$$\{T_1\}_{N\times 1} = \begin{Bmatrix} 1 \\ 0 \\ -\left[\|\overline{\underline{\overline{D}}}\|\right]^{-1} \{\overline{\underline{\overline{D}}}_1\} \\ 0 \\ 0 \end{Bmatrix},$$

$$\{T_2\}_{N\times 1} = \begin{Bmatrix} 0 \\ 1 \\ -\left[\|\overline{\underline{\overline{D}}}\|\right]^{-1} \{\overline{\underline{\overline{D}}}_2\} \\ 0 \\ 0 \end{Bmatrix},$$

$$\{T_3\}_{N\times 1} = \begin{Bmatrix} 0 \\ 0 \\ -\left[\|\overline{\underline{\overline{D}}}\|\right]^{-1} \{\overline{\underline{\overline{D}}}_{N-1}\} \\ 1 \\ 0 \end{Bmatrix},$$

$$\{T_4\}_{N\times 1} = \begin{Bmatrix} 0 \\ 0 \\ -\left[\|\overline{\underline{\overline{D}}}\|\right]^{-1} \{\overline{\underline{\overline{D}}}_N\} \\ 0 \\ 1 \end{Bmatrix},$$

$$\{S\}_{4\times 1} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}.$$

So, the general solution of the Bernoulli-beam is governed with an explicit and simple formulation as below:

$$\{y\} = \{Q\} + [T]\{S\}. \tag{30}$$

Now, we can easily implement the homogeneous and non-homogeneous boundary conditions exactly at the boundary points by satisfying them into the general solution in Eq. (30). Then, by solving the resulting algebraic equation system, the unknown constants will be evaluated. Finally, the particular solution of each boundary condition will be obtained by substituting the value of unknown constants back into the general solution of Eq. (30).

6. Implementation of boundary conditions and obtaining particular solutions

There is no difference as to which different boundary condition is implemented, because the procedures for implementation of all of them are the same as in the present method. In this section, we will govern the particular solution for a cantilever beam as an example, and the other conditions can be implemented as the same procedure.

For a cantilever beam, the non-homogeneous boundary conditions are:

$$\begin{cases} \text{Clamped at } x = 0: \\ y_1 = \beta_1, \qquad \theta_1 = \sum_{j=1}^{N} C_{1j}^{(1)} y_j = \beta_2 \\ \\ \text{Free at } x = L: \\ V_N = -\text{EI}_N \sum_{j=1}^{N} C_{Nj}^{(3)} y_j = \gamma_1 \\ M_N = -\text{EI}_N \sum_{j=1}^{N} C_{Nj}^{(2)} y_j = \gamma_2 \end{cases} \tag{31}$$

where β_1, β_2, γ_1 and γ_2 indicate the deflection at $x = 0$, the slope at $x = 0$, the concentrated lateral load at $x = L$ and the concentrated moment at $x = L$, respectively.

By writing these four equations for boundary conditions in matrix form, we have:

$$\begin{cases} y_1 = \beta_1 \\ \langle A_1\rangle\{y\} = \beta_2 \\ \langle C_N\rangle\{y\} = -\frac{\gamma_1}{\text{EI}_N} \\ \langle B_N\rangle\{y\} = -\frac{\gamma_2}{\text{EI}_N} \end{cases} \tag{32}$$

Note that $\{A_k\}$ and $\langle A_k\rangle$ indicate the kth column and kth row of matrix A.

By substituting these four equations into Eq. (30), an algebraic equation system consisting of four equations will be obtained that provides the value of four unknown constants in matrix S:

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ -\frac{\gamma_1}{\mathrm{EI}_N} \\ -\frac{\gamma_2}{\mathrm{EI}_N} \end{Bmatrix}_{4\times 1} = \begin{Bmatrix} Q_1 \\ \langle A_1\rangle\{Q\} \\ \langle C_N\rangle\{Q\} \\ \langle B_N\rangle\{Q\} \end{Bmatrix}_{4\times 1} + \begin{bmatrix} \langle T_1\rangle \\ \langle A_1\rangle[T] \\ \langle C_N\rangle[T] \\ \langle B_N\rangle[T] \end{bmatrix}_{4\times 4} \{S\}_{4\times 1} \Rightarrow \{S\} = \begin{bmatrix} \langle T_1\rangle \\ \langle A_1\rangle[T] \\ \langle C_N\rangle[T] \\ \langle B_N\rangle[T] \end{bmatrix}^{-1} \left[\begin{Bmatrix} \beta_1 \\ \beta_2 \\ -\frac{\gamma_1}{\mathrm{EI}_N} \\ -\frac{\gamma_2}{\mathrm{EI}_N} \end{Bmatrix} - \begin{Bmatrix} Q_1 \\ \langle A_1\rangle\{Q\} \\ \langle C_N\rangle\{Q\} \\ \langle B_N\rangle\{Q\} \end{Bmatrix} \right]. \quad (33)$$

Last, by substituting matrix S into the general solution in Eq. (30), the particular solution for a cantilever beam with non-homogeneous boundary conditions will be governed in Eq. (33).

It is noted that the general solution is the same for each boundary condition, but matrix S and its resultant particular solution differs for each one of the boundary conditions. Here, we provide the particular solution for a cantilever beam with non-homogeneous boundary conditions as an example, and the particular solution of other types of boundary condition can be obtained in a similar procedure. Note that for the purpose of using the proposed formulations, we need just to calculate the following matrices: A, B, C, T and Q, where A, B and C are the weighting coefficient matrices. So, just two matrices (T and Q) are new in the formulations, which can easily be derived from other existing matrices.

7. Results and discussion

In this paper, the proposed SBCGS method is applied to the GDQ analysis of a Bernoulli-beam with a variety of boundary conditions. The analytical solutions for the Bernoulli-beam equation will be used to validate this approach, and a comparison will be made between SBCGS and SBCGE for the accuracy of their results, in any combination of simply supported and clamped boundary conditions. For this aim, a Bernoulli-beam with constant flexural rigidity is subjected under a uniform lateral load per unit of length with different boundary conditions. The coordinates of grid points are chosen as Chebyshev-Gauss-Lobatto points [24,26]:

$$X_i = \frac{L}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right], \qquad i = 1, 2, \cdots, N. \quad (34)$$

The results are presented in Table 1. It shows the accuracy of this approach for a variety of boundary conditions with only nine grid points. The accuracy of the SBCGE method results is also illustrated.

Table 1 indicates that SBCGS is a general approach to implementing any boundary condition, and its accuracy is in the range of other accurate methods like SBCGE, because it satisfies boundary conditions exactly at boundary points. So, SBCGS agrees very well with analytical results and makes a very inconsiderable percentage of error. Also, virtual storage, running time and computational effort were in acceptable ranges in comparison with other methods. The error percent between SBCGS (or SBCGE) results and exact analytical results (Err.%) in Table 1 is defined as:

$$\text{Err. \%} = \text{relative diff. with exact result \%} = 100 \times \left| \frac{\text{SBCGS (or SBCGE) result -exact analytical result}}{\text{exact analytical result}} \right|. \quad (35)$$

Note that error percentage values presented in Table 1 for each boundary condition are the maximum value in the whole domain of that case.

Table 2 shows that GDQ analysis of a cantilever beam using the SBCGS method provides very good results with a low number of mesh points for each type of Chebyshev or uniform grid spacing. It also indicates that Chebyshev grid spacing provides more accurate results than a uniform one. However, by increasing the number of mesh points, the accuracy of the SBCGS method decreases, due to an increase in the amount of computational effort.

Table 1. Err.% for two approach results (SBCGS and SBCGE) in the GDQ analysis of Bernouli-beam with different boundary conditions ($N = 9$).

		BCs.					
		S-S	**C-C**	**S-C**	**G-C**	**F-C**	**G-S**
Error %	**SBCGS**	<5E-12	<9E-13	<5E-13	<2E-11	<2E-9	<1E-10
	SBCGE	<5E-12	<5E-13	<6E-12	-	-	-

Table 2. Err.% for GDQ analysis of cantilever beam with different method and number of grid spacing.

Number of grid points	Err.%	
	Chebyshev grids	Uniform grids
6	<6E-12	<3E-11
7	<5E-11	<2E-10
8	<2E-10	<6E-11
9	<2E-9	<2E-9
10	<2E-9	<5E-10
11	<2E-8	<4E-8
12	<3E-8	<5E-8
13	<3E-8	<4E-6
14	<2E-8	<3E-6
15	<8E-8	<1E-5
16	<2E-7	<4E-5

8. Conclusions

In this paper, a new approach is proposed for implementing any homogeneous and non-homogeneous boundary condition in the GDQ analysis of beams. Like an analytical solution of a differential equation, this approach firstly governs a general solution for the GDQ discretized equations by assuming n unknown constants (n is the order of differential equation) as known answers to change the coefficients matrix to a full rank matrix. It then obtains the general solution, which contains n unknown constants, by pre-multiplying the inverse of the coefficient matrix to both sides of the equation. Finally, it evaluates the amount of unknown constant by satisfying n boundary conditions in the general solution, and so, the particular solution for any boundary condition will be obtained. It is, therefore, referred to as Satisfying Boundary Conditions in General Solution (SBCGS). In addition, in this paper, a general solution for GDQ discretized equations of a Bernoulli-beam is presented and the particular solution for a cantilever beam (Clamped-Free boundary conditions) is also governed by the SBCGS method. Note that the general solution is written in the form of a simple and explicit matrix equation to further simplify its format. The accuracy of the results for a variety of boundary conditions, in comparison with exact analytical results for SBCGS and SBCGE, is presented in Table 1, and it is shown in Table 2 that the accuracy of the SBCGS method decreases by increasing the number of mesh points. Finally, results show that the SBCGS approach works very well with any boundary conditions and the relative errors are very small and negligible. Also, the SBCGS approach has some high grade advantages, as follows: easiness and generality of use for any linear differential equation with proper boundary conditions; no difference between the implementation procedure of any homogeneous and non-homogeneous boundary condition; satisfying boundary conditions exactly at boundary points; high accuracy and less computational effort, and, finally, by using the SBCGS approach, the GDQ method will be expandable for analyzing continuous beams as a result of the implementation of non-homogeneous boundary conditions.

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